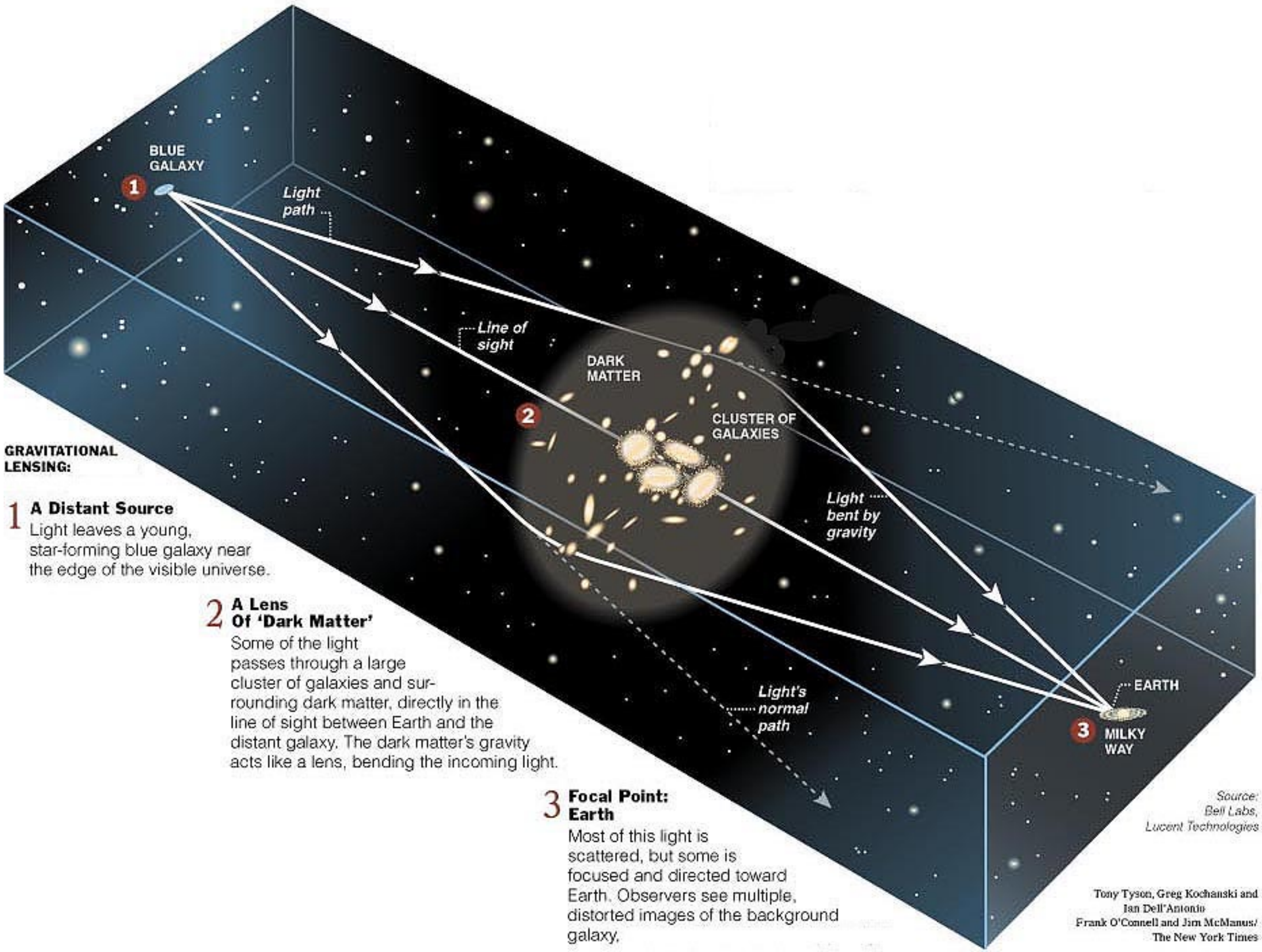
Alexander Knebe, *Universidad Autonoma de Madrid*


BLUE GALAXY
1
Light path
Line of sight
DARK MATTER
2
CLUSTER OF GALAXIES
Light bent by gravity
Light's normal path
EARTH
3
MILKY WAY
GRAVITATIONAL LENSING:
1 A Distant Source
Light leaves a young, star-forming blue galaxy near the edge of the visible universe.
2 A Lens Of 'Dark Matter'
Some of the light passes through a large cluster of galaxies and surrounding dark matter, directly in the line of sight between Earth and the distant galaxy. The dark matter's gravity acts like a lens, bending the incoming light.
3 Focal Point: Earth
Most of this light is scattered, but some is focused and directed toward Earth. Observers see multiple, distorted images of the background galaxy.
Source: Bell Labs, Lucent Technologies
Tony Tyson, Greg Kochanski and Ian Dell'Antonio
Frank O'Connell and Jim McManus/ The New York Times

...and now some theory!

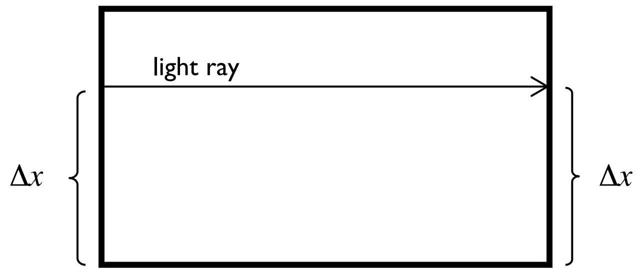
- theory
  - the basics of lensing…
  - some sample lenses…

- theory
  - the basics of lensing…
    - the lens equation
    - the lensing potential
    - critical surface mass density
    - magnification
    - caustics and critical curves
    - distortion
    - mass-sheet degeneracy
  - some sample lenses…

- theory
  - the basics of lensing…
    - the lens equation
    - the lensing potential
    - critical surface mass density
    - magnification
    - caustics and critical curves
    - distortion
    - mass-sheet degeneracy
  - some sample lenses…
    - point mass
    - extended mass
    - singular isothermal sphere

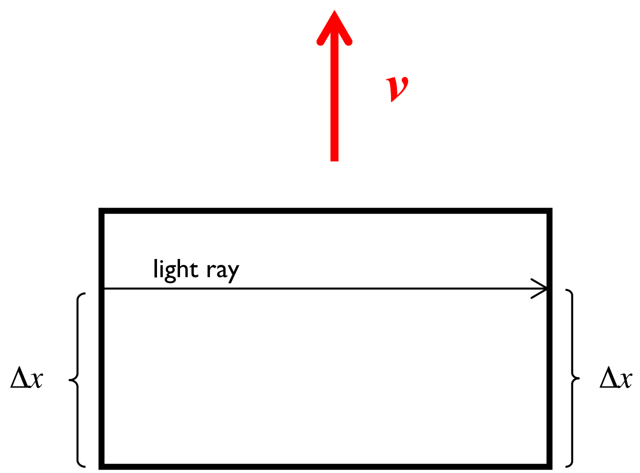
- theory
  - **the basics of lensing…**
    - the lens equation
    - the lensing potential
    - critical surface mass density
    - magnification
    - caustics and critical curves
    - distortion
    - mass-sheet degeneracy
  - some sample lenses…
    - point mass
    - extended mass
    - singular isothermal sphere

- lensing in general
  - laboratory at rest:



▪ lensing in general

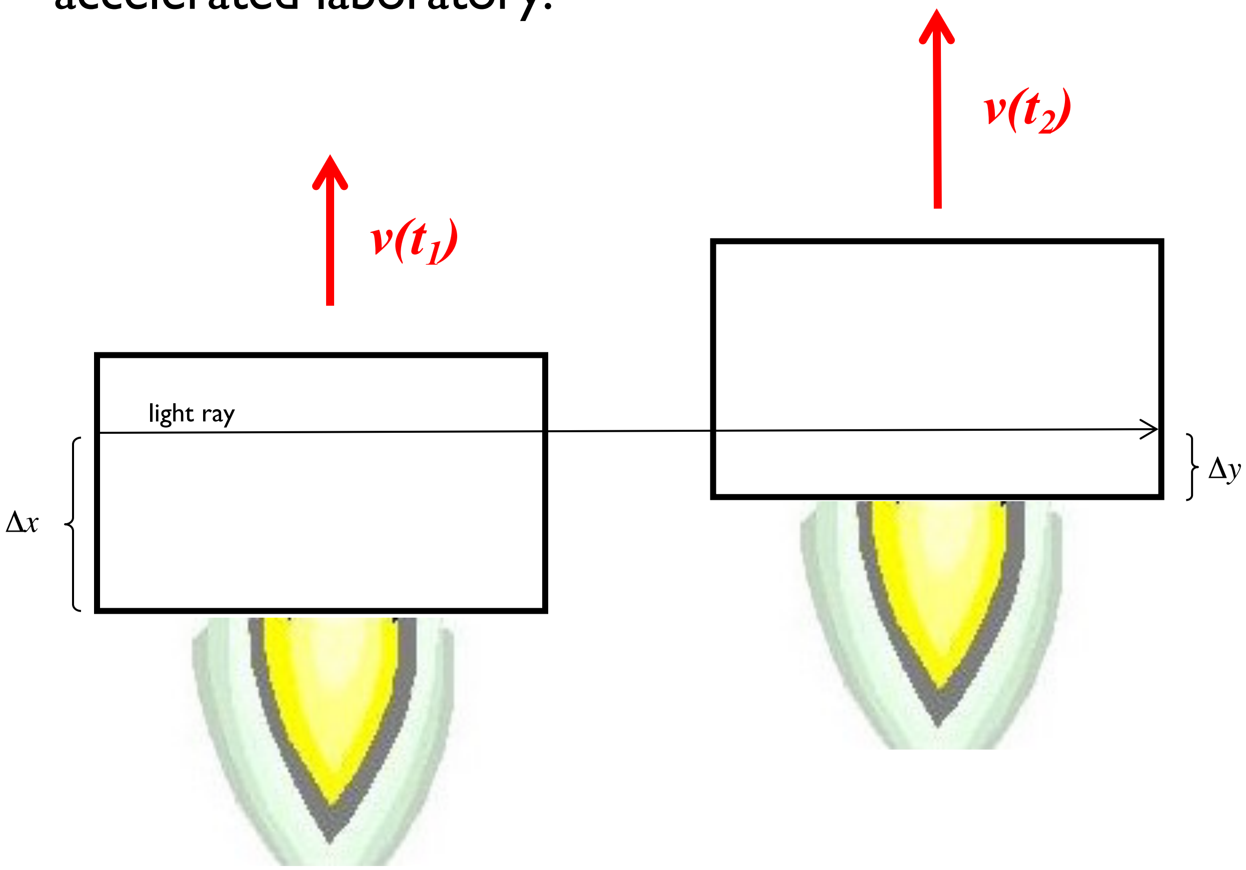
- laboratory at constant velocity:



Newton's 1. Law:
law of inertia

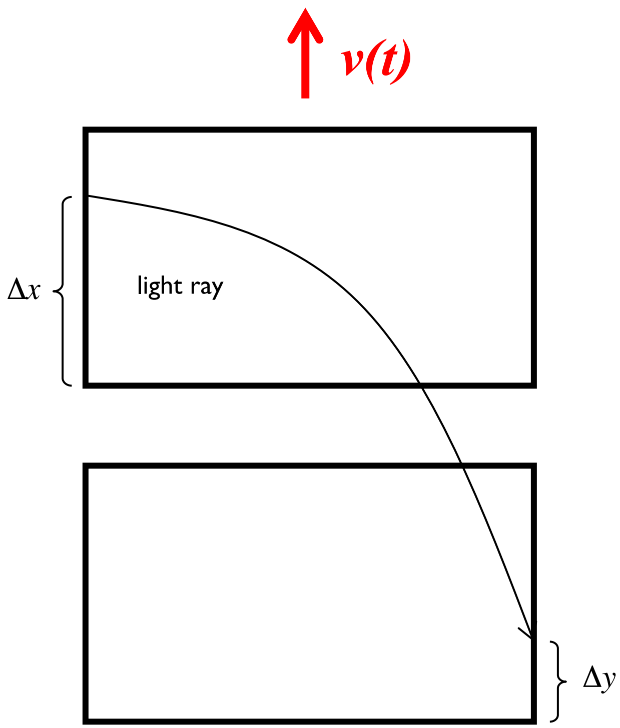
- lensing in general

  - accelerated laboratory:



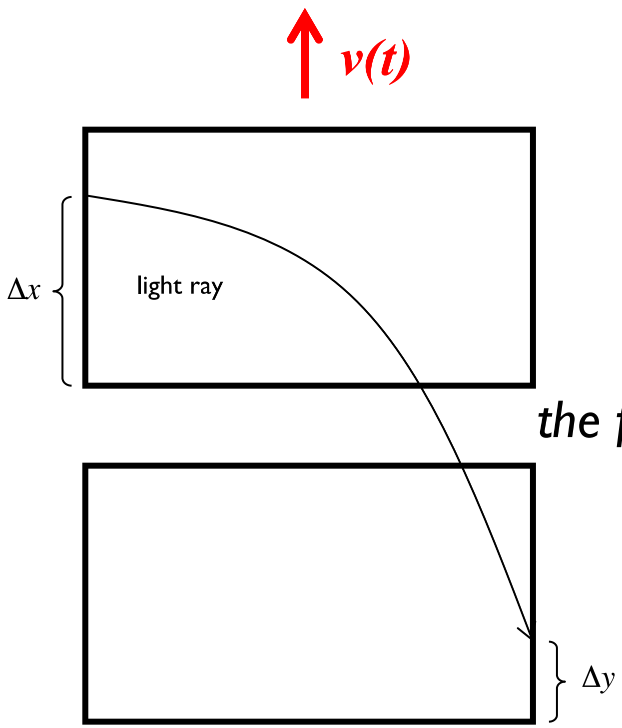
- lensing in general

  - accelerated laboratory:



- lensing in general

  - accelerated laboratory:

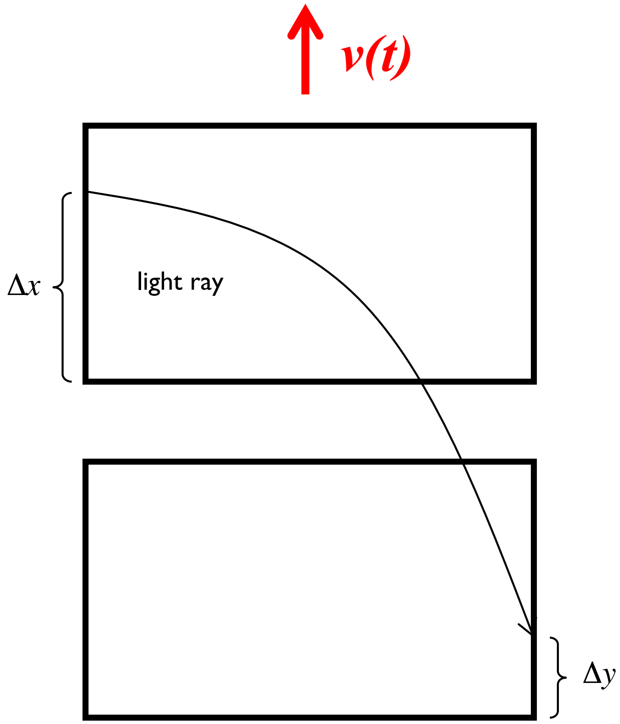


strong equivalence principle:*
the forces of gravitation and acceleration are equivalent.

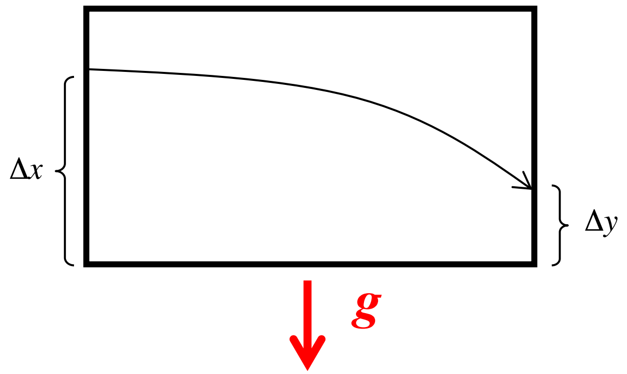
*see Cosmology lecture #1

- lensing in general

  - accelerated laboratory:

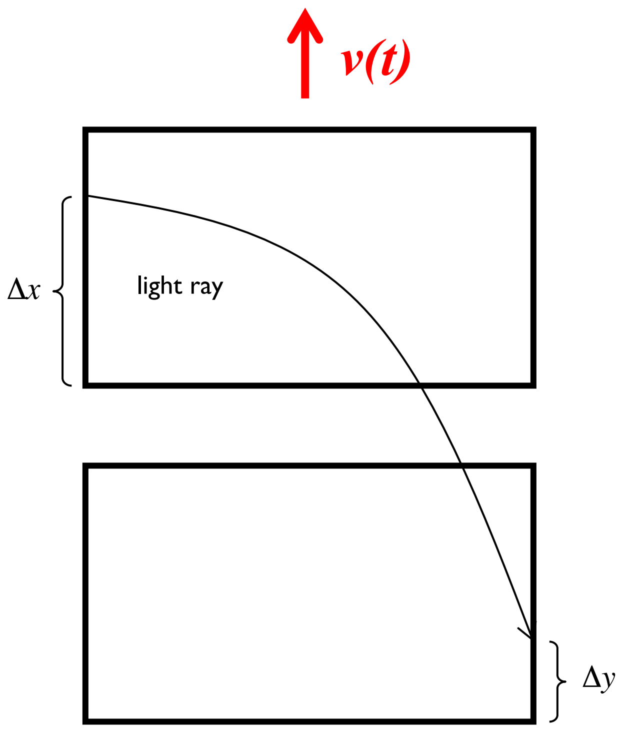


  - laboratory in gravity field:

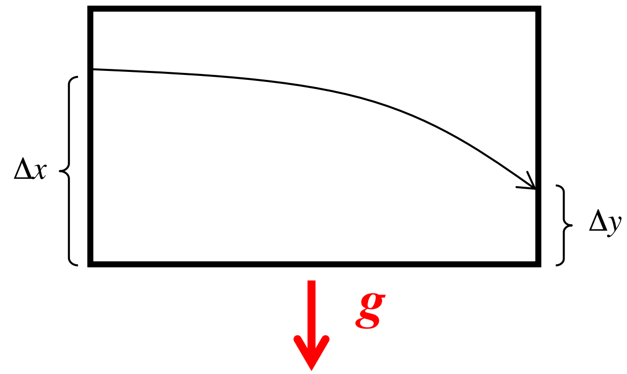


▪ lensing in general

- accelerated laboratory:



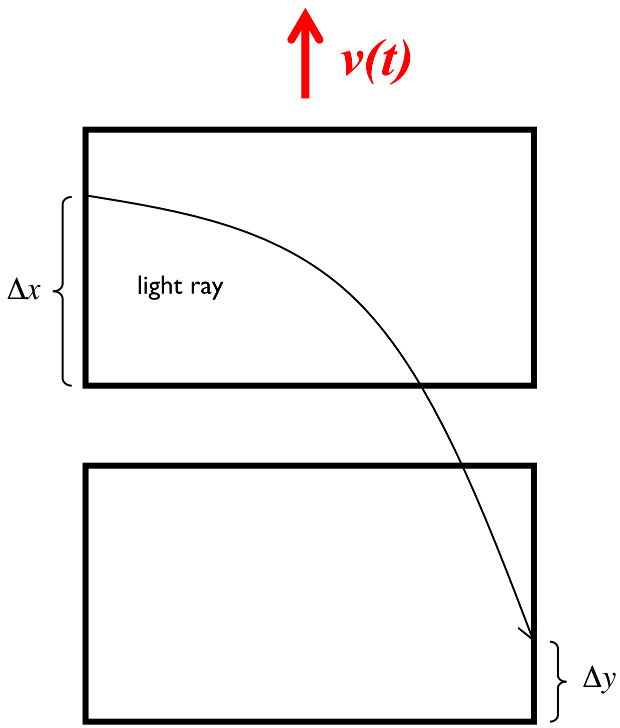
- laboratory in gravity field:



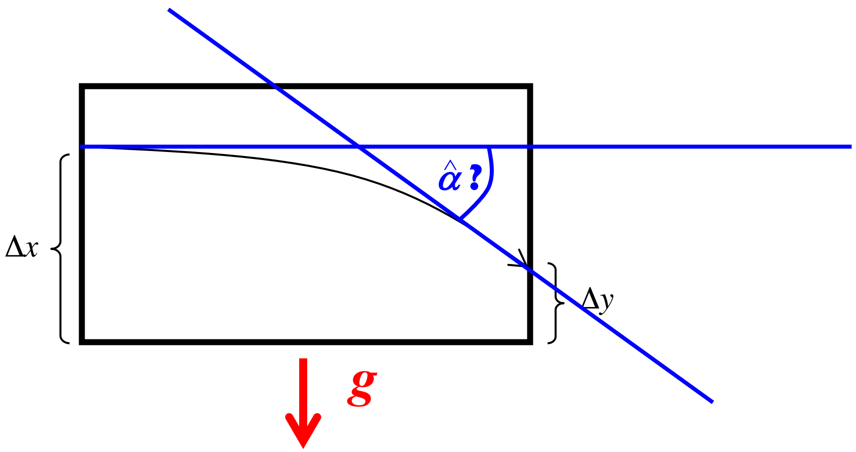
light ray also feels gravity!

- lensing in general

  - accelerated laboratory:

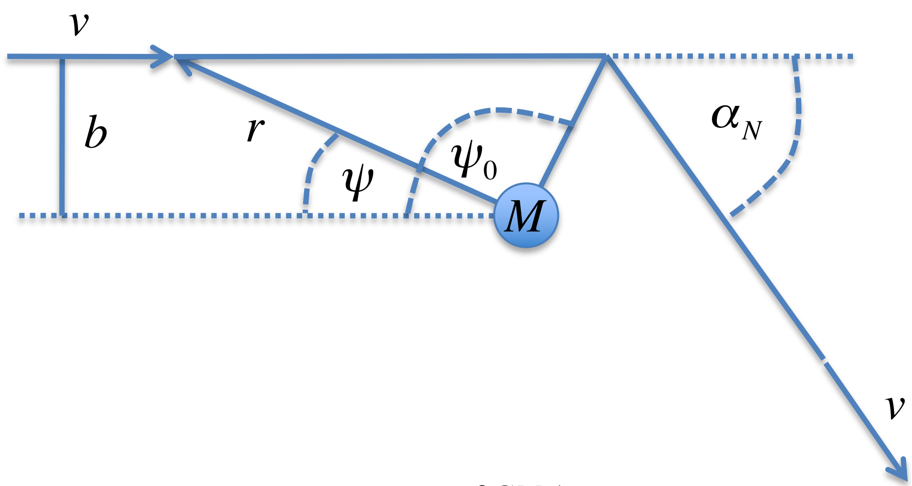


  - laboratory in gravity field:



light ray also feels gravity!

- analogy to gravity ***(exercise)***



$$\alpha_N = \frac{2GM}{c^2}\frac{1}{b}$$

- analogy to optics

- analogy to optics

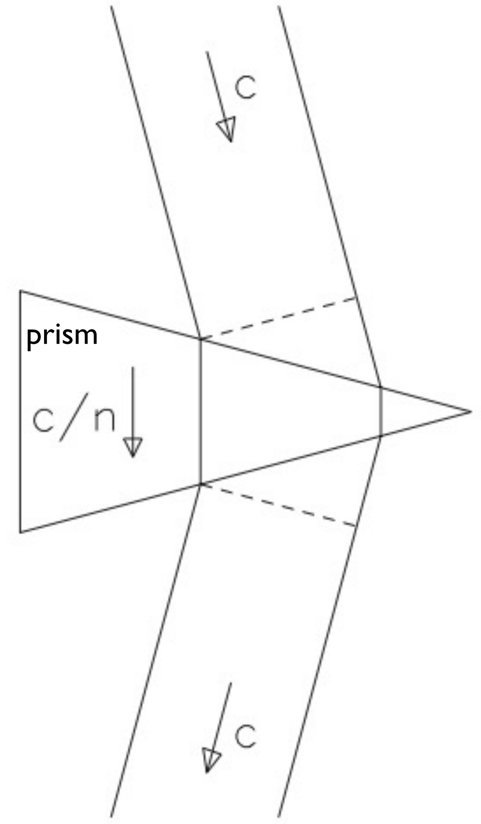


figure taken from Narayan & Bartelmann (1995)

- analogy to optics

  - effective index of refraction (in optics)

$$n = \frac{c}{v}$$

the refractive index $n > 1$ of the glass in the prism reduces the effective speed of light…

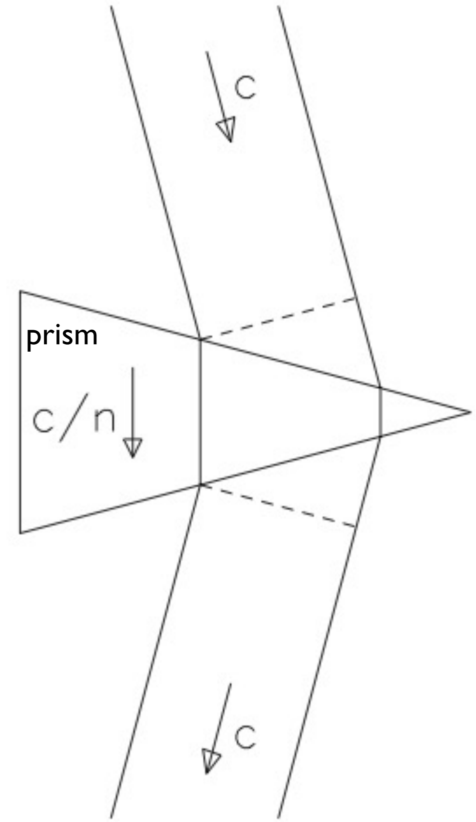
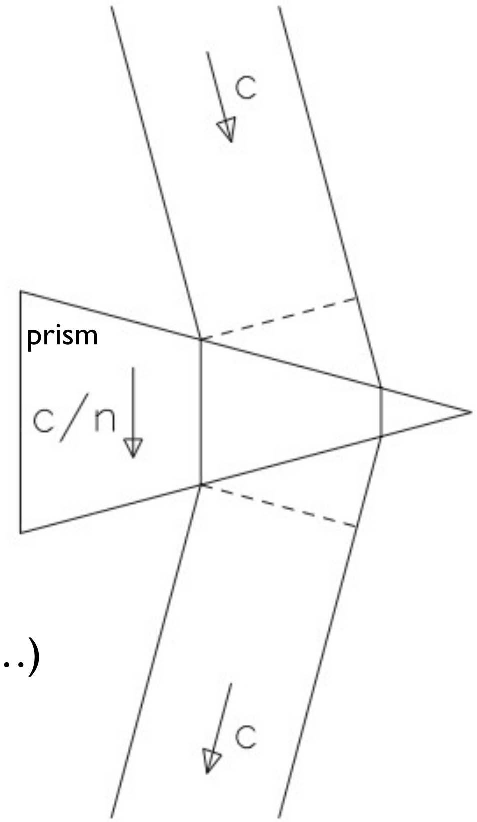


figure taken from Narayan & Bartelmann (1995)

- analogy to optics
  - effective index of refraction (in optics)

$$n = \frac{c}{v}$$

the refractive index $n > 1$ of the glass in the prism reduces the effective speed of light…



  - effective index of refraction (in gravity, post-Newtonian…)

$$n = \frac{c}{v}$$

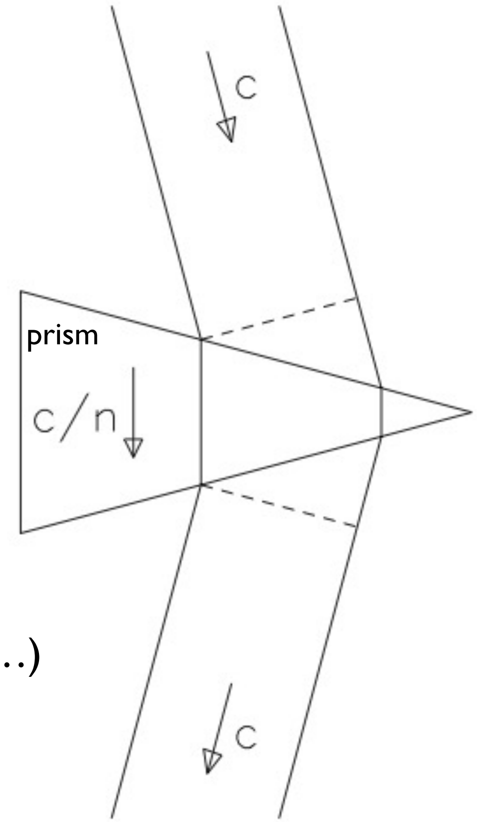
figure taken from Narayan & Bartelmann (1995)

▪ analogy to optics

- effective index of refraction (in optics)

$$n = \frac{c}{v}$$

the refractive index $n > 1$ of the glass in the prism reduces the effective speed of light…



- effective index of refraction (in gravity, post-Newtonian…)

$$n = \frac{c}{v} = ?$$ *any idea?*

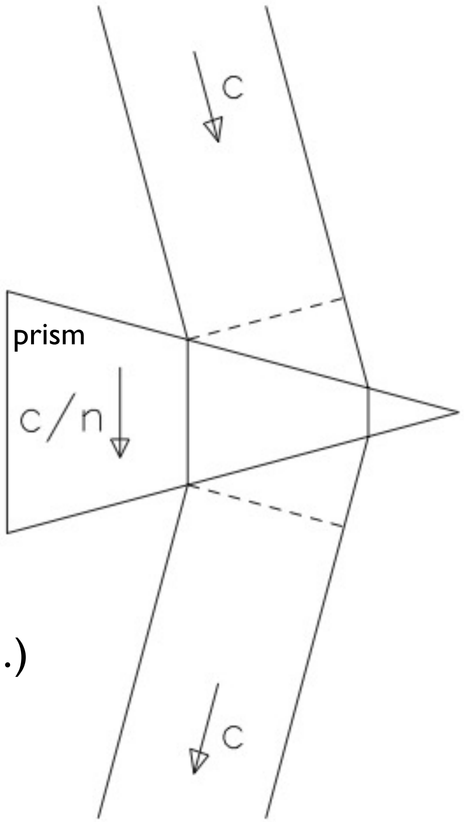
figure taken from Narayan & Bartelmann (1995)

- analogy to optics

  - effective index of refraction (in optics)

$$n = \frac{c}{v}$$

the refractive index $n > 1$ of the glass in the prism reduces the effective speed of light…



  - effective index of refraction (in gravity, post-Newtonian…)

$$n = \frac{c}{v} = ?$$ *any idea?*

we need to somehow calculate v*…*

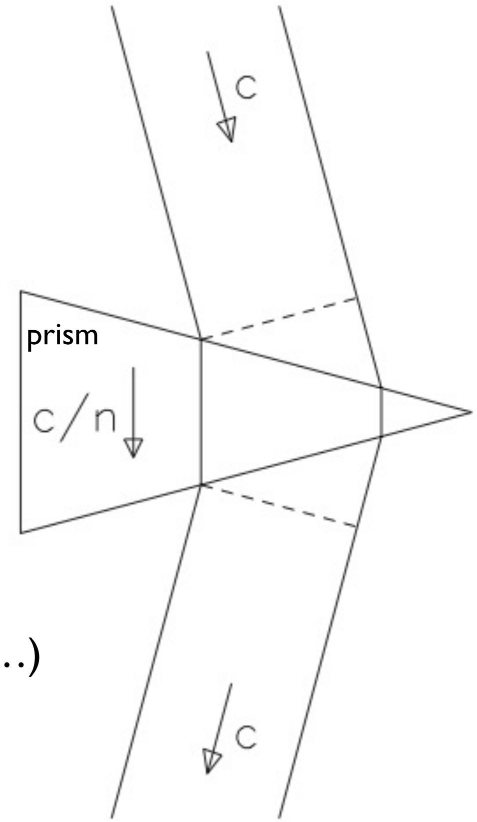
figure taken from Narayan & Bartelmann (1995)

- analogy to optics

  - effective index of refraction (in optics)

$$n = \frac{c}{v}$$

the refractive index $n > 1$ of the glass in the prism reduces the effective speed of light…



  - effective index of refraction (in gravity, post-Newtonian…)

$$n = \frac{c}{v} = ?$$

Schwarzschild metric

$$0 = ds^2 = \left(1 + \frac{2}{c^2}\Phi\right) c^2 dt^2 - \left(1 - \frac{2}{c^2}\Phi\right) dl^2$$

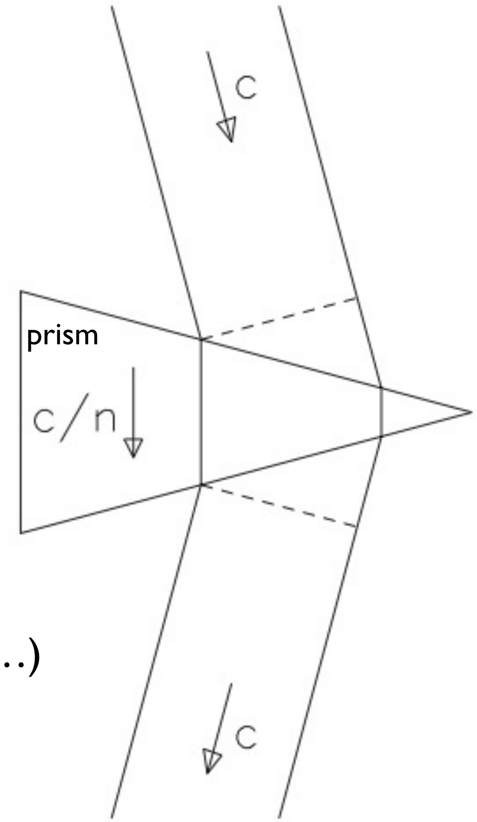
figure taken from Narayan & Bartelmann (1995)

▪ analogy to optics

- effective index of refraction (in optics)

$$n = \frac{c}{v}$$

the refractive index $n > 1$ of the glass in the prism reduces the effective speed of light…



- effective index of refraction (in gravity, post-Newtonian…)

$$n = \frac{c}{v} \quad = ?$$

Schwarzschild metric

$$0 = ds^2 = \left(1 + \frac{2}{c^2}\Phi\right) c^2 dt^2 - \left(1 - \frac{2}{c^2}\Phi\right) dl^2$$

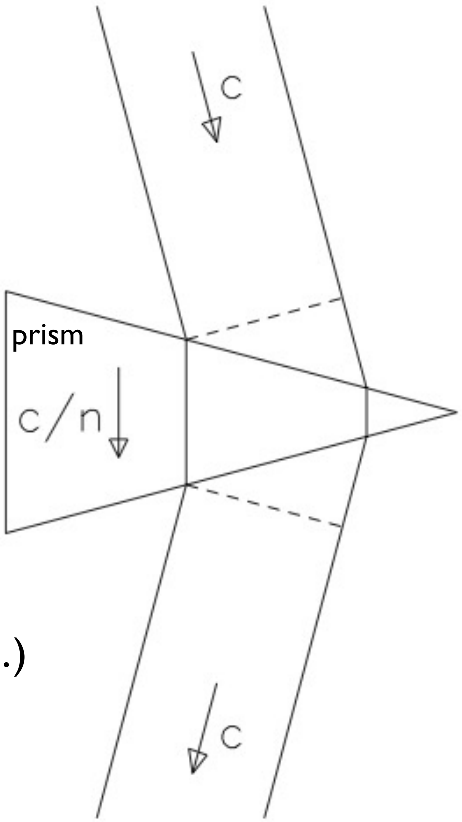
$$\Rightarrow \quad v = \frac{dl}{dt}$$

figure taken from Narayan & Bartelmann (1995)

- analogy to optics
  - effective index of refraction (in optics)

$$n = \frac{c}{v}$$

the refractive index $n > 1$ of the glass in the prism reduces the effective speed of light…



  - effective index of refraction (in gravity, post-Newtonian…)

$$n = \frac{c}{v} = ?$$

Schwarzschild metric

$$0 = ds^2 = \left(1 + \frac{2}{c^2}\Phi\right) c^2 dt^2 - \left(1 - \frac{2}{c^2}\Phi\right) dl^2$$

$$\Rightarrow \quad v = \frac{dl}{dt} = c\sqrt{\frac{1 + \frac{2}{c^2}\Phi}{1 - \frac{2}{c^2}\Phi}} \approx c\left(1 + \frac{2}{c^2}\Phi\right)$$

$$\Rightarrow \quad \frac{c}{v} \approx \left(1 + \frac{2}{c^2}\Phi\right)^{-1} \approx \left(1 - \frac{2}{c^2}\Phi\right)$$

figure taken from Narayan & Bartelmann (1995)

- analogy to optics

  - effective index of refraction (in optics)

$$n = \frac{c}{v}$$

the refractive index $n > 1$ of the glass in the prism reduces the effective speed of light…

  - effective index of refraction (in gravity, post-Newtonian…)

$$n = 1 - \frac{2}{c^2}\Phi$$

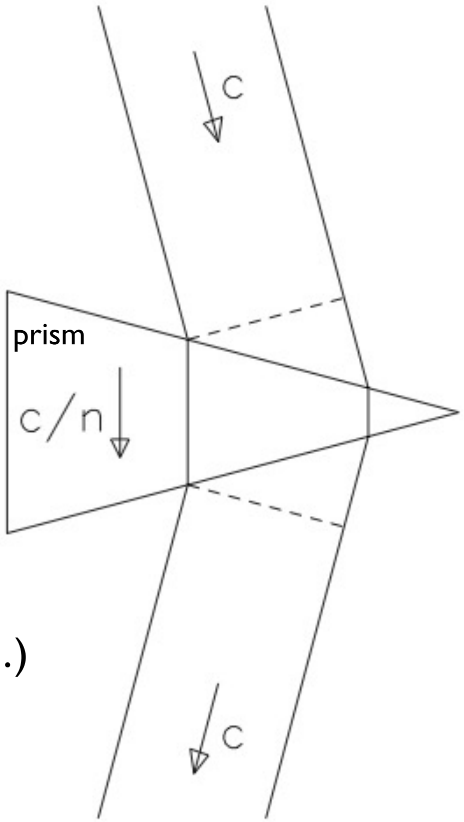


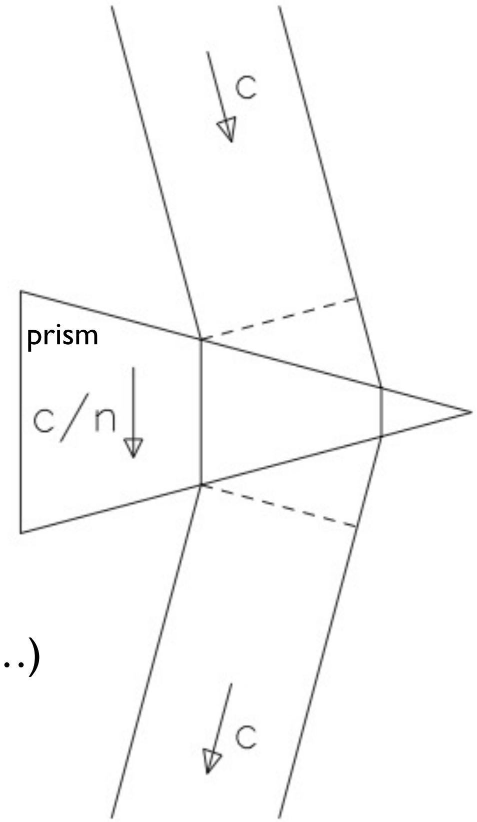
figure taken from Narayan & Bartelmann (1995)

▪ analogy to optics

- effective index of refraction (in optics)

$$n = \frac{c}{v}$$

the refractive index $n > 1$ of the glass in the prism reduces the effective speed of light…



- effective index of refraction (in gravity, post-Newtonian…)

$$n = 1 - \frac{2}{c^2}\Phi$$

Note:

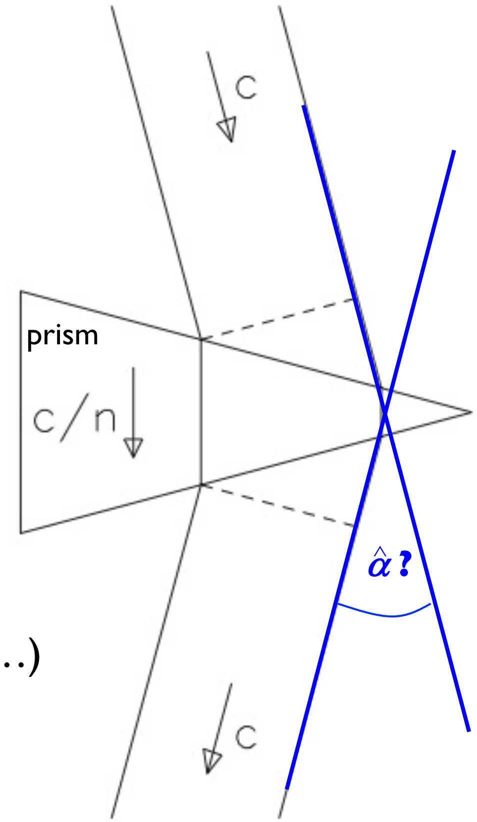
- gravitational lensing is achromatic!
- $\Phi < 0$ is the Newtonian potential

figure taken from Narayan & Bartelmann (1995)

- analogy to optics
  - effective index of refraction (in optics)

$$n = \frac{c}{v}$$

the refractive index $n > 1$ of the glass in the prism reduces the effective speed of light…



  - effective index of refraction (in gravity, post-Newtonian…)

$$n = 1 - \frac{2}{c^2}\Phi$$

Note:

- gravitational lensing is achromatic!
- $\Phi < 0$ is the Newtonian potential

figure taken from Narayan & Bartelmann (1995)

- analogy to optics
  - deflection angle (in optics)

$$\hat{\alpha} = -\int \nabla_{\perp} n \; dz$$

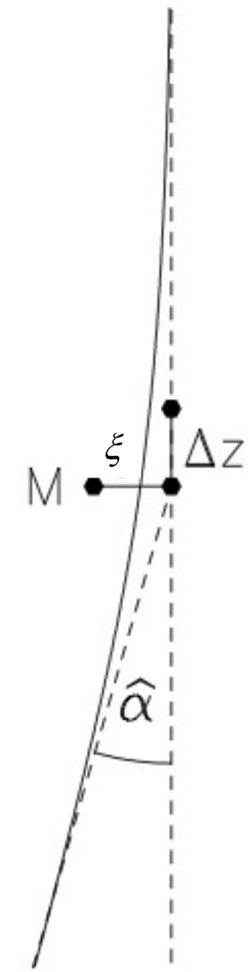
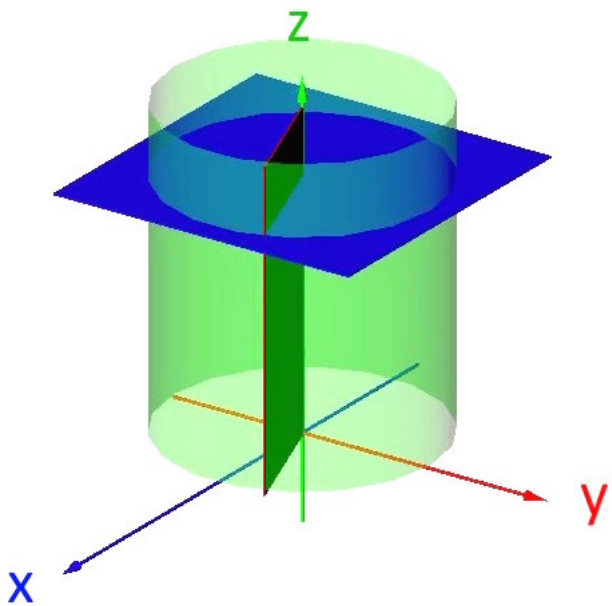


figure taken from Narayan & Bartelmann (1995)

- analogy to optics

  - deflection angle (in optics)

$$\hat{\alpha} = -\int \nabla_{\perp} n \; dz$$



$$\xi^2 = x^2 + y^2$$

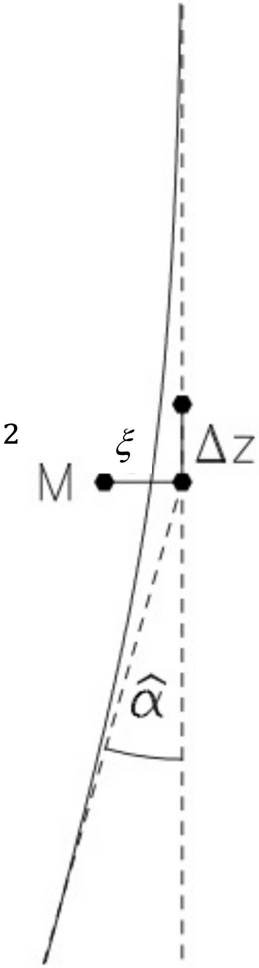


figure taken from Narayan & Bartelmann (1995)

- analogy to optics

  - deflection angle (in optics)

$$\hat{\alpha} = -\int \nabla_{\perp} n \;\; dz$$

  - deflection angle (in gravity)

$$\hat{\alpha} = -\int \nabla_{\perp} n \;\; dz$$

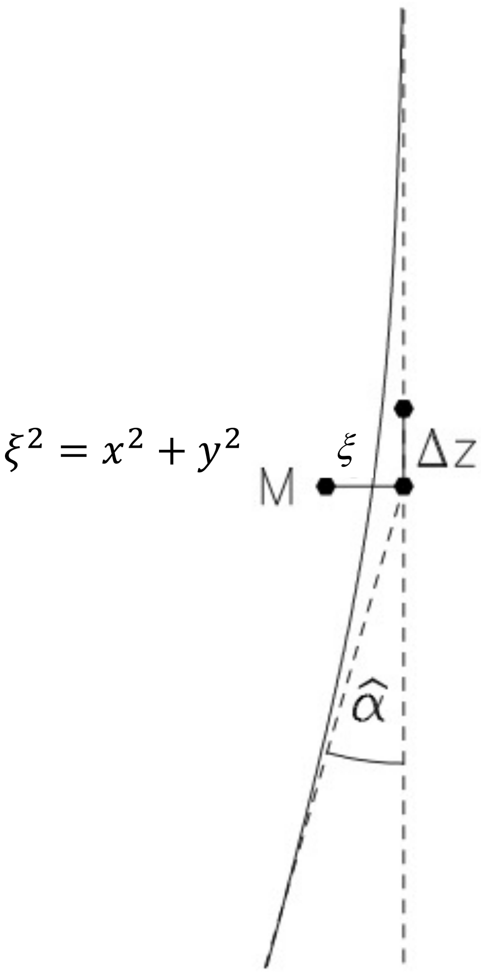


figure taken from Narayan & Bartelmann (1995)

- analogy to optics

  - deflection angle (in optics)

$$\hat{\alpha} = -\int \nabla_{\perp} n \ \ dz$$

  - deflection angle (in gravity)

$$\hat{\alpha} = -\int \nabla_{\perp} n \ \ dz = \frac{2}{c^2} \int \nabla_{\xi} \Phi(\xi, z) \ \ dz$$

$$n = 1 - \frac{2}{c^2}\Phi$$

$$\xi^2 = x^2 + y^2$$

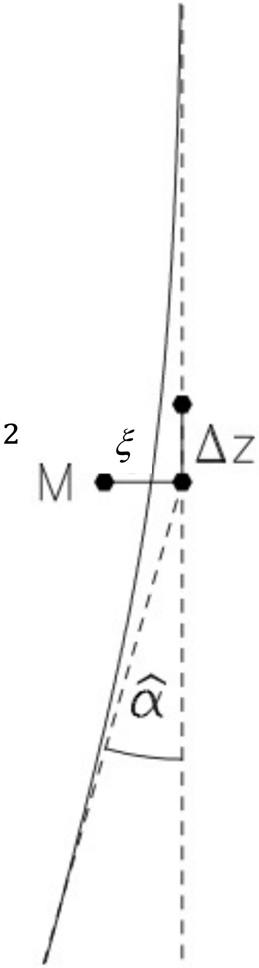


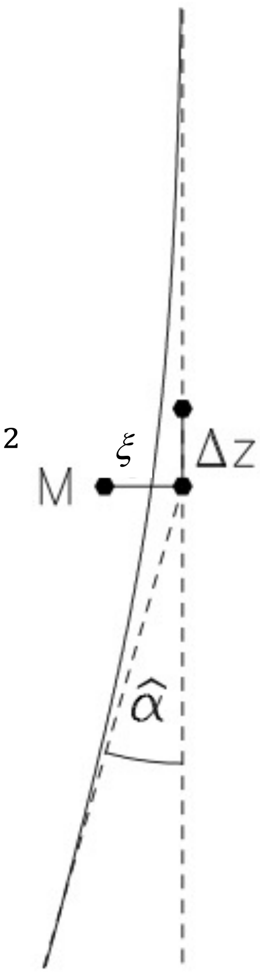
figure taken from Narayan & Bartelmann (1995)

- gravitational lensing

  - effective index of refraction (in gravity, post-Newtonian…)

$$n = 1 - \frac{2}{c^2}\Phi$$

$$\xi^2 = x^2 + y^2$$

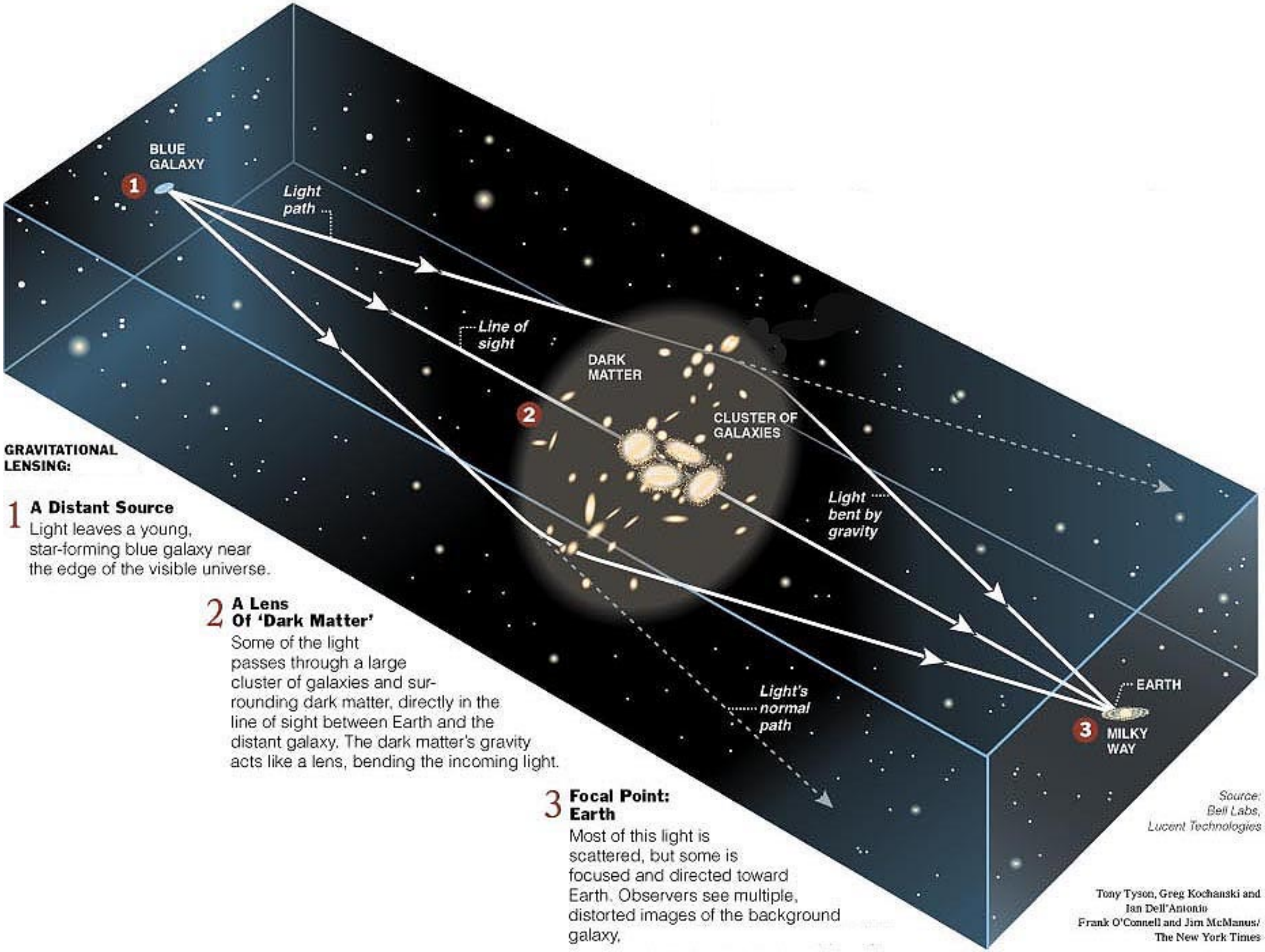


  - deflection angle (in gravity)

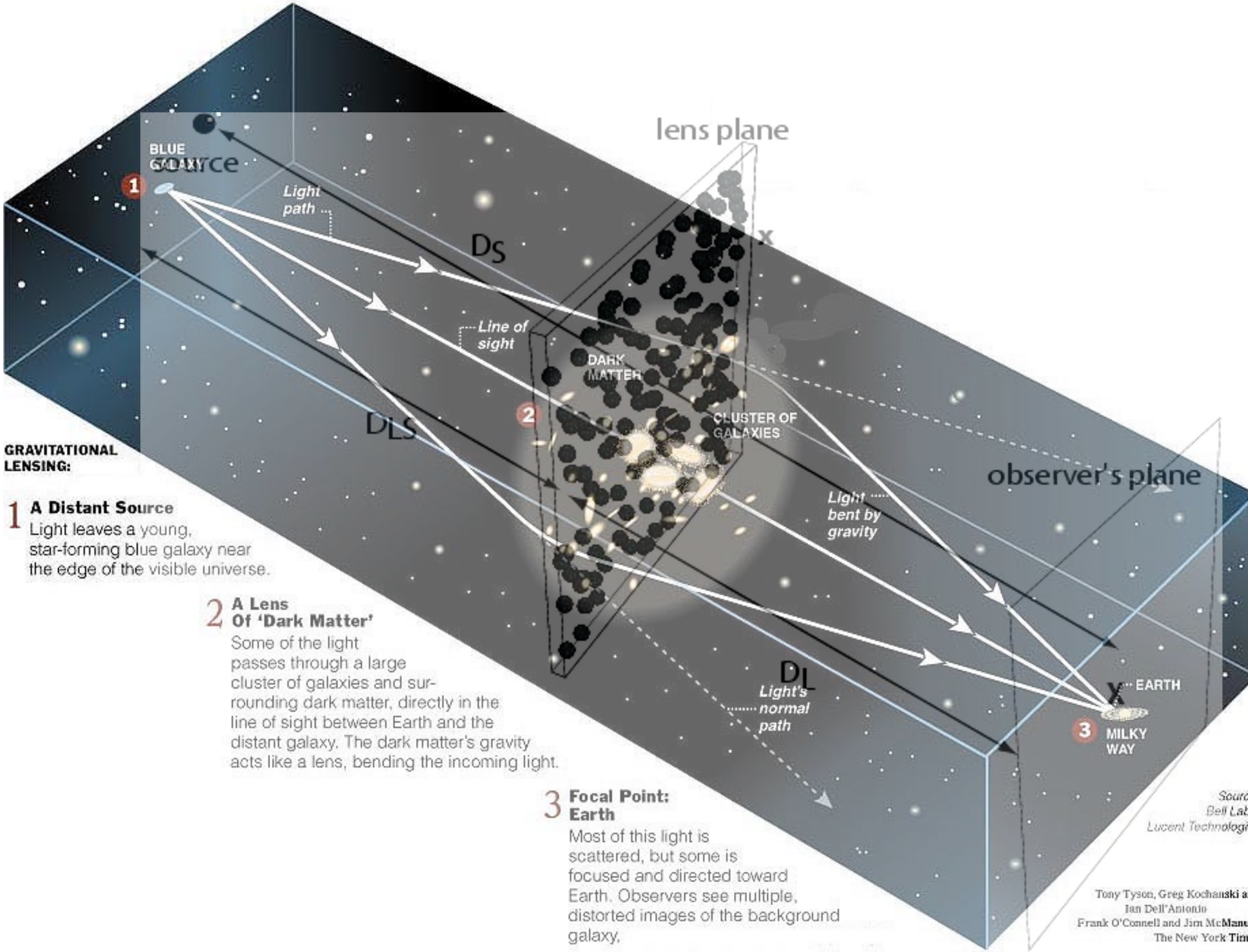
$$\hat{\alpha} = -\int \nabla_{\perp} n \; dz = \frac{2}{c^2}\int \nabla_{\xi}\Phi(\xi, z) \; dz$$

figure taken from Narayan & Bartelmann (1995)

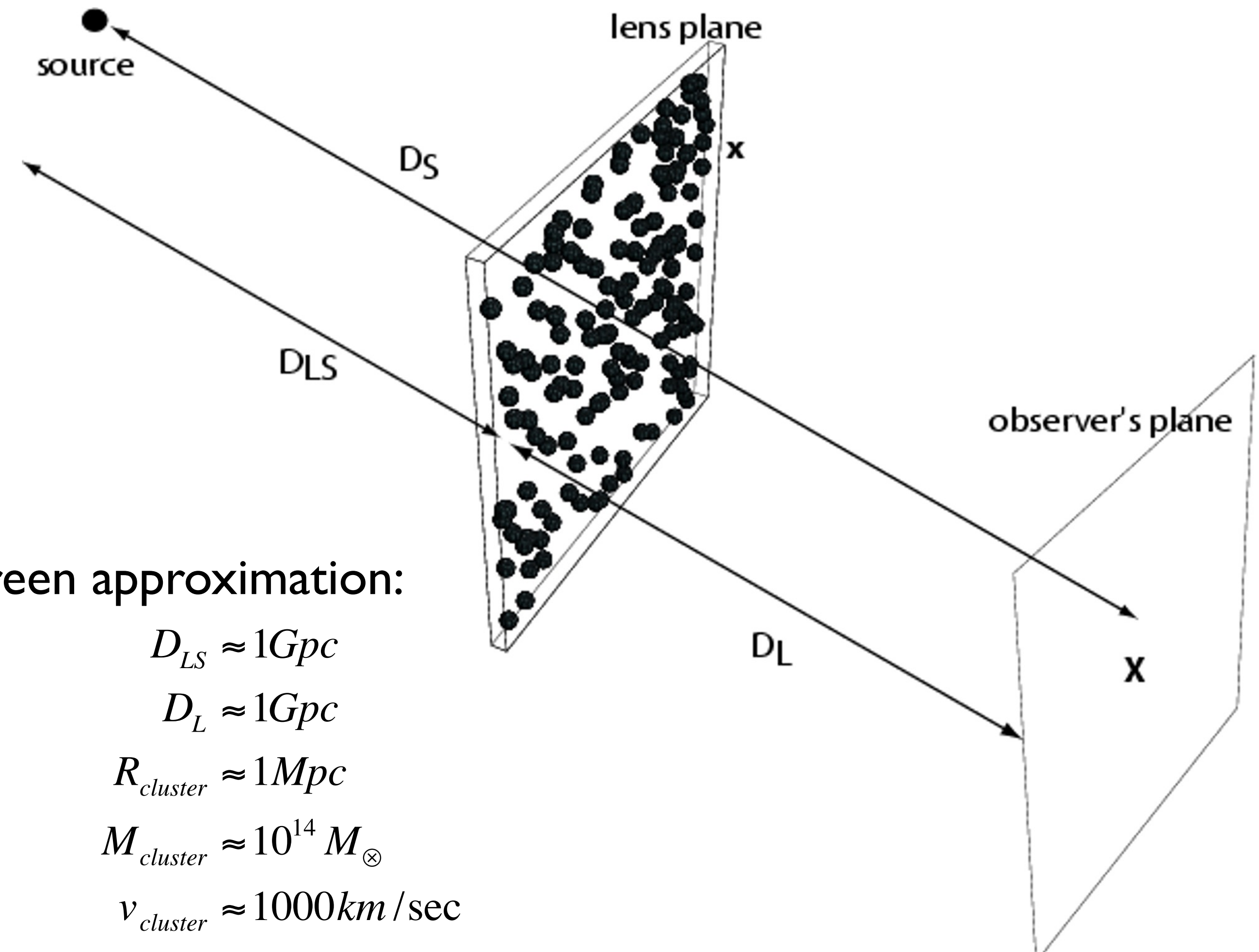
- gravitational lensing - assumptions



- gravitational lensing - assumptions



- gravitational lensing - assumptions



thin screen approximation:

$$D_{LS} \approx 1Gpc$$
$$D_L \approx 1Gpc$$
$$R_{cluster} \approx 1Mpc$$
$$M_{cluster} \approx 10^{14} M_{\otimes}$$
$$v_{cluster} \approx 1000 km/\sec$$

▪ gravitational lensing - assumptions

- deflection angles are small

$$\hat{\alpha} << 1$$

- matter inhomogeneities causing lensing are local perturbations:

$$|\Phi| << c^2$$

$$v_{lens} << c$$

thin screen approximation:

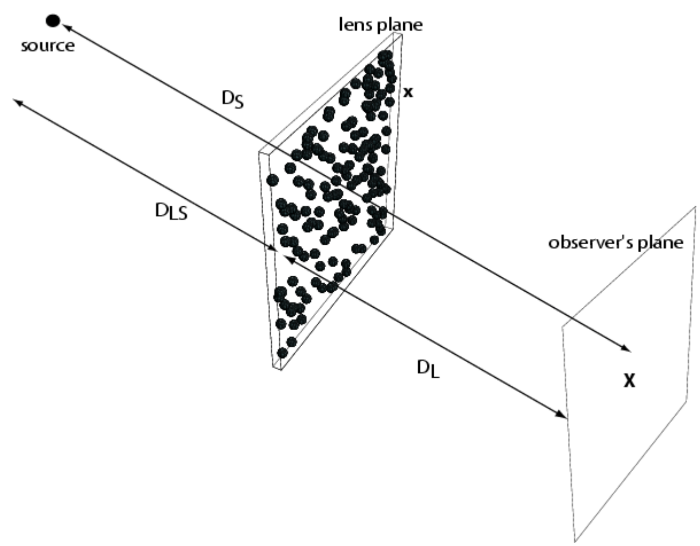
$$D_{LS} \approx 1Gpc$$

$$D_L \approx 1Gpc$$

$$R_{cluster} \approx 1Mpc$$

$$M_{cluster} \approx 10^{14} M_{\otimes}$$

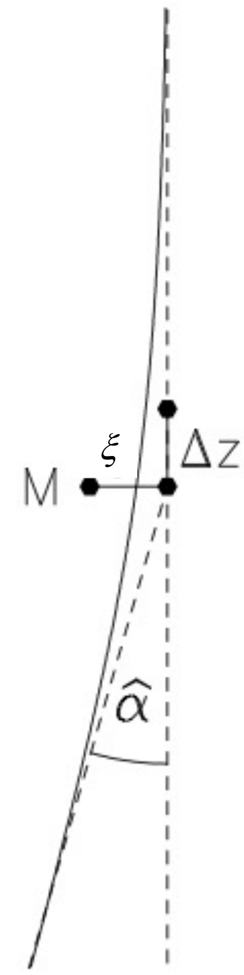
$$v_{cluster} \approx 1000 km/\text{sec}$$



- theory
  - the basics of lensing…
    - **the lens equation**
    - the lensing potential
    - critical surface mass density
    - magnification
    - caustics and critical curves
    - distortion
    - mass-sheet degeneracy
  - some sample lenses…
    - point mass
    - extended mass
    - singular isothermal sphere

- the lens equation

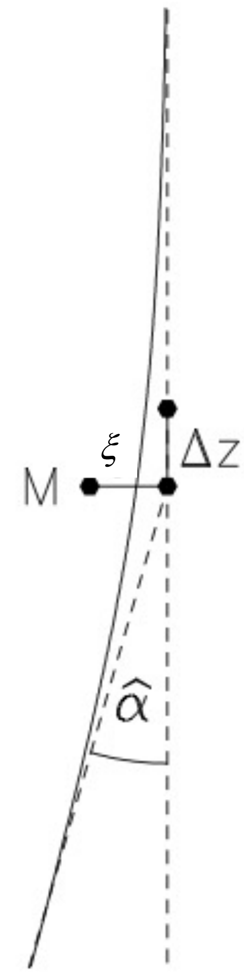
$$\hat{\alpha} = -\int \nabla_{\perp} n \;\; dz = \frac{2}{c^2} \int \nabla_{\xi} \Phi(\xi, z) \;\; dz$$



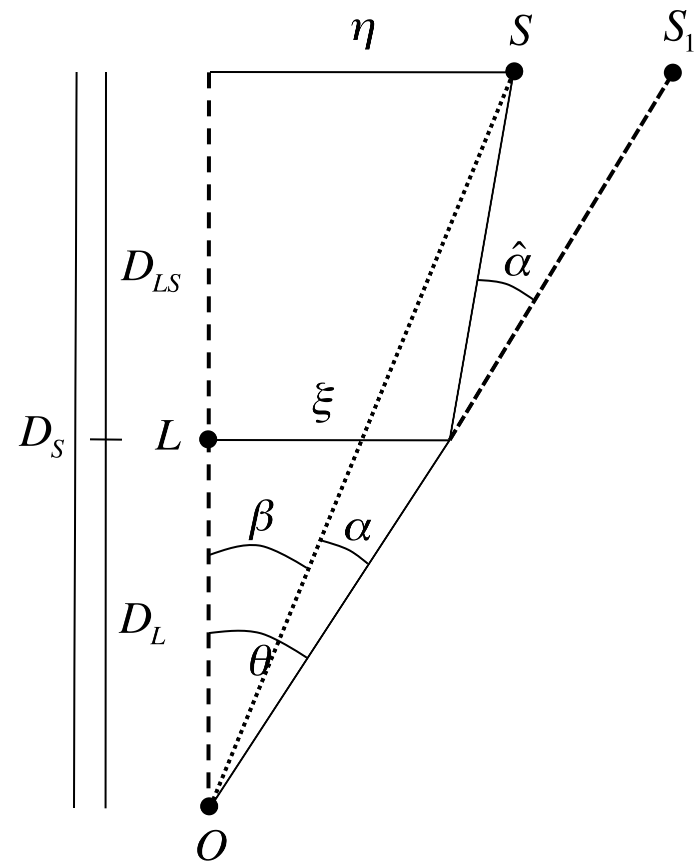
- the lens equation

how to relate to anything we can observe?

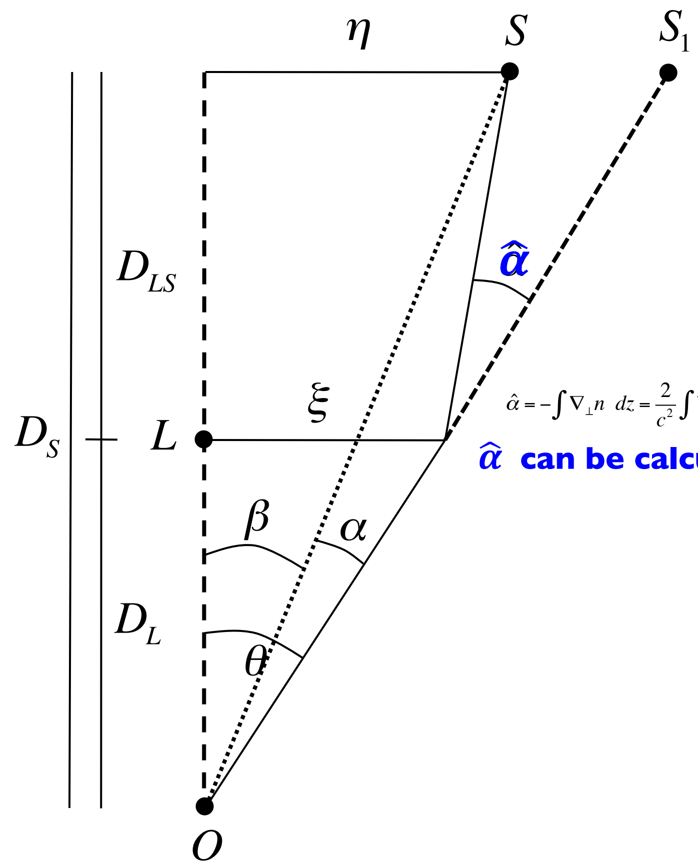
$$\hat{\alpha} = -\int \nabla_{\perp} n \;\; dz = \frac{2}{c^2} \int \nabla_{\xi} \Phi(\xi, z) \;\; dz$$



- the lens equation



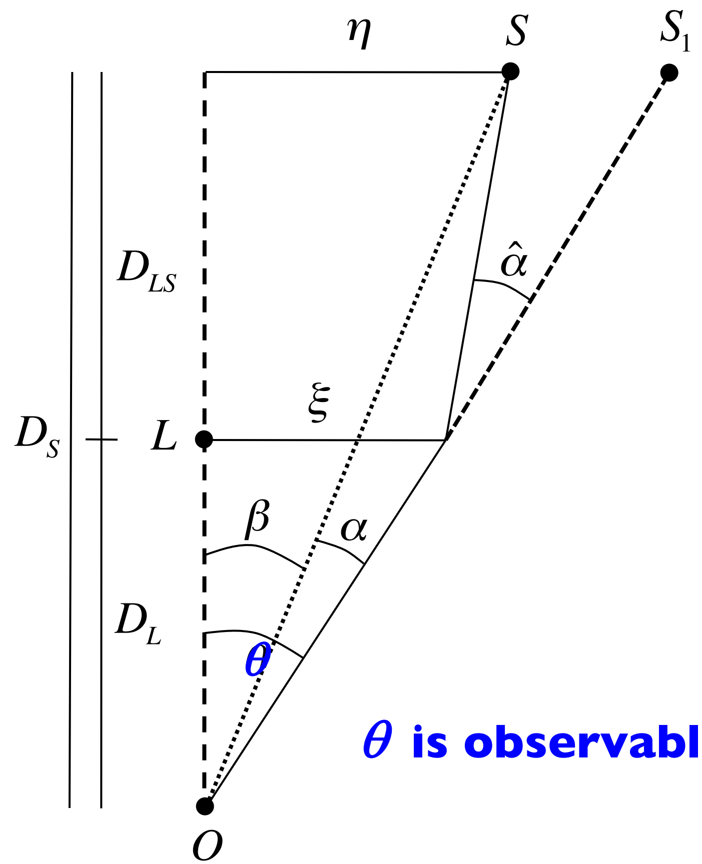
- the lens equation



$$\hat{\alpha} = -\int \nabla_\perp n \; dz = \frac{2}{c^2}\int \nabla_\xi \Phi(\xi, z) \; dz$$

$\hat{\alpha}$ can be calculated!

- the lens equation

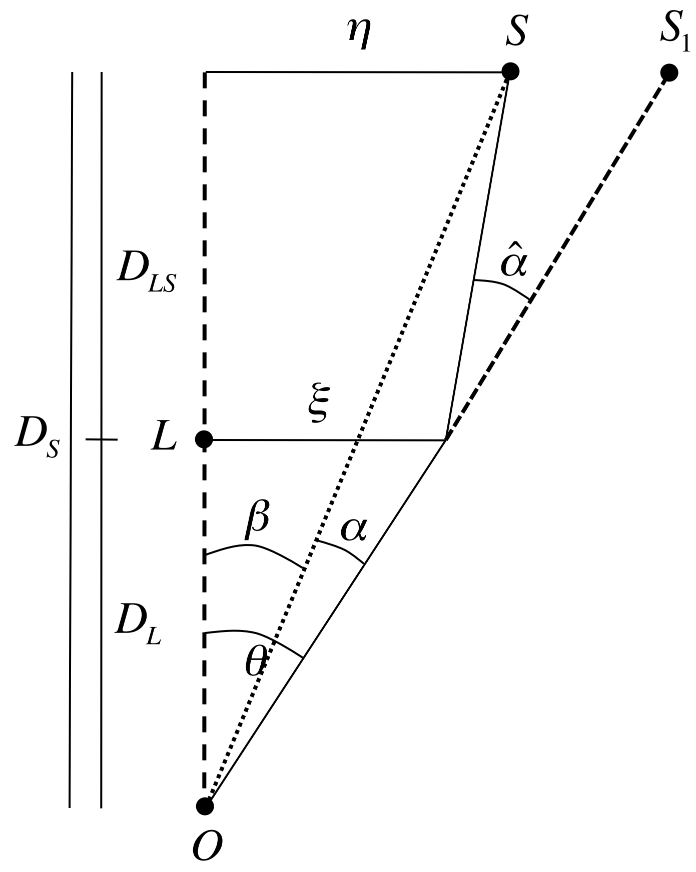


θ is observable!

- the lens equation

$$\beta = \theta - \frac{D_{LS}}{D_S}\hat{\alpha}(\theta)$$

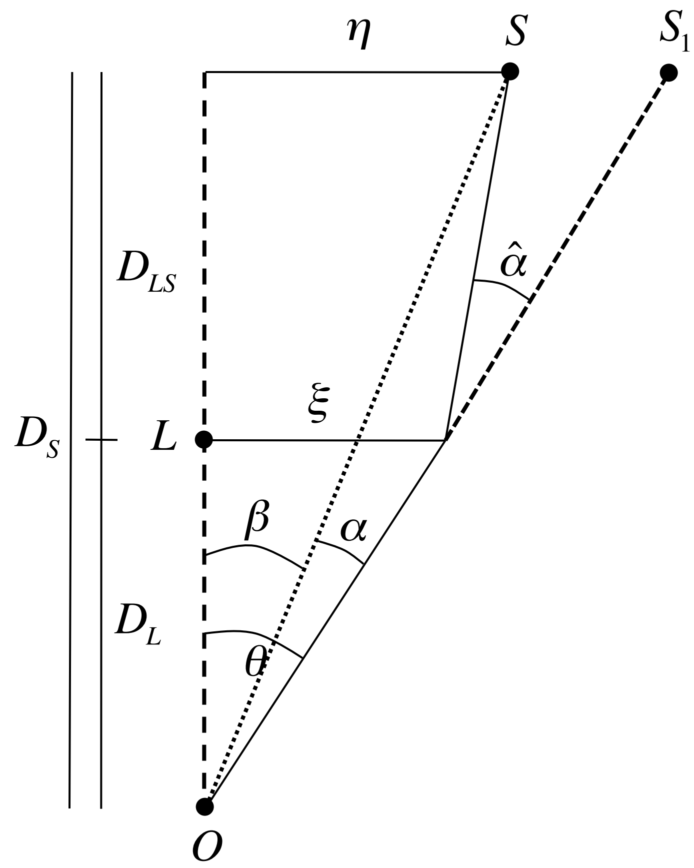
(exercise)



- the lens equation

$$\beta = \theta - \frac{D_{LS}}{D_S}\hat{\alpha}(\theta)$$

(exercise)

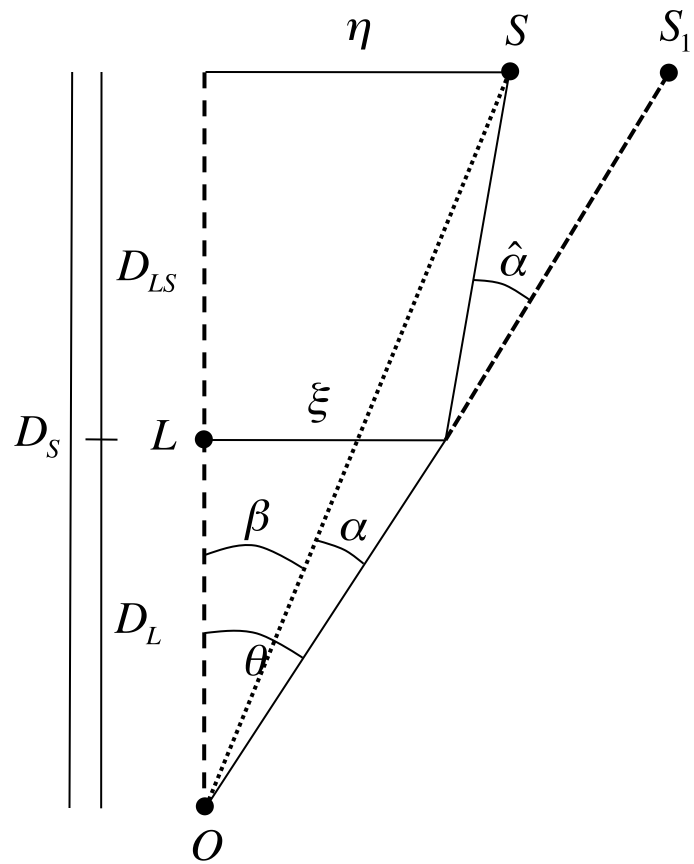


- reduced deflection angle:

$$\alpha = \frac{D_{LS}}{D_S}\hat{\alpha}$$

- the lens equation

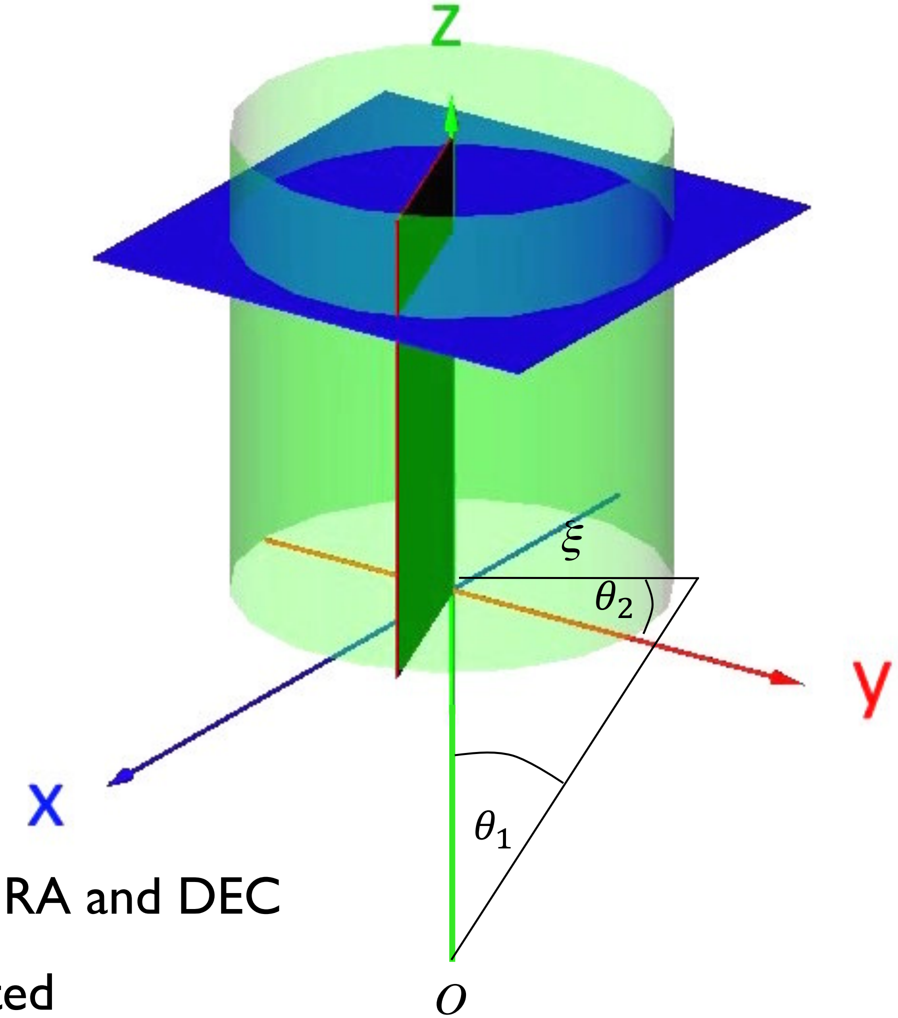
$$\beta = \theta - \alpha(\theta)$$



- measured: θ
- wanted: β
- needed: $\alpha(\theta)$

- the lens equation

$$\beta = \theta - \alpha(\theta)$$



important note:

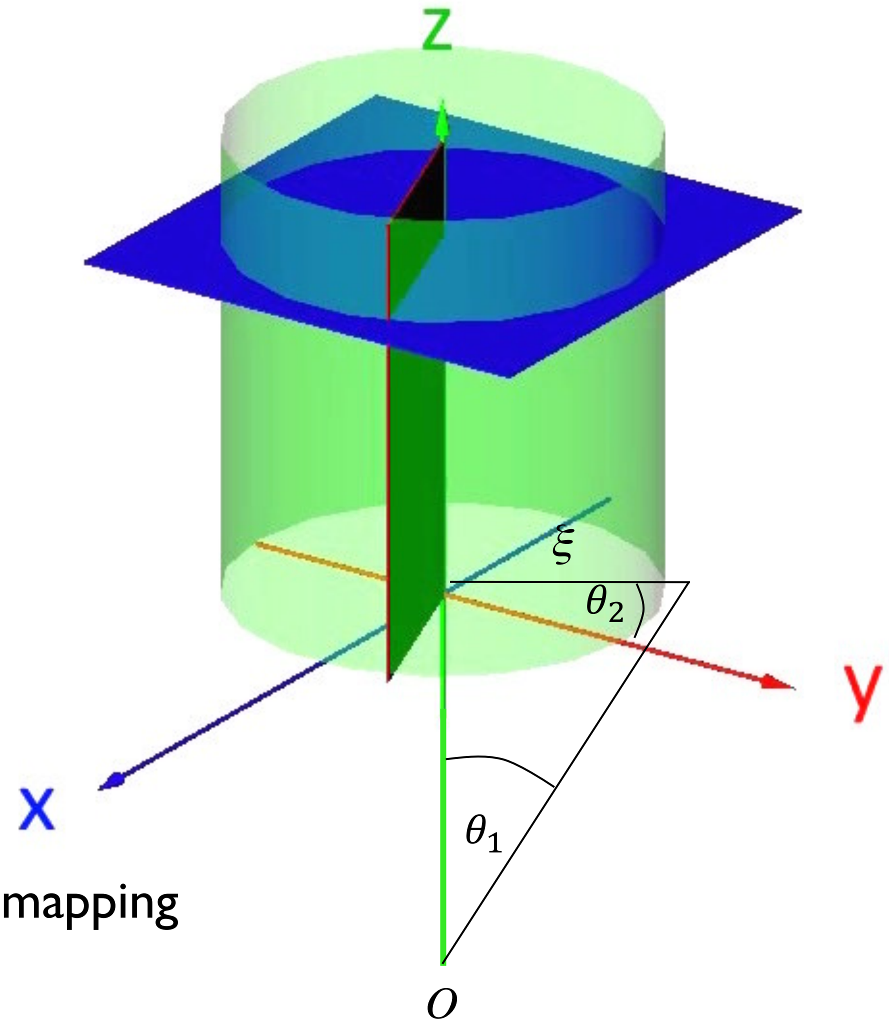
- all angles are in fact 2D, i.e.

$$\vec{\theta} = (\theta_1, \theta_2)$$
$$\vec{\beta} = (\beta_1, \beta_2)$$

- in Astronomy the two angles are RA and DEC
- for the setup used here, we rotated the system so that it becomes 1D

- the lens equation

$$\beta = \theta - \alpha(\theta)$$



important notes:

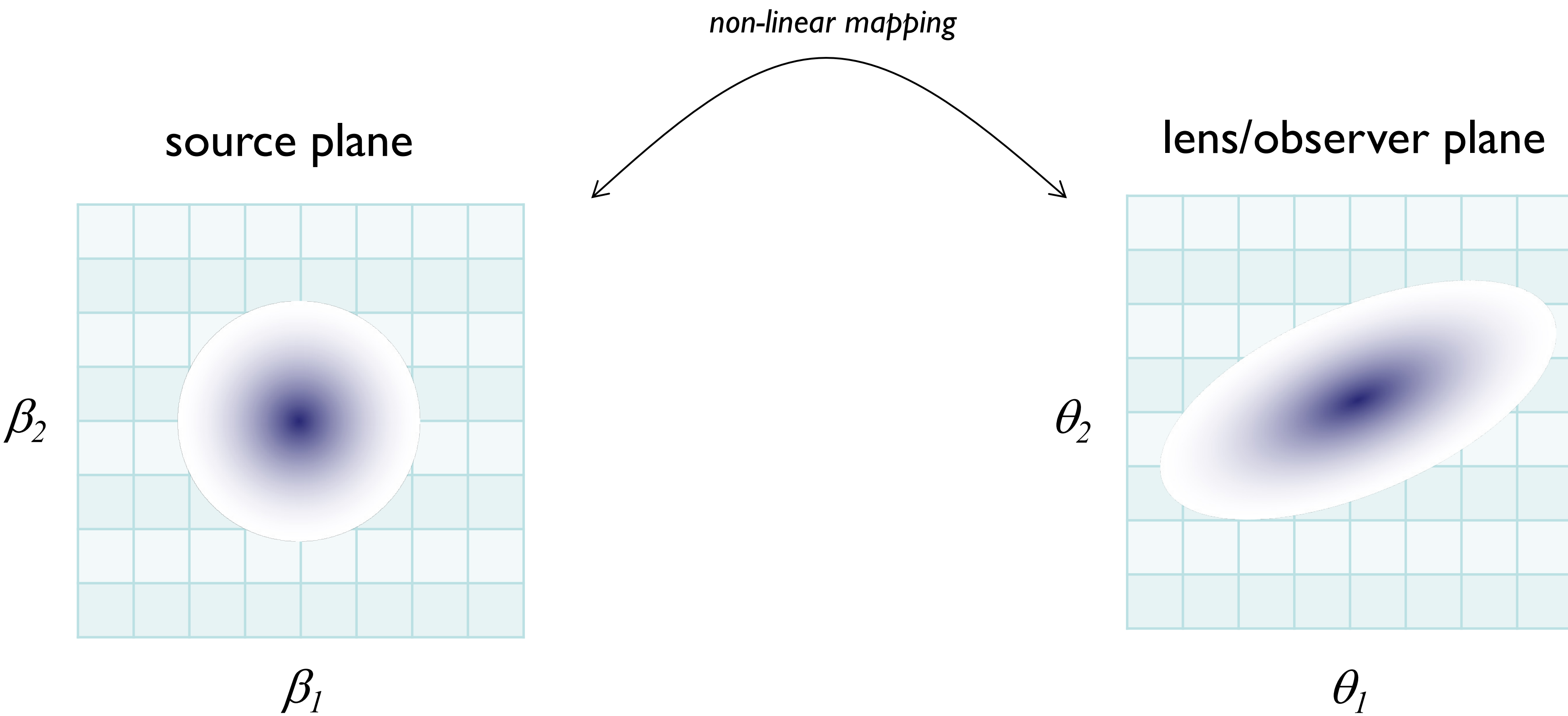
- all angles are in fact 2D, i.e.

$$\vec{\theta} = (\theta_1, \theta_2)$$
$$\vec{\beta} = (\beta_1, \beta_2)$$

- the lens equation describes a 2D mapping

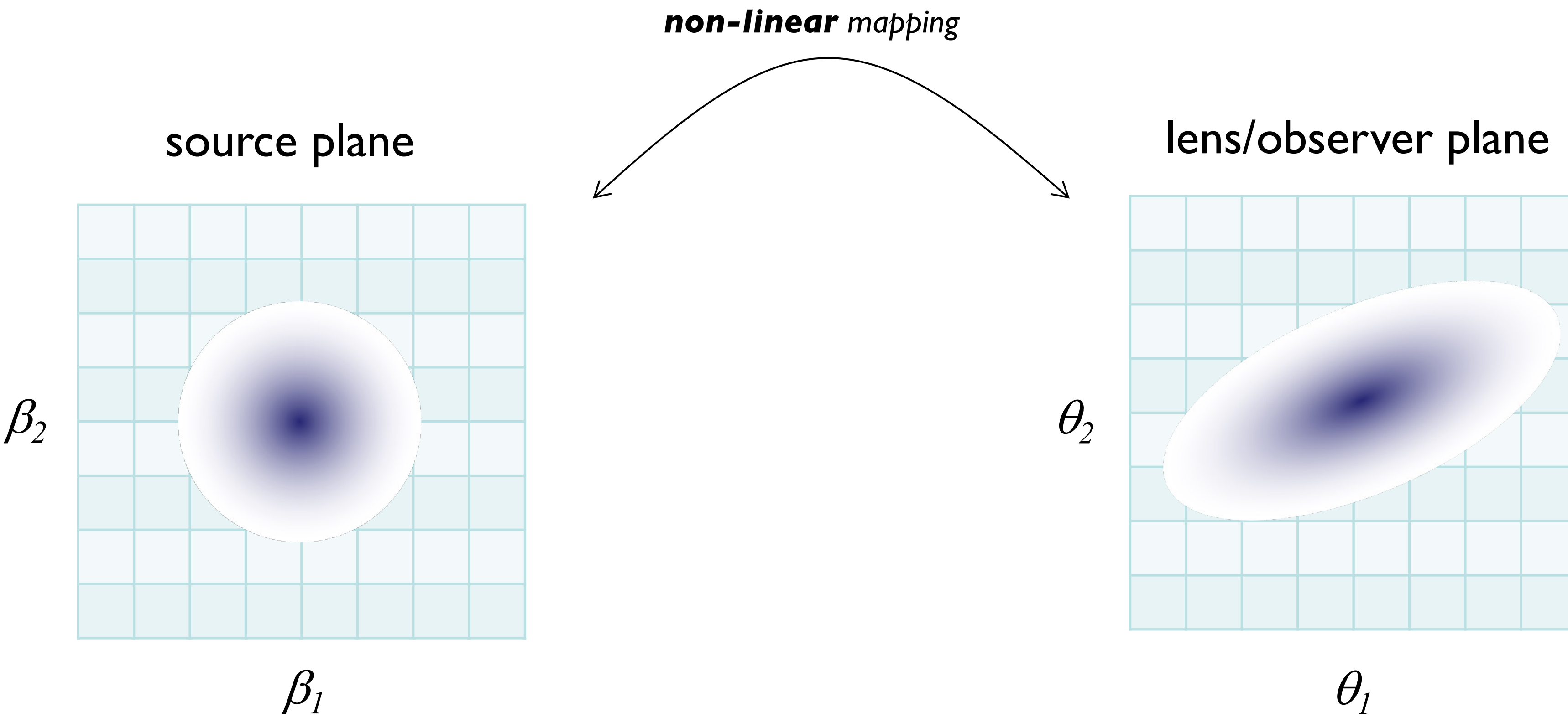
- the lens equation

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$



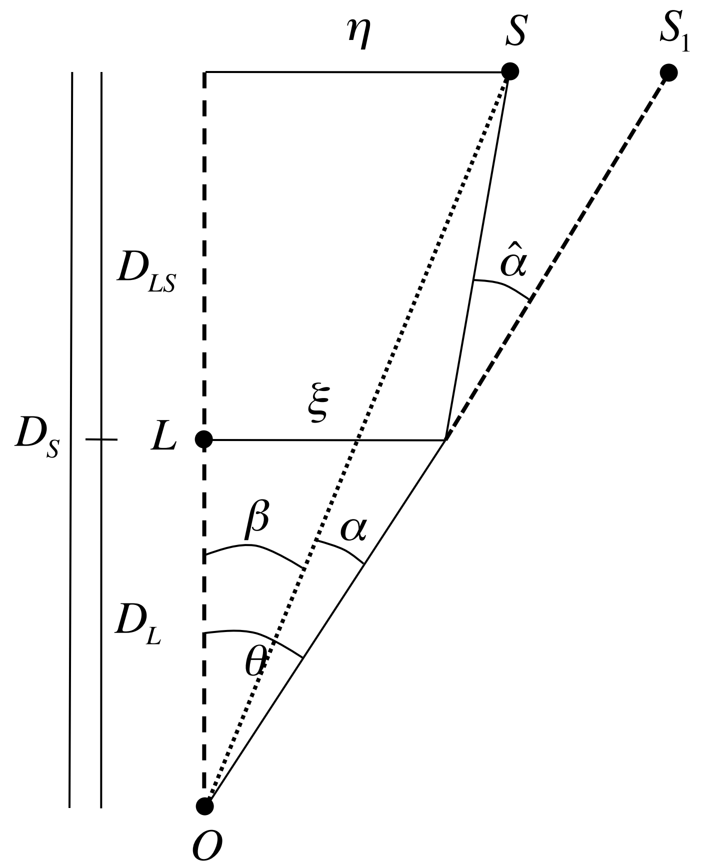
- the lens equation

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta})$$



- the lens equation

$$\beta = \theta - \alpha(\theta)$$



important notes:

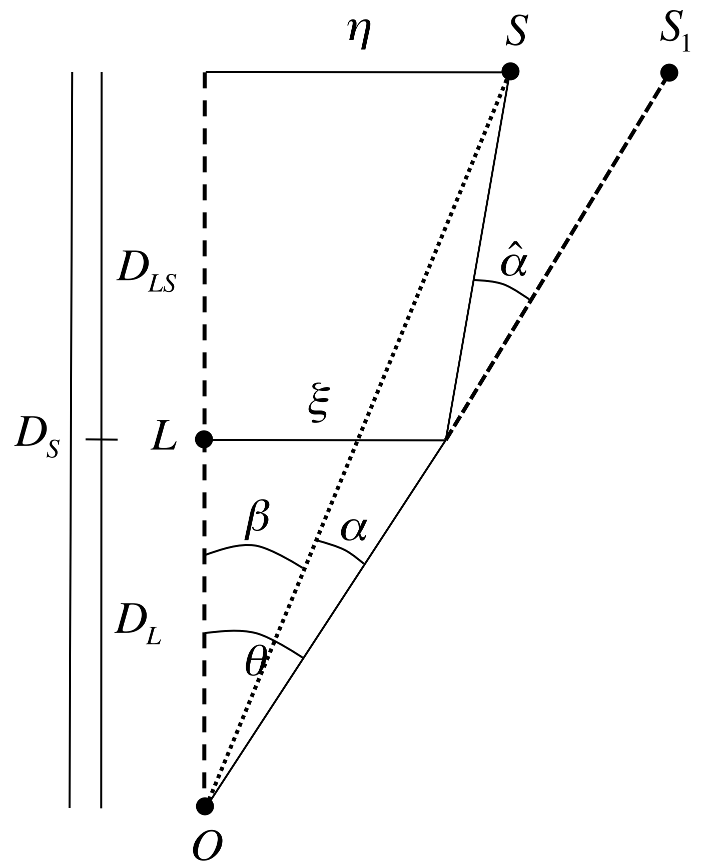
- all angles are in fact 2D, i.e.

$$\vec{\theta} = (\theta_1, \theta_2)$$
$$\vec{\beta} = (\beta_1, \beta_2)$$

- the lens equation describes a 2D mapping
- in general a non-linear equation → multiple images!

- the lens equation

$$\beta = \theta - \alpha(\theta)$$



important notes:

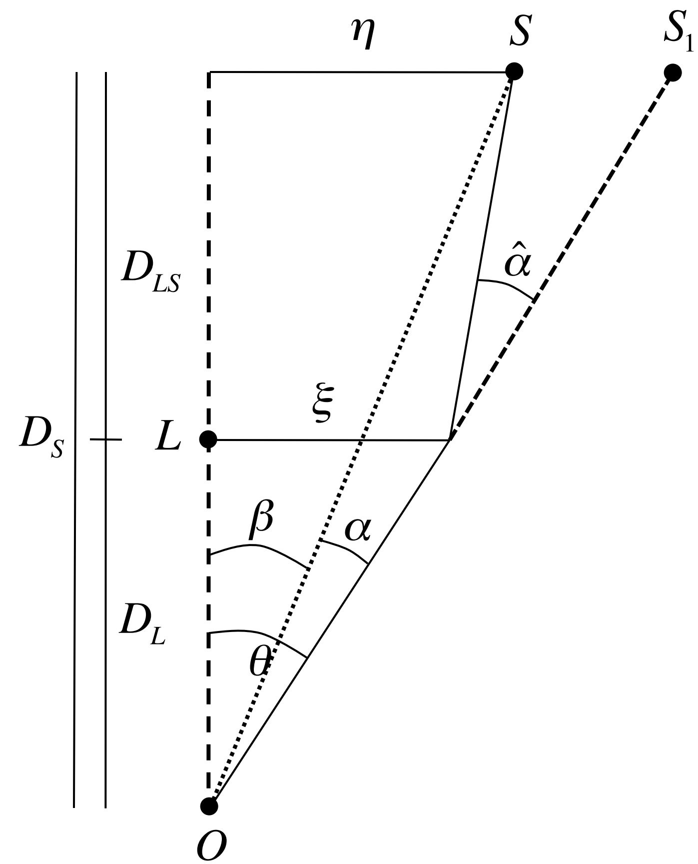
- all angles are in fact 2D, i.e.

$$\vec{\theta} = (\theta_1, \theta_2)$$
$$\vec{\beta} = (\beta_1, \beta_2)$$

- the lens equation describes a 2D mapping
- in general a non-linear equation → multiple images!
- based upon the assumption that "separation = angle x distance"

- the lens equation

$$\beta = \theta - \alpha(\theta)$$



<u>important notes:</u>

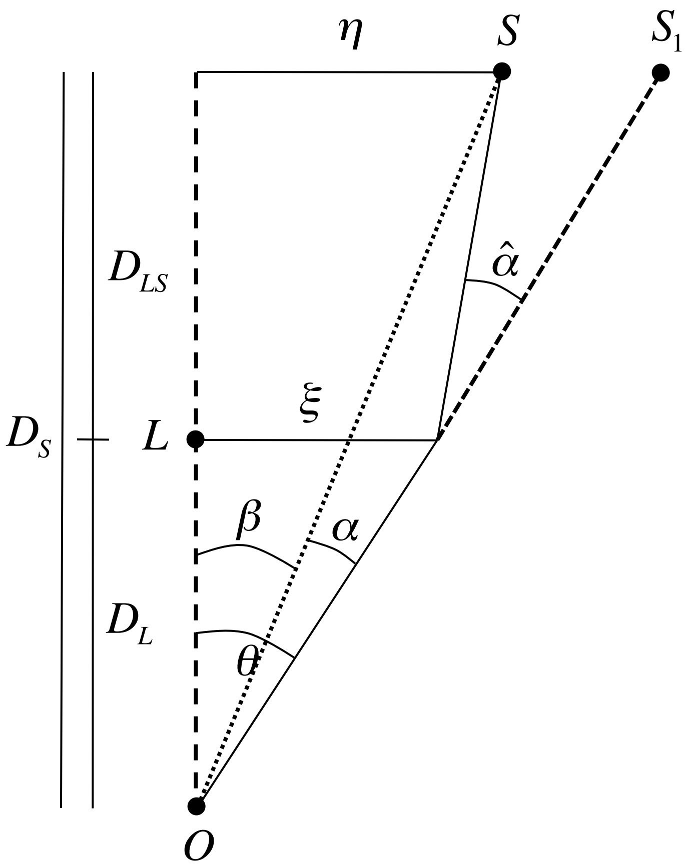
- all angles are in fact 2D, i.e.

$$\vec{\theta} = (\theta_1, \theta_2)$$
$$\vec{\beta} = (\beta_1, \beta_2)$$

- the lens equation describes a 2D mapping
- in general a non-linear equation → multiple images!
- based upon the assumption that "separation = angle x distance"
- in general $D_S \neq D_L + D_{LS}$

- the lens equation

$$\beta = \theta - \alpha(\theta)$$



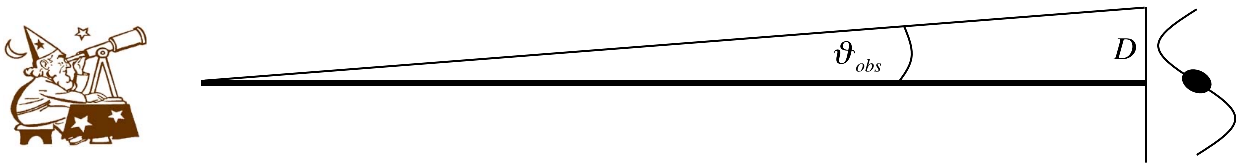
<u>important notes:</u>

- all angles are in fact 2D, i.e.

$$\vec{\theta} = (\theta_1, \theta_2)$$
$$\vec{\beta} = (\beta_1, \beta_2)$$

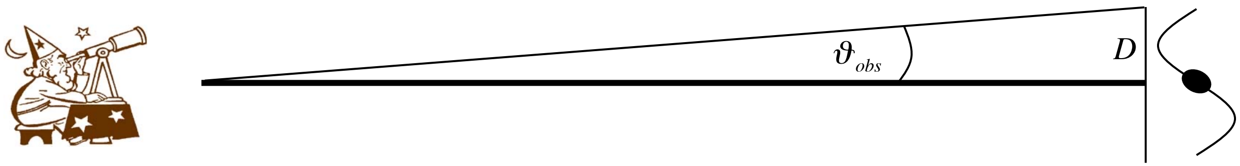
- the lens equation describes a 2D mapping
- in general a non-linear equation → multiple images!
- based upon the assumption that "separation = angle x distance"
- in general $D_S \neq D_L + D_{LS}$ => ***what distances are these?***

- the lens equation uses angular diameter distances d_A:



$$\vartheta_{obs} \stackrel{!}{=} \frac{D}{d_A}$$

- the lens equation uses angular diameter distances d_A:



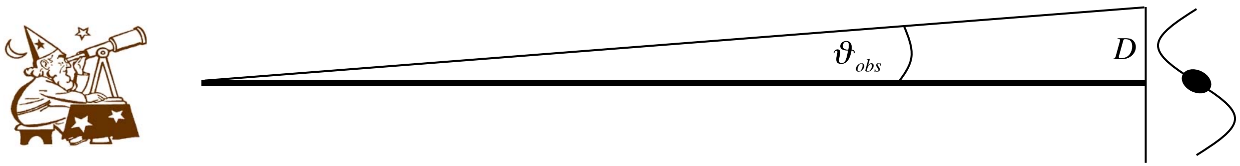
$$D = R(t_E) x_E \int_0^{\vartheta_E} d\vartheta = R(t_E) x_E \vartheta_E$$

($R(t_E)$ because of "galaxy size at time of emission")

$$\vartheta_{obs} \equiv \vartheta_E$$

$$\vartheta_{obs} \stackrel{!}{=} \frac{D}{d_A} \qquad \Rightarrow \qquad \boxed{d_A = \frac{D}{\vartheta_{obs}} = R(t_E) x_E}$$

- the lens equation uses angular diameter distances d_{A}:



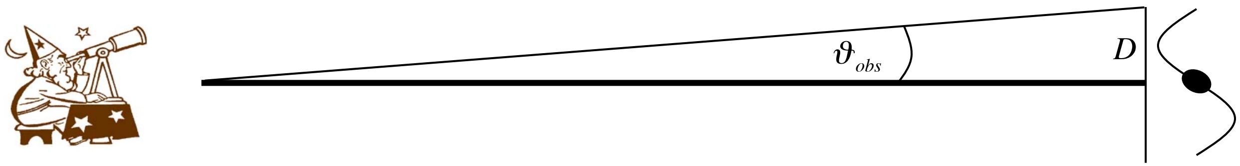
$$D = R(t_E)x_E \int_0^{\vartheta_E} d\vartheta = R(t_E)x_E\vartheta_E$$

($R(t_E)$ because of "galaxy size at time of emission")

$$\vartheta_{obs} \equiv \vartheta_E$$

$$\vartheta_{obs} \overset{!}{=} \frac{D}{d_A} \quad \Rightarrow \quad \boxed{d_A = \frac{D}{\vartheta_{obs}} = R(t_E)x_E} \; ?$$

- the lens equation uses angular diameter distances d_A:

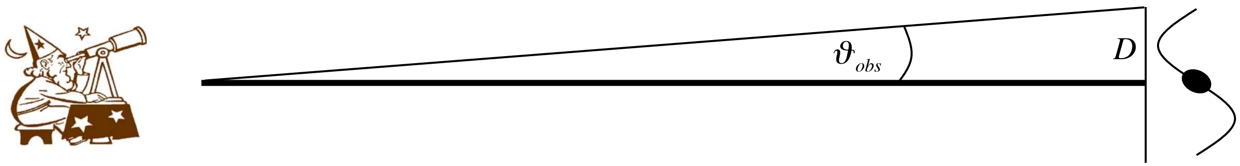


$$d_c = \frac{c}{H_0}\int_0^{z_E} \frac{1}{E(z)}\,dz \qquad E^2(z) = \sum_i \Omega_{i,0}(1+z)^{3(1+w_i)}$$

$$x_E = \begin{cases} \frac{1}{R_0}\, d_c & ;k=0 \\ \frac{1}{R_0}\frac{c}{H_0\sqrt{|\Omega_{k,0}|}}\sin\left(\frac{\sqrt{|\Omega_{k,0}|}H_0}{c}d_c\right) & ;k=1 \\ \frac{1}{R_0}\frac{c}{H_0\sqrt{|\Omega_{k,0}|}}\sinh\left(\frac{\sqrt{|\Omega_{k,0}|}H_0}{c}d_c\right) & ;k=-1 \end{cases}$$

$$d_A = \frac{D}{\vartheta_{obs}} = \frac{R(t)}{R_0} R_0 x_E$$

- the lens equation uses angular diameter distances d_A:



$$d_c = \frac{c}{H_0}\int_0^{z_E}\frac{1}{E(z)}\,dz \qquad E^2(z) = \sum_i \Omega_{i,0}\left(1+z\right)^{3(1+w_i)}$$

$$x_E = \begin{cases} \dfrac{1}{R_0} & d_c & ;k=0 \\ \dfrac{1}{R_0}\dfrac{c}{H_0\sqrt{|\Omega_{k,0}|}}\sin & \left(\dfrac{\sqrt{|\Omega_{k,0}|}H_0}{c}d_c\right) & ;k=1 \\ \dfrac{1}{R_0}\dfrac{c}{H_0\sqrt{|\Omega_{k,0}|}}\sinh & \left(\dfrac{\sqrt{|\Omega_{k,0}|}H_0}{c}d_c\right) & ;k=-1 \end{cases}$$

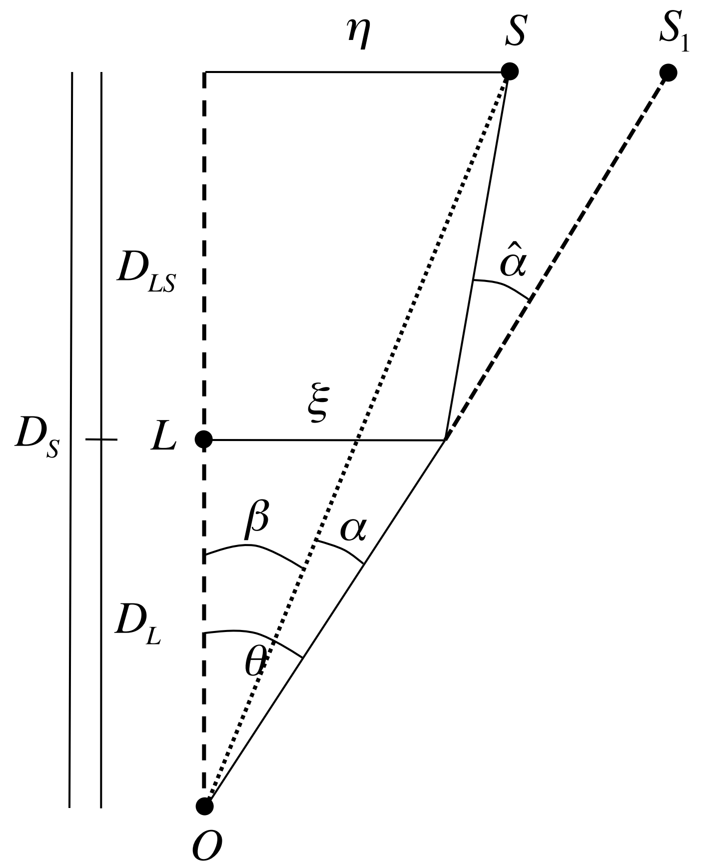
can be measured observationally

$$d_A = \boxed{\frac{D}{\vartheta_{obs}}} = \frac{R(t)}{R_0}R_0\boxed{x_E}$$

can be calculated

- the lens equation

$$\beta = \theta - \alpha(\theta)$$



important notes:

- all angles are in fact 2D, i.e.

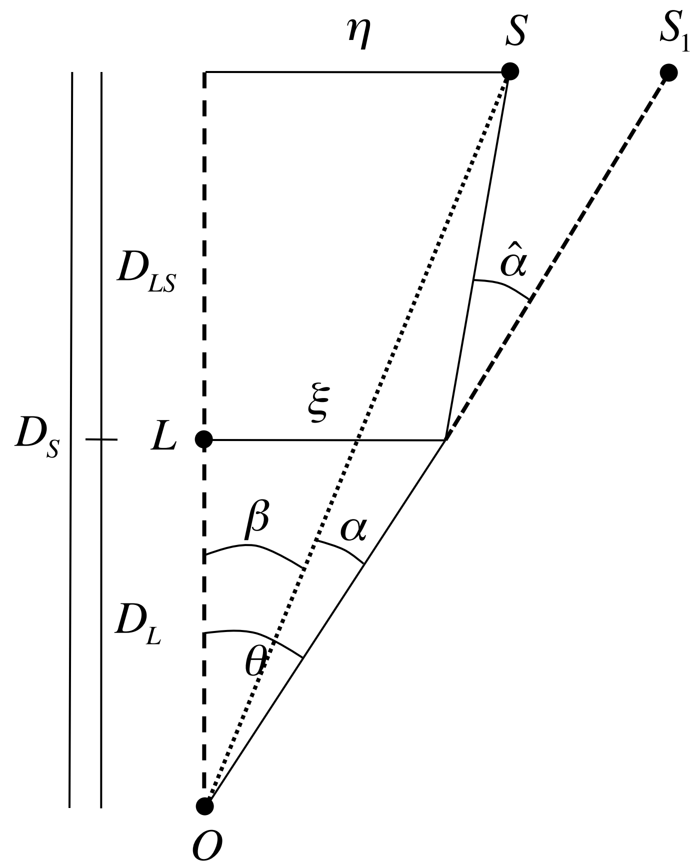
$$\vec{\theta} = (\theta_1, \theta_2)$$
$$\vec{\beta} = (\beta_1, \beta_2)$$

- the lens equation describes a 2D mapping
- in general a non-linear equation → multiple images!
- based upon the assumption that "separation = angle x distance"
- in general $D_S \neq D_L + D_{LS}$ => ***angular diameter distances!***

- the lens equation

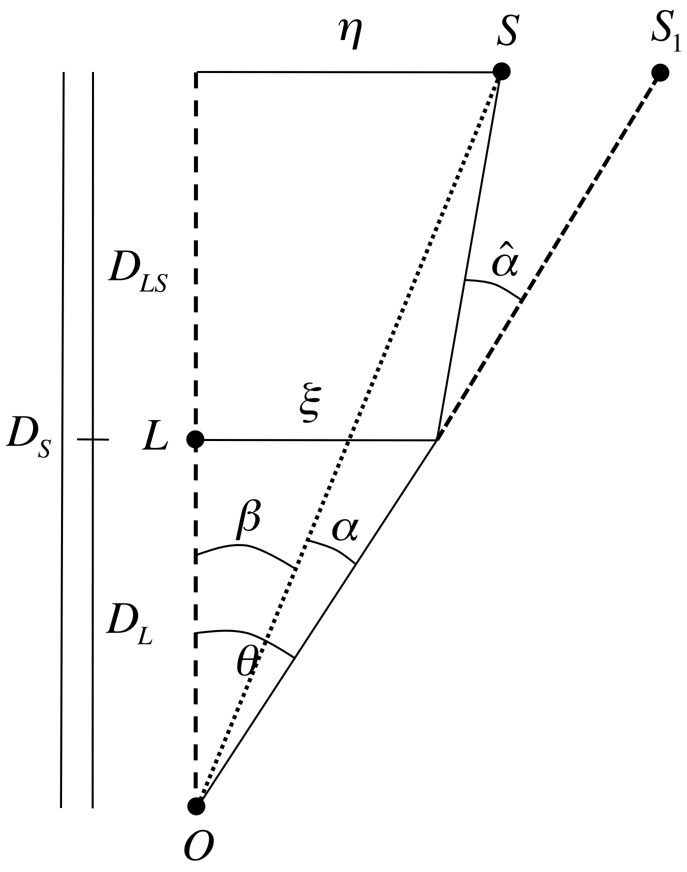
$$\beta = \theta - \alpha(\theta)$$

- measured: θ
- wanted: β
- needed: $\alpha(\theta)$



- the lens equation

$$\beta = \theta - \alpha(\theta)$$

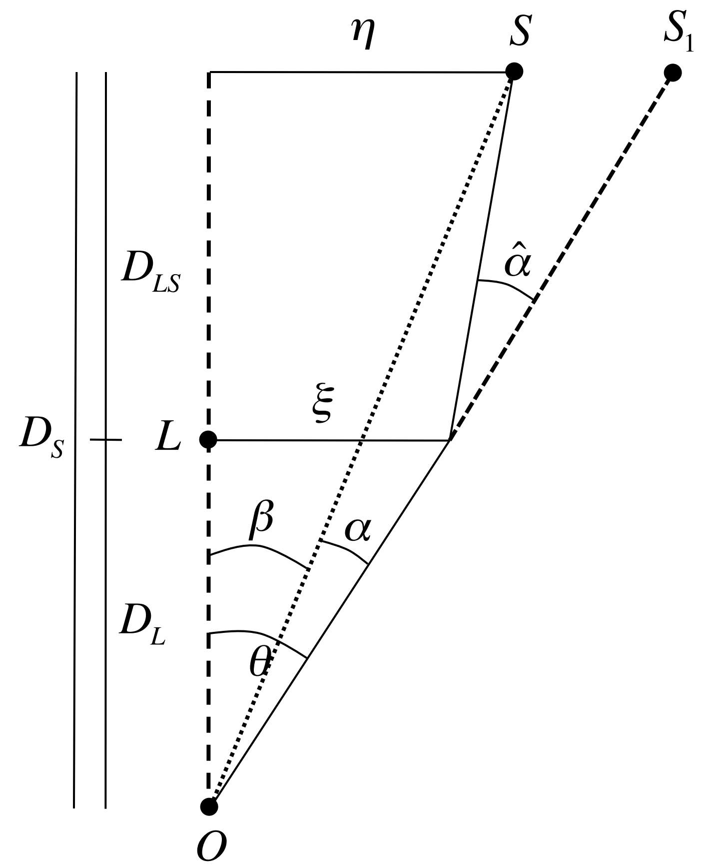


- measured: θ
- wanted: β
- **needed:** $\alpha(\theta)$

related to particulars of lens

- the lens equation

$$\beta = \theta - \alpha(\theta)$$



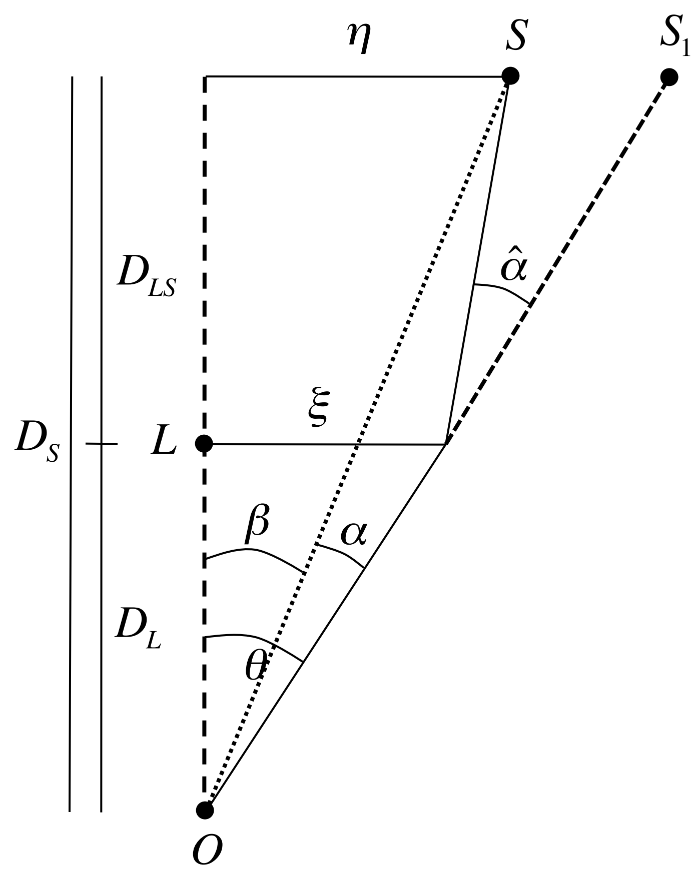
- measured: θ
- wanted: β
- **needed:** $\alpha(\theta)$

related to particulars of lens

$$\alpha = \frac{D_{LS}}{D_S}\hat{\alpha}, \qquad \hat{\alpha} = \frac{2}{c^2}\int \nabla_\xi \Phi(\xi, z)dz$$

- the lens equation

$$\beta = \theta - \alpha(\theta)$$



- measured: θ
- wanted: β
- **needed:** $\alpha(\theta)$

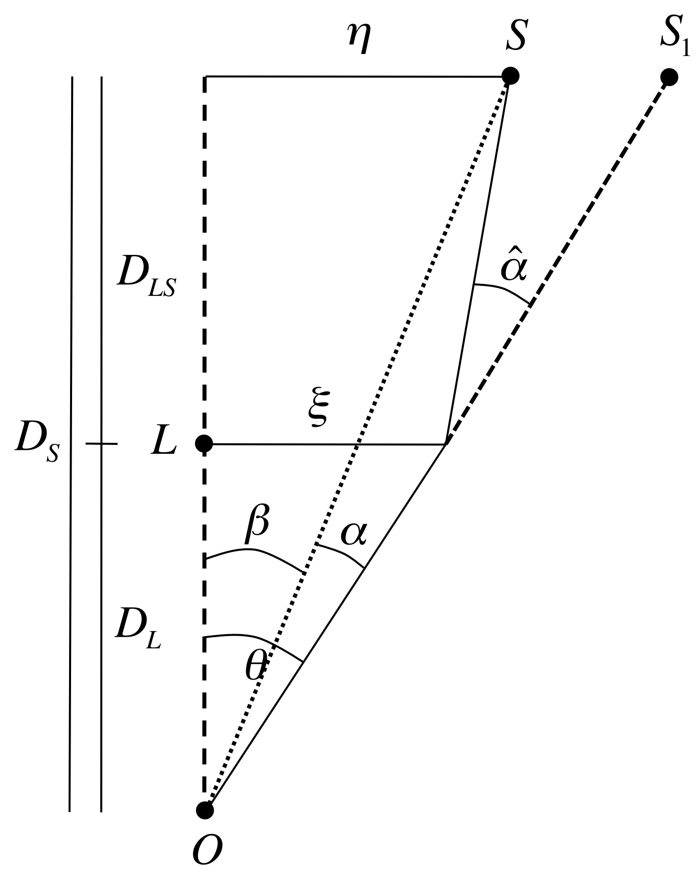
related to particulars of lens

$$\alpha = \frac{D_{LS}}{D_S}\hat{\alpha}, \qquad \hat{\alpha} = \frac{2}{c^2}\int \nabla_\xi \Phi(\xi, z)\, dz$$

projected gravitational potential of lens

- the lens equation

$$\beta = \theta - \alpha(\theta)$$



- measured: θ
- wanted: β
- **needed:** $\alpha(\theta)$

related to particulars of lens

$$\alpha = \frac{D_{LS}}{D_S}\hat{\alpha}, \qquad \hat{\alpha} = \frac{2}{c^2}\int \nabla_\xi \Phi(\xi, z)\,dz$$

projected gravitational potential of lens: the "lensing potential" →

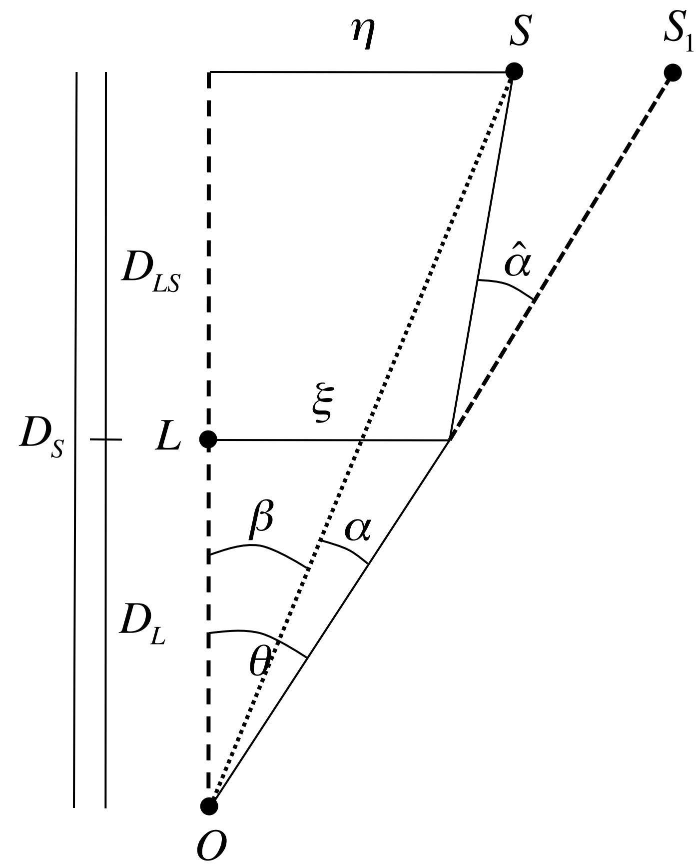
- theory
  - the basics of lensing…
    - the lens equation
    - **the lensing potential**
    - critical surface mass density
    - magnification
    - caustics and critical curves
    - distortion
    - mass-sheet degeneracy
  - some sample lenses…
    - point mass
    - extended mass
    - singular isothermal sphere

- the lensing potential

<u>analogy to gravity:</u>

force = ∇(potential)

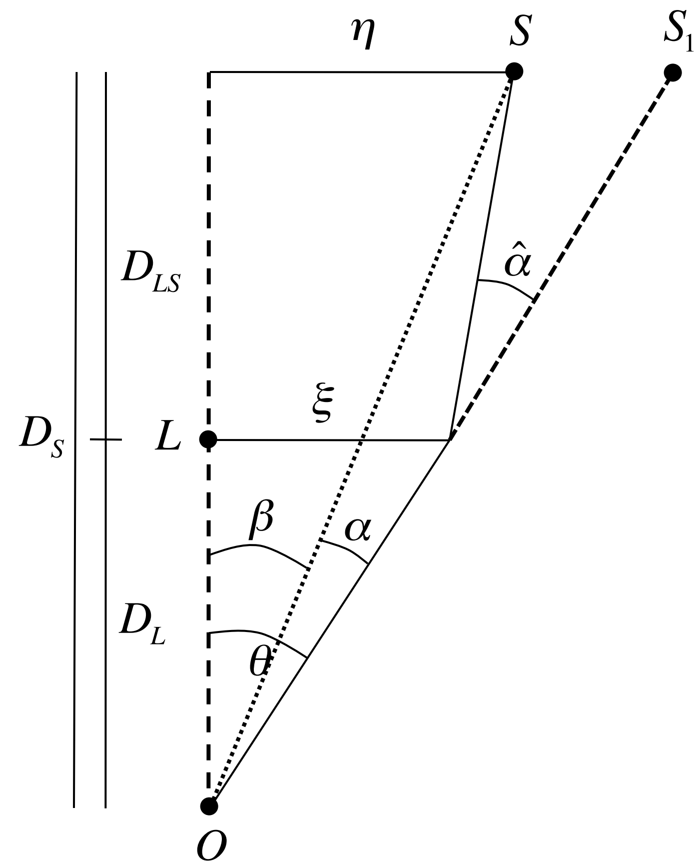
deflection angle = ∇(lensing potential)



- the lensing potential

$$\beta = \theta - \alpha(\theta)$$

"optics": $\hat{\alpha} = -\int \nabla_{\perp} n \ dz$

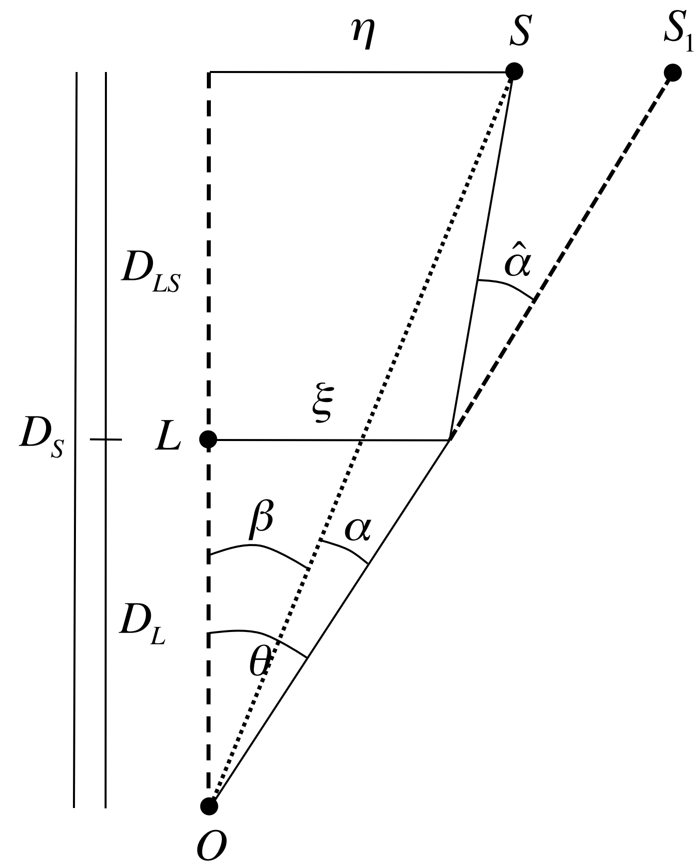


- the lensing potential

$$\beta = \theta - \alpha(\theta)$$

"optics": $$\hat{\alpha} = -\int \nabla_{\perp} n \; dz$$

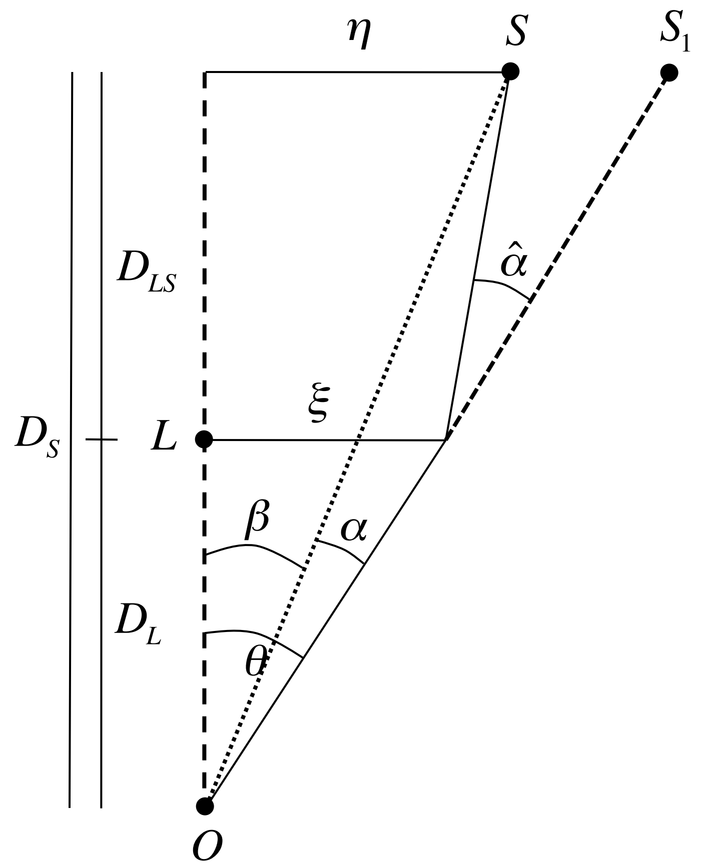
GR: $$n = 1 - \frac{2}{c^2}\Phi$$



- the lensing potential

$$\beta = \theta - \alpha(\theta)$$

$$\hat{\alpha} = -\int \nabla_{\perp} n \;\; dz$$
$$= \frac{2}{c^2}\int \nabla_{\xi}\Phi(\xi) \;\; dz$$

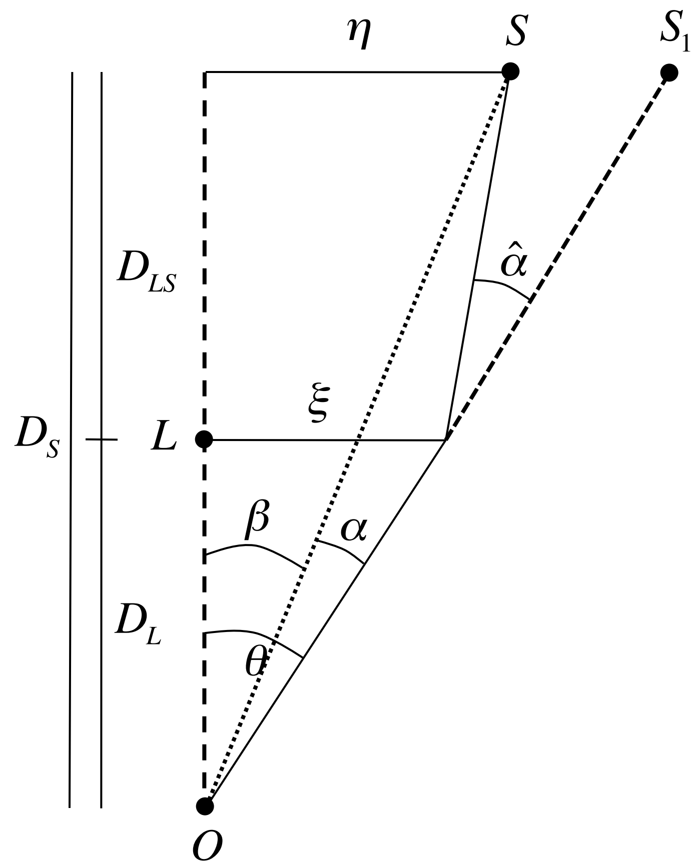


- the lensing potential

$$\beta = \theta - \alpha(\theta)$$

$$\hat{\alpha} = -\int \nabla_{\perp} n \; dz$$
$$= \frac{2}{c^2} \int \nabla_{\xi} \Phi(\xi) \; dz$$

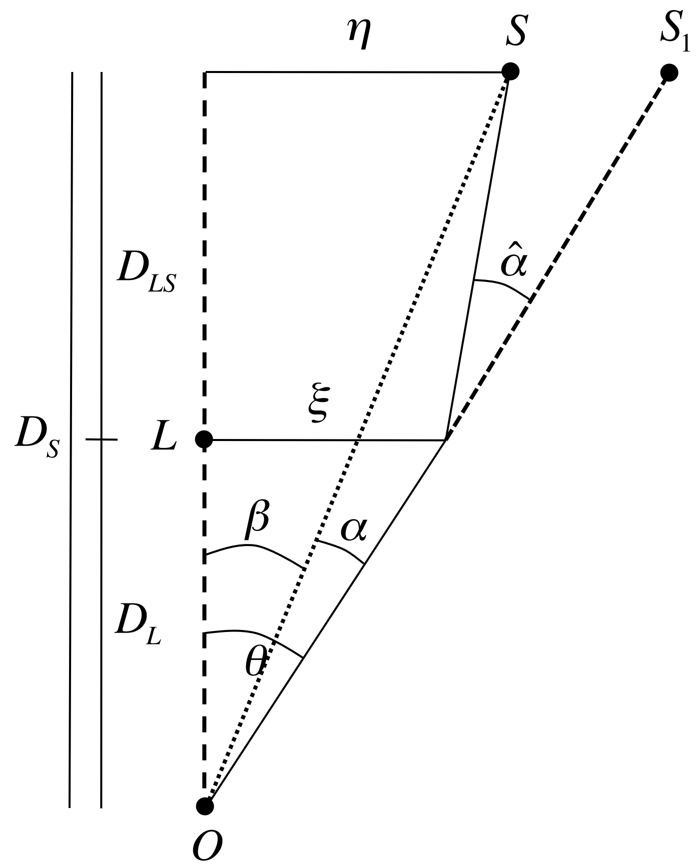
$$\alpha = \frac{D_{LS}}{D_S} \hat{\alpha}$$
$$= \frac{2}{c^2} \frac{D_{LS}}{D_S} \int \nabla_{\xi} \Phi(\xi, z) \; dz$$



- the lensing potential

$$\beta = \theta - \alpha(\theta)$$

$$\alpha = \frac{2}{c^2}\int \frac{D_{LS}}{D_S}\nabla_\xi \Phi(\xi,z)\ dz$$

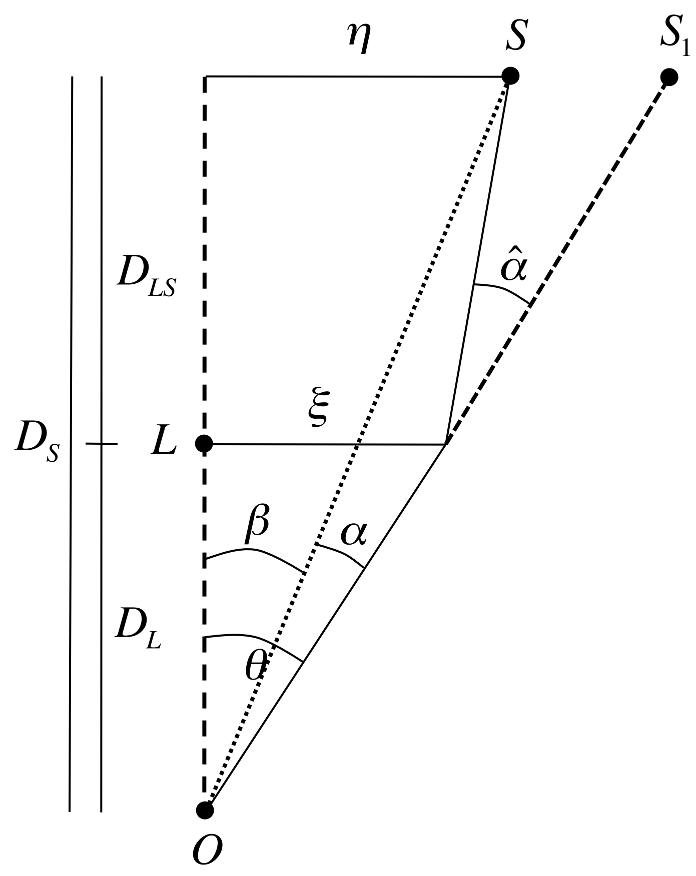


- the lensing potential

$$\beta = \theta - \alpha(\theta)$$

$$\alpha = \frac{2}{c^2} \int \frac{D_{LS}}{D_S} \nabla_\xi \Phi(\xi, z) \; dz$$

not really useful...



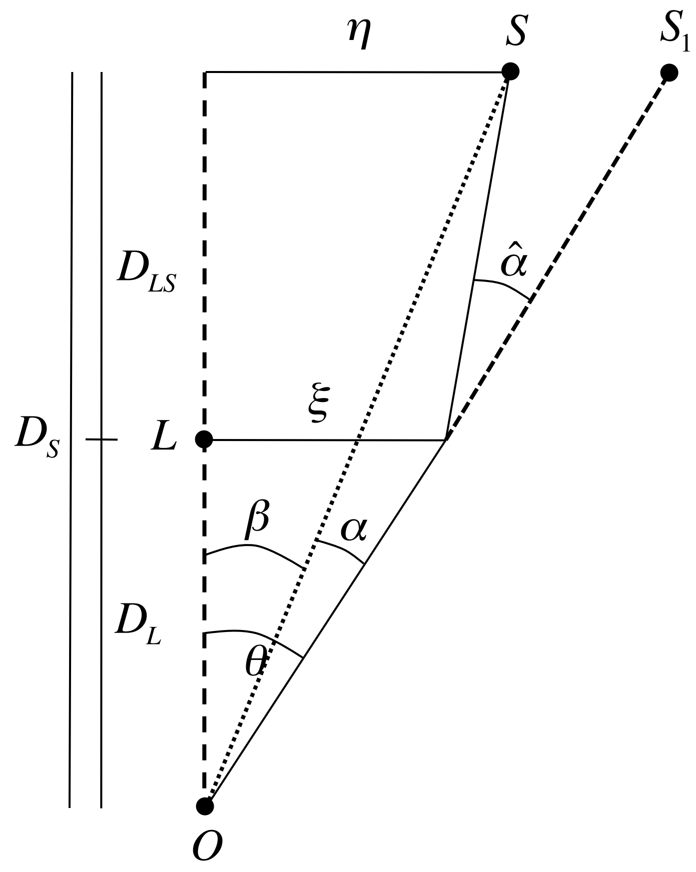
- the lensing potential

$$\beta = \theta - \alpha(\theta)$$

$$\alpha = \frac{2}{c^2}\int \frac{D_{LS}}{D_S} \nabla_\xi \Phi(\xi, z) \; dz$$

not really useful...

...but: $\xi = D_L \theta$



▪ the lensing potential

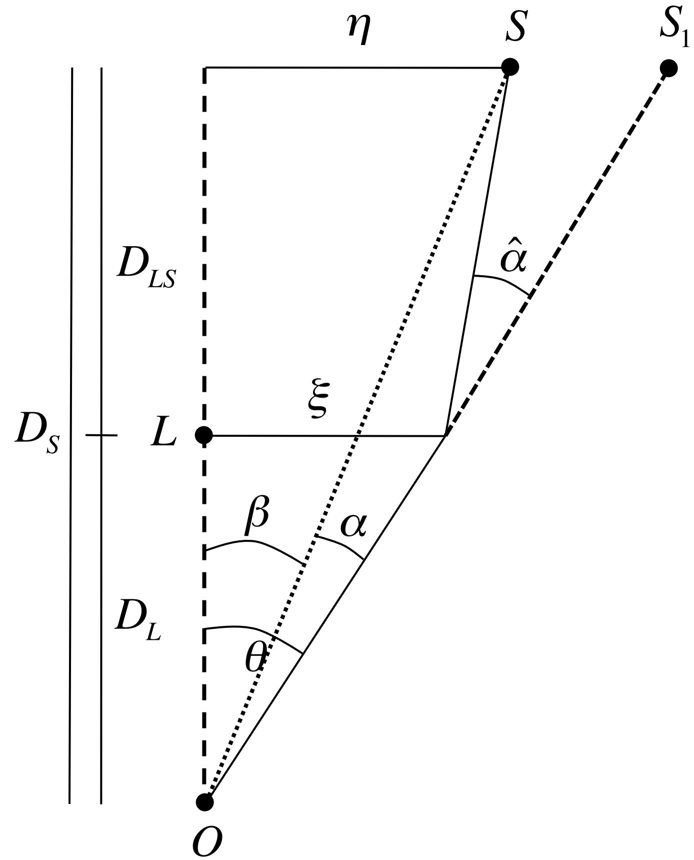
$$\beta = \theta - \alpha(\theta)$$

$$\alpha = \frac{2}{c^2}\int \frac{D_{LS}}{D_S}\nabla_\xi \Phi(\xi,z)\;\; dz$$

$$\xi = D_L \theta$$

$$= \frac{2}{c^2}\int \frac{D_{LS}}{D_S D_L}\nabla_\theta \Phi(\theta,z) dz$$

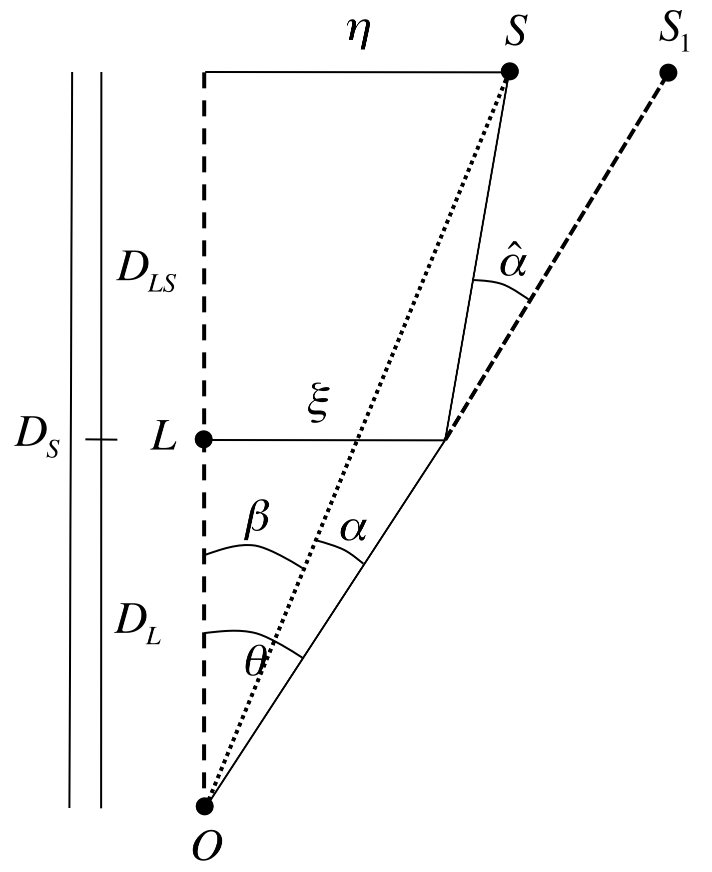
$$= \nabla_\theta \frac{2}{c^2}\int \frac{D_{LS}}{D_S D_L}\Phi(\theta,z) dz$$



- the lensing potential

$$\beta = \theta - \alpha(\theta)$$

$$\alpha = \nabla_\theta \frac{2}{c^2} \int \frac{D_{LS}}{D_S D_L} \Phi(\theta, z) dz$$



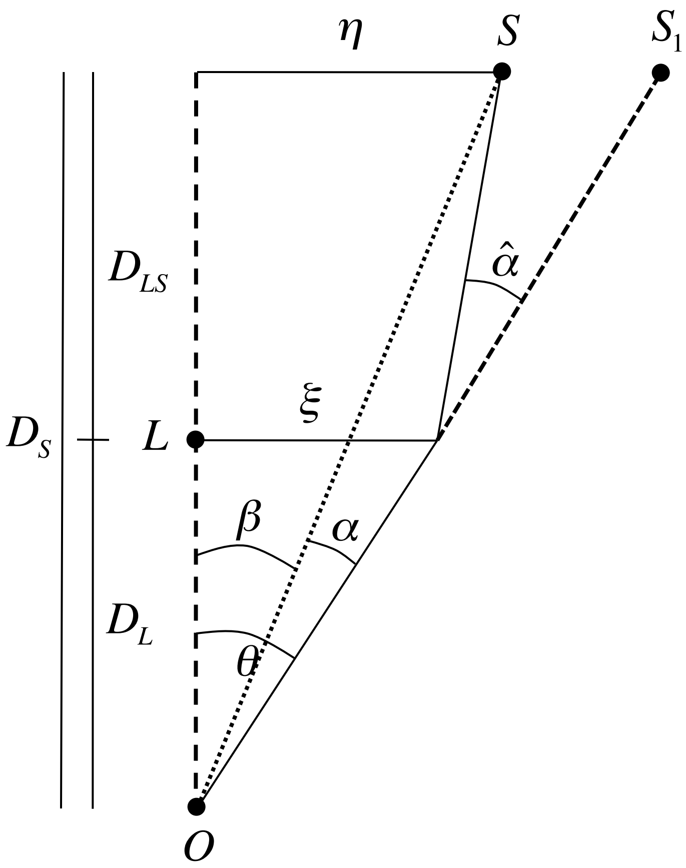
- the lensing potential

$$\beta = \theta - \alpha(\theta)$$

$$\alpha = \nabla_\theta \frac{2}{c^2} \int \frac{D_{LS}}{D_S D_L} \Phi(\theta, z) dz$$

definition of "lensing potential"

$$\varphi(\theta) = \frac{2}{c^2} \int \frac{D_{LS}}{D_S D_L} \Phi(\theta, z) dz$$

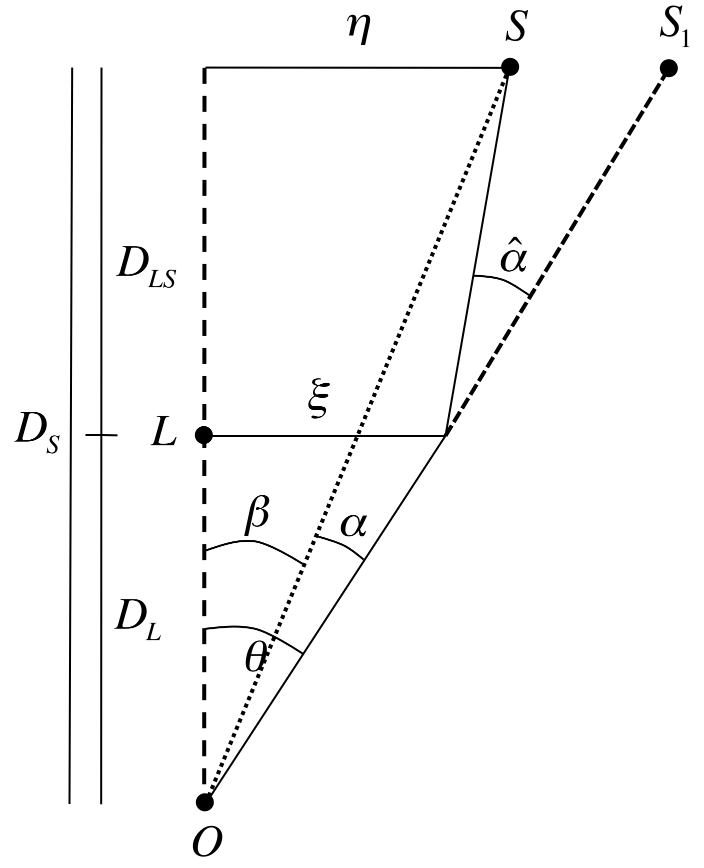


- the lensing potential

$$\beta = \theta - \alpha(\theta)$$

$$\alpha(\theta) = \nabla_\theta \varphi(\theta)$$

$$\varphi(\theta) = \frac{2}{c^2} \int \frac{D_{LS}}{D_S D_L} \Phi(\theta, z)\, dz$$



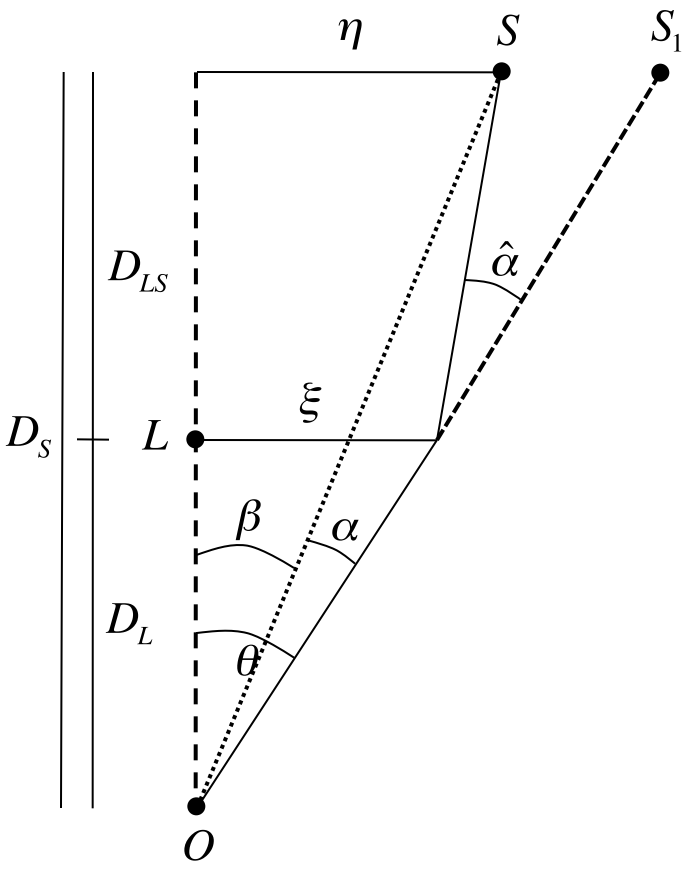
- the lensing potential

$$\beta = \theta - \alpha(\theta)$$

$$\alpha(\theta) = \nabla_{\theta}\varphi(\theta)$$

$$\varphi(\theta) = \frac{2}{c^2}\int \frac{D_{LS}}{D_S D_L}\Phi(\theta,z)\,dz$$

3D potential projected into 2D along line-of-sight!



- the lensing potential

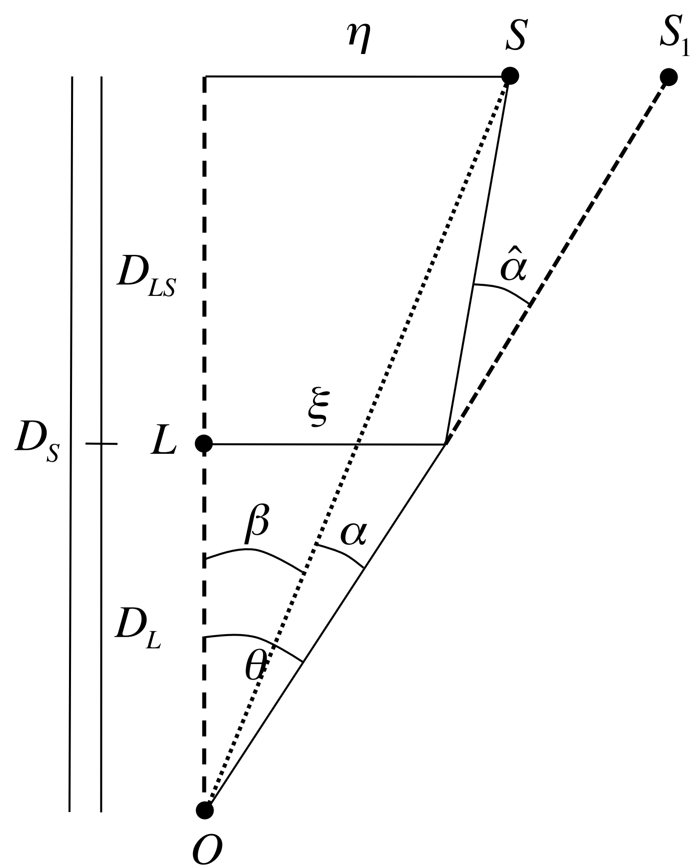
$$\beta = \theta - \alpha(\theta)$$

$$\alpha(\theta) = \nabla_\theta \varphi(\theta)$$

$$\varphi(\theta) = \frac{2}{c^2} \int \frac{D_{LS}}{D_S D_L} \Phi(\theta, z)\, dz$$

Note: $\varphi(\theta)$ is dimensionless!

3D potential projected into 2D along line-of-sight!

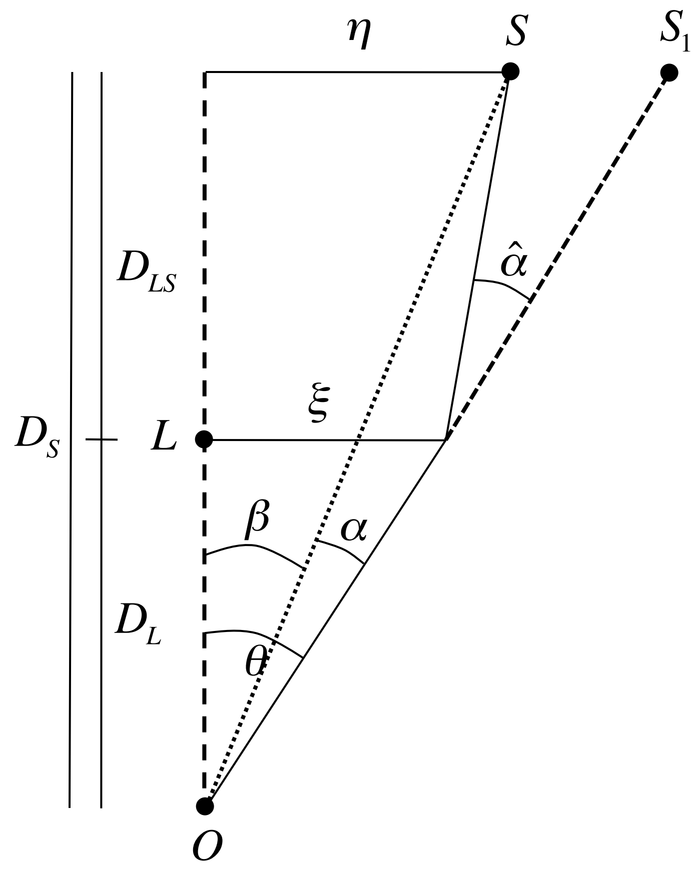


- the lensing potential

$$\varphi(\theta) = \frac{2}{c^2} \int \frac{D_{LS}}{D_S D_L} \Phi(\theta, z) dz$$

$$\nabla_\theta \varphi(\theta) = \alpha(\theta)$$

(by definition)



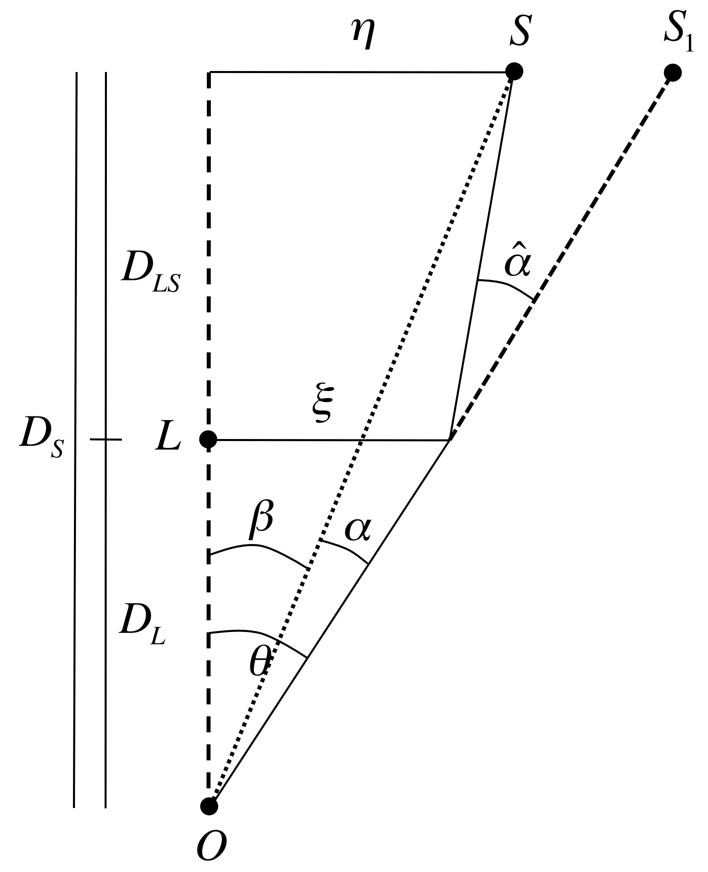
- the lensing potential

$$\varphi(\theta) = \frac{2}{c^2}\int \frac{D_{LS}}{D_S D_L}\Phi(\theta,z)\,dz$$

the knowledge of the lensing potential allows to calculate all deflection angles...

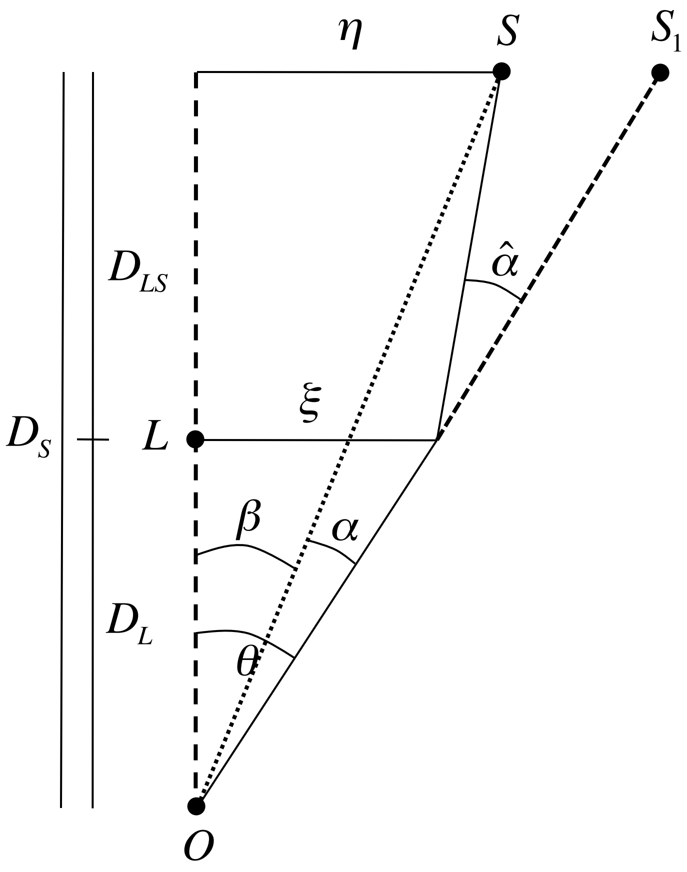
$$\nabla_\theta \varphi(\theta) = \alpha(\theta)$$

(by definition)



- the lensing potential

$$\varphi(\theta) = \frac{2}{c^2} \int \frac{D_{LS}}{D_S D_L} \Phi(\theta, z) dz$$



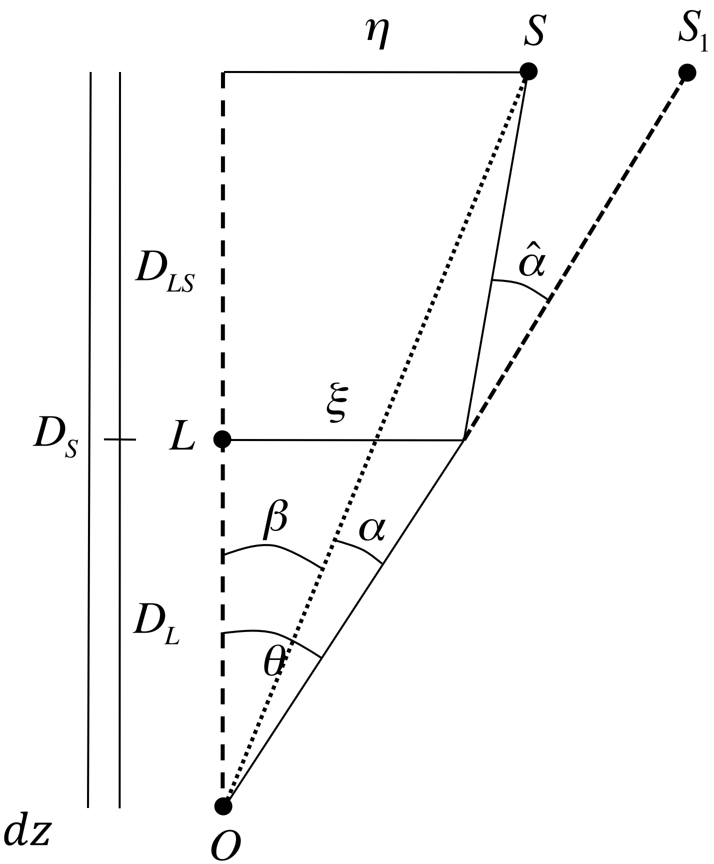
the knowledge of the lensing potential allows to calculate all deflection angles...

$$\nabla_\theta \varphi(\theta) = \alpha(\theta)$$ **(by definition)**

...and the lensing potential is related to the projected surface mass density $\Sigma(\theta)$ of the lens!

- the lensing potential

$$\varphi(\theta) = \frac{2}{c^2} \int \frac{D_{LS}}{D_S D_L} \Phi(\theta, z) dz$$



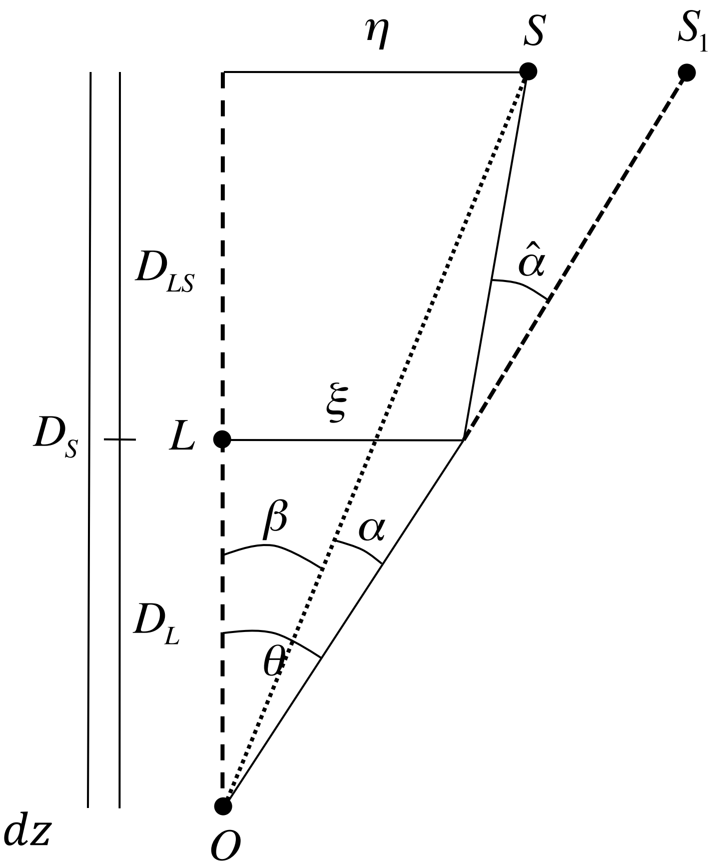
$$\nabla_\theta \varphi(\theta) = \alpha(\theta)$$ ***(by definition)***

projected surface mass density $\Sigma(\theta) = \int \rho(\theta, z) dz$

$$\nabla_\theta^2 \varphi(\theta) = 2 \frac{\Sigma(\theta)}{\Sigma_{crit}}$$ ***(exercise)***

- the lensing potential

$$\varphi(\theta) = \frac{2}{c^2}\int \frac{D_{LS}}{D_S D_L}\Phi(\theta,z)dz$$



$$\nabla_\theta \varphi(\theta) = \alpha(\theta)$$ ***(by definition)***

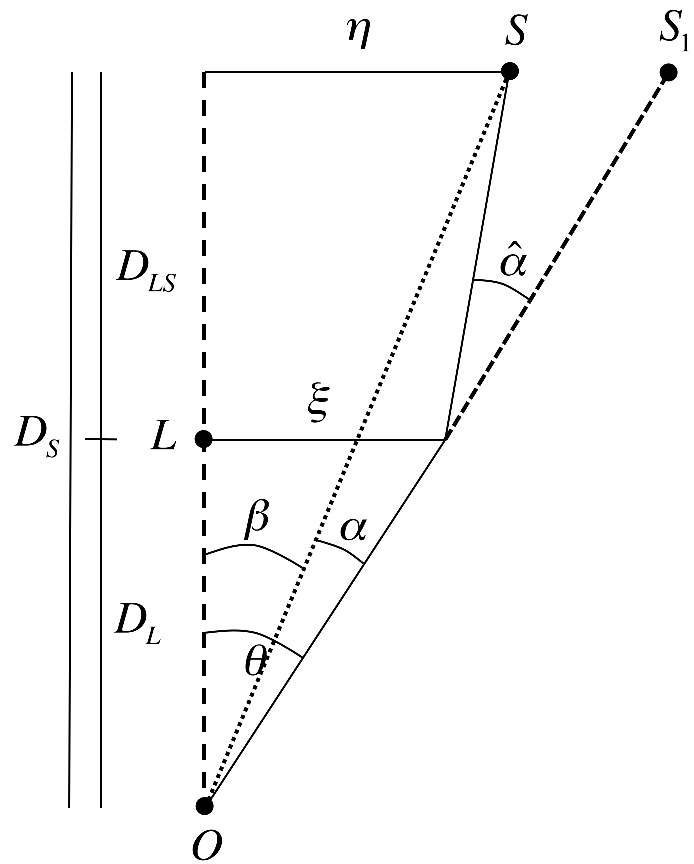
projected surface mass density $\Sigma(\theta) = \int \rho(\theta, z)dz$

$$\nabla_\theta^2 \varphi(\theta) = 2\frac{\Sigma(\theta)}{\Sigma_{crit}}$$ ***(exercise)***

$$\Sigma_{crit} = \frac{c^2}{4\pi G}\frac{D_S}{D_{LS}D_L}$$ critical surface mass density (mere geometry dependence…)

- the lensing potential

$$\nabla_\theta^2 \varphi(\theta) = 2\frac{\Sigma(\theta)}{\Sigma_{crit}}$$



$$\alpha(\theta) = \nabla_\theta \varphi(\theta)$$

$$\beta = \theta - \alpha(\theta)$$

- the lensing potential

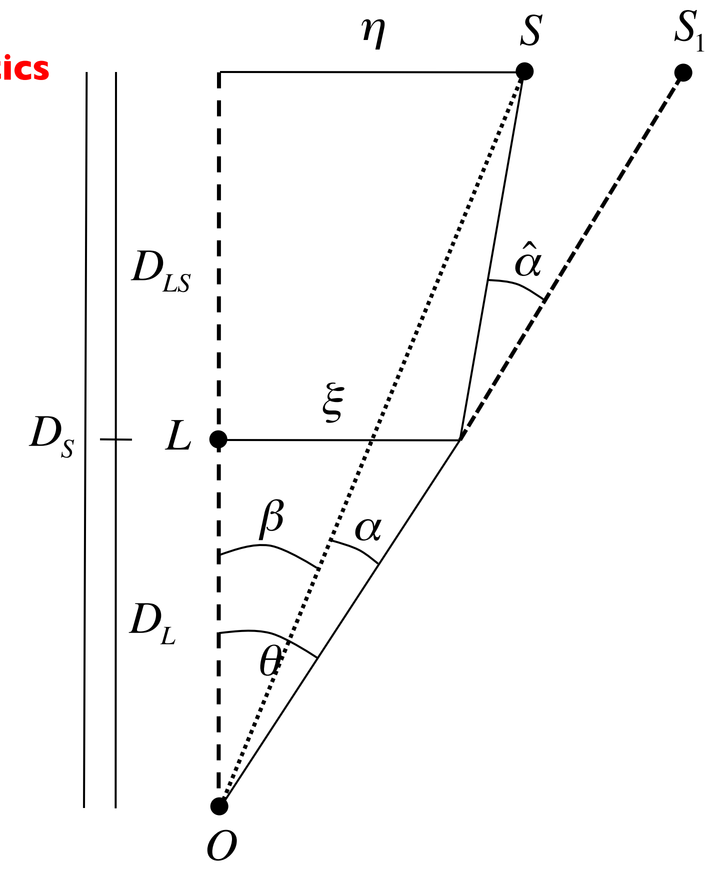
$$\nabla_\theta^2 \varphi(\theta) = 2\frac{\Sigma(\theta)}{\Sigma_{crit}}$$

lens characteristics

$$\Sigma(\theta) = \int \rho(\theta, z)\, dz$$

$$\alpha(\theta) = \nabla_\theta \varphi(\theta)$$

$$\beta = \theta - \alpha(\theta)$$



- the lensing potential

$$\nabla_\theta^2 \varphi(\theta) = 2\frac{\Sigma(\theta)}{\Sigma_{crit}}$$

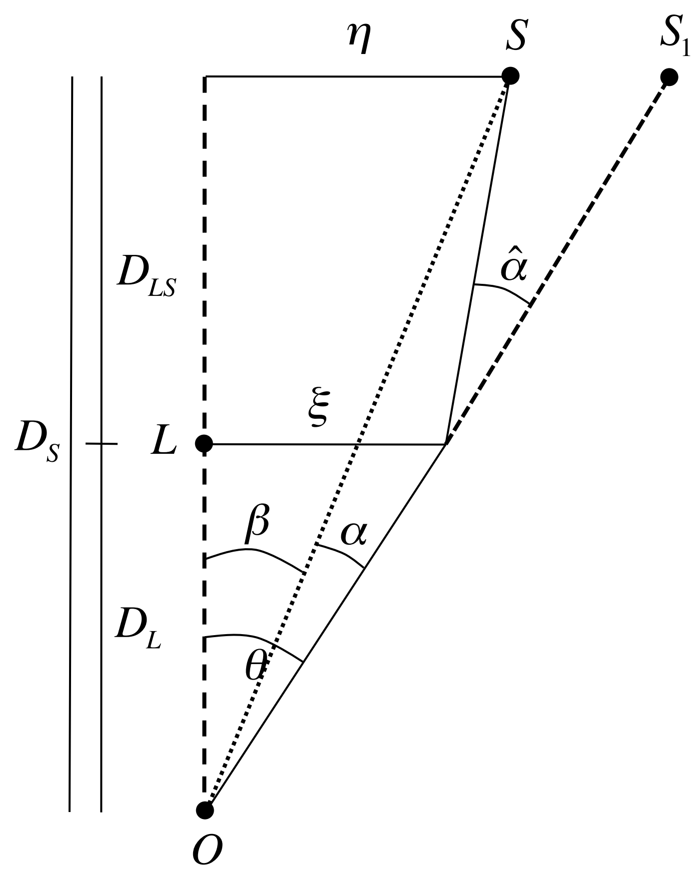
geometry

$$\Sigma(\theta) = \int \rho(\theta, z)dz$$

$$\Sigma_{crit} = \frac{c^2}{4\pi G}\frac{D_S}{D_{LS}D_L}$$

$$\alpha(\theta) = \nabla_\theta \varphi(\theta)$$

$$\beta = \theta - \alpha(\theta)$$



- the lensing potential

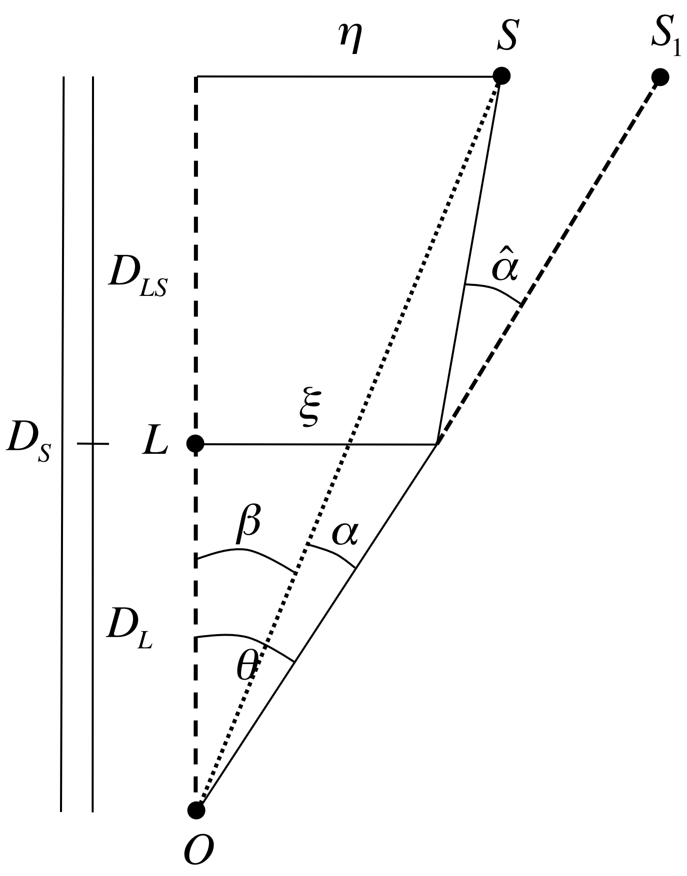
$$\nabla^2_\theta \varphi(\theta) = 2\frac{\Sigma(\theta)}{\Sigma_{crit}}$$

$$\Sigma(\theta) = \int \rho(\theta, z)dz$$

$$\Sigma_{crit} = \frac{c^2}{4\pi G}\frac{D_S}{D_{LS}D_L}$$

$$\alpha(\theta) = \nabla_\theta \varphi(\theta)$$

$$\beta = \theta - \alpha(\theta)$$



critical mass surface density →

- theory
  - the basics of lensing…
    - the lens equation
    - the lensing potential
    - **critical surface mass density**
    - magnification
    - caustics and critical curves
    - distortion
    - mass-sheet degeneracy
  - some sample lenses…
    - point mass
    - extended mass
    - singular isothermal sphere

- critical surface mass density

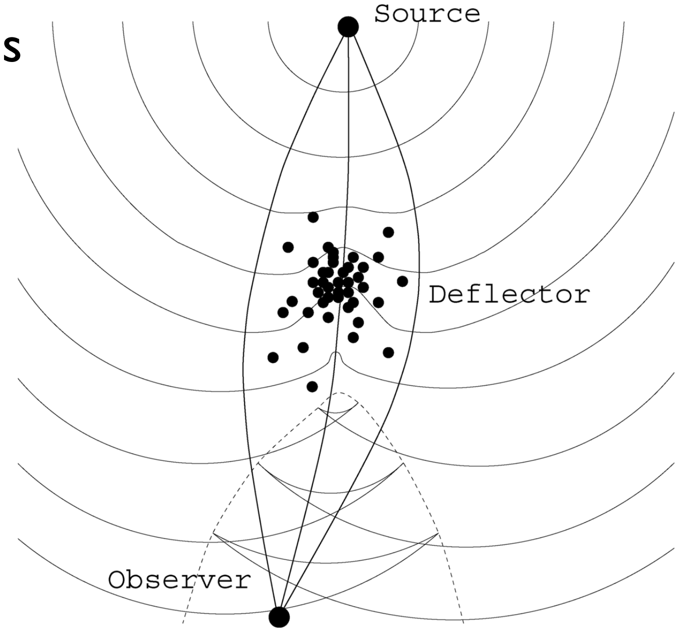
$$\Sigma_{crit} = \frac{c^2}{4\pi G}\frac{D_S}{D_{LS}D_L}$$

  - depends *only* on distances to source and lens

- critical surface mass density

$$\Sigma_{crit} = \frac{c^2}{4\pi G}\frac{D_S}{D_{LS}D_L}$$

- depends *only* on distances to source and lens



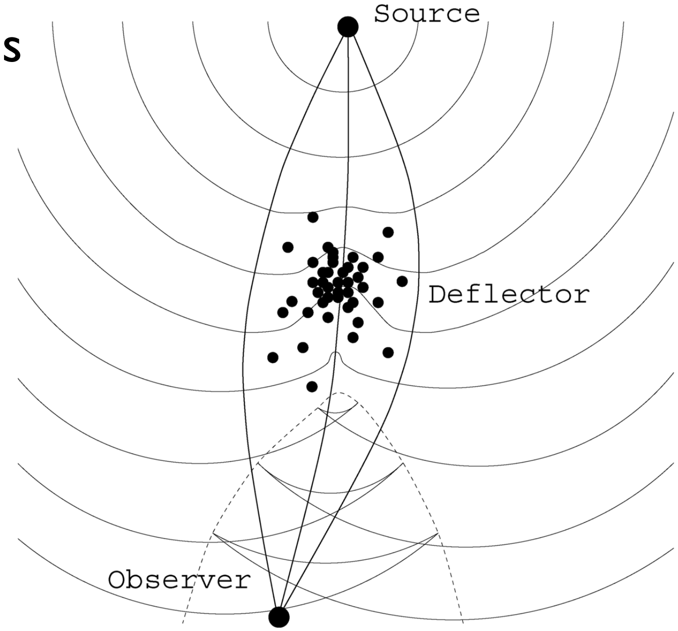
- critical surface mass density

$$\Sigma_{crit} = \frac{c^2}{4\pi G}\frac{D_S}{D_{LS}D_L}$$

- depends *only* on distances to source and lens
- separates 'weak' from 'strong' lenses:

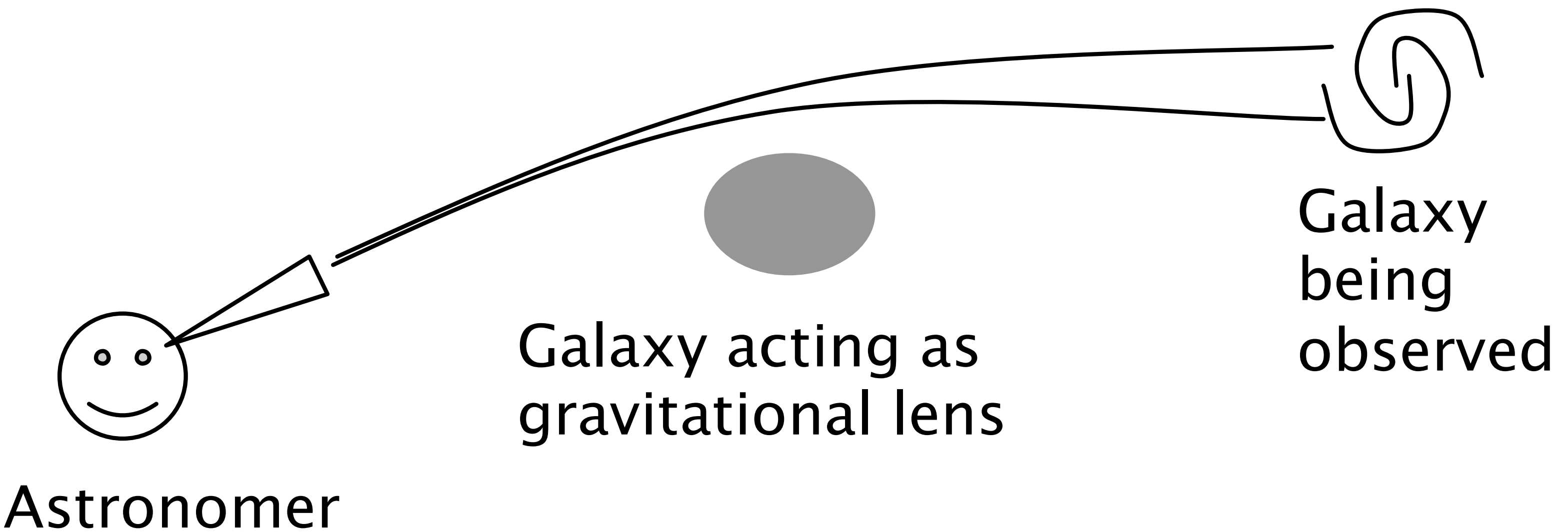
$\Sigma > \Sigma_{crit}$ => multiple images possible

$\Sigma < \Sigma_{crit}$ => only distortions

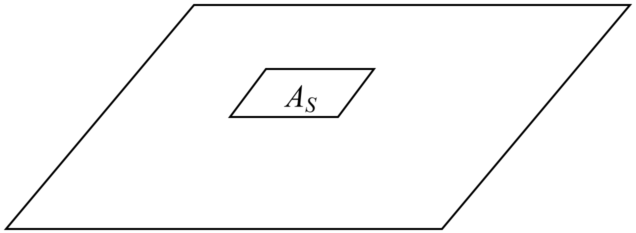


- theory
  - the basics of lensing…
    - the lens equation
    - the lensing potential
    - critical surface mass density
    - **magnification**
    - caustics and critical curves
    - distortion
    - mass-sheet degeneracy
  - some sample lenses…
    - point mass
    - extended mass
    - singular isothermal sphere

- magnification



(Bernard Schutz, http://www.gravityfromthegroundup.org) lensing preserves surface brightness*

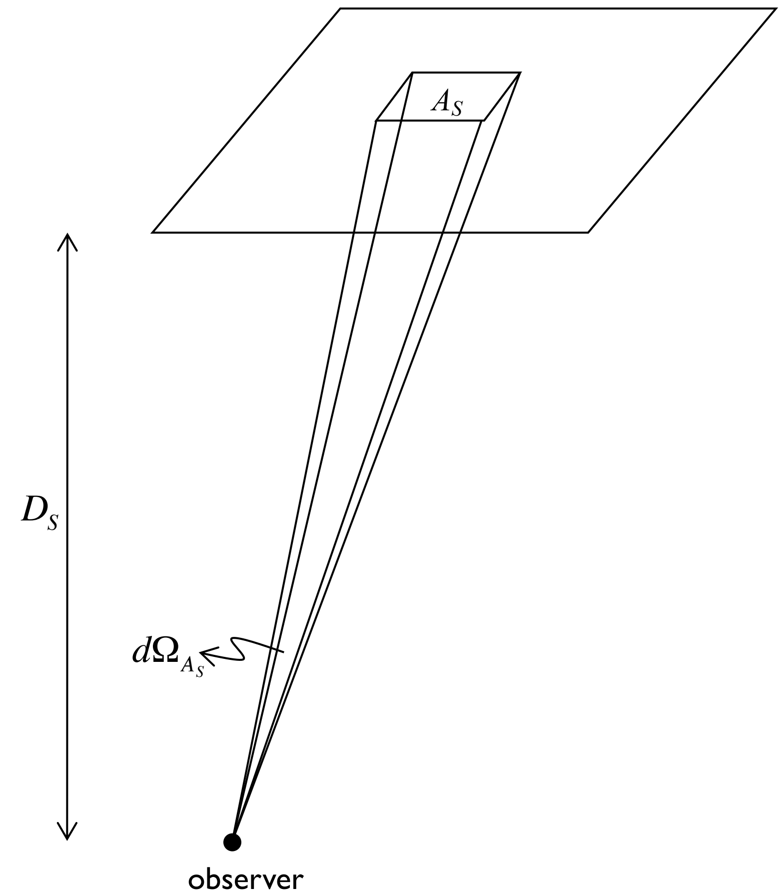


- magnification
    - differential deflection of light-rays

*surface brightness = apparent brightness per unit ***angular*** area

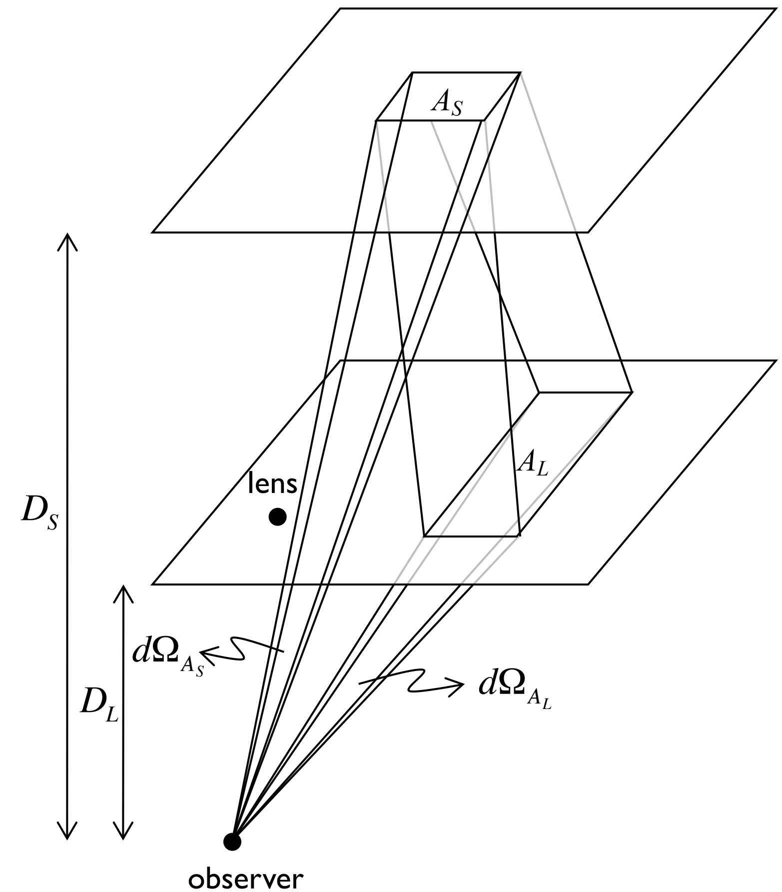
- magnification
  - differential deflection of light-rays

lensing preserves surface brightness



- magnification
  - differential deflection of light-rays

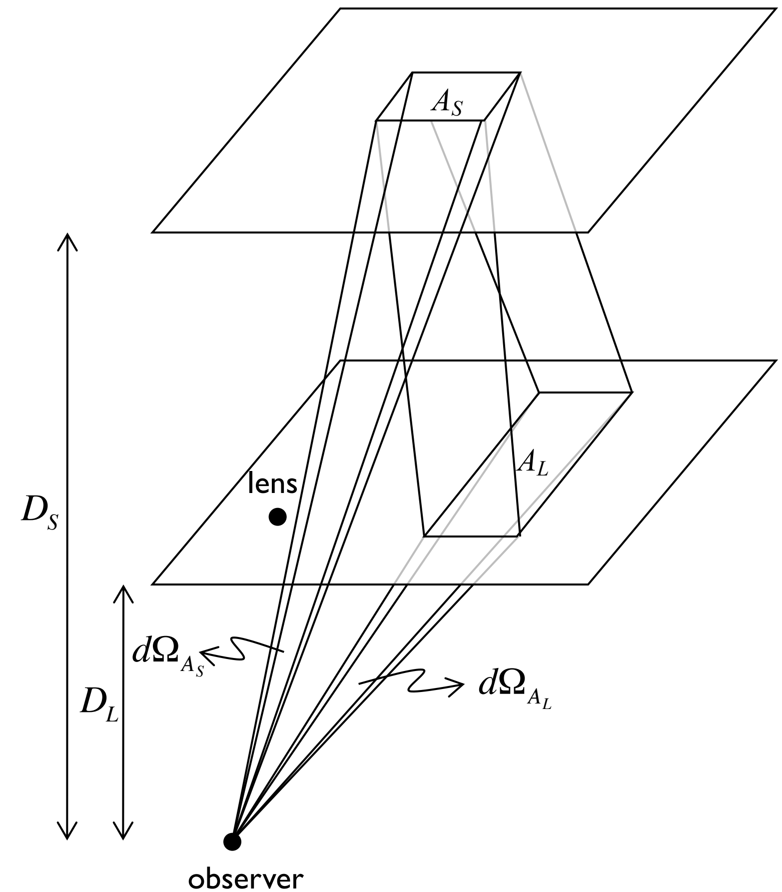
lensing preserves surface brightness



- magnification
  - differential deflection of light-rays

$$d\Omega_{A_S} \neq d\Omega_{A_L}$$

lensing preserves surface brightness



▪ magnification

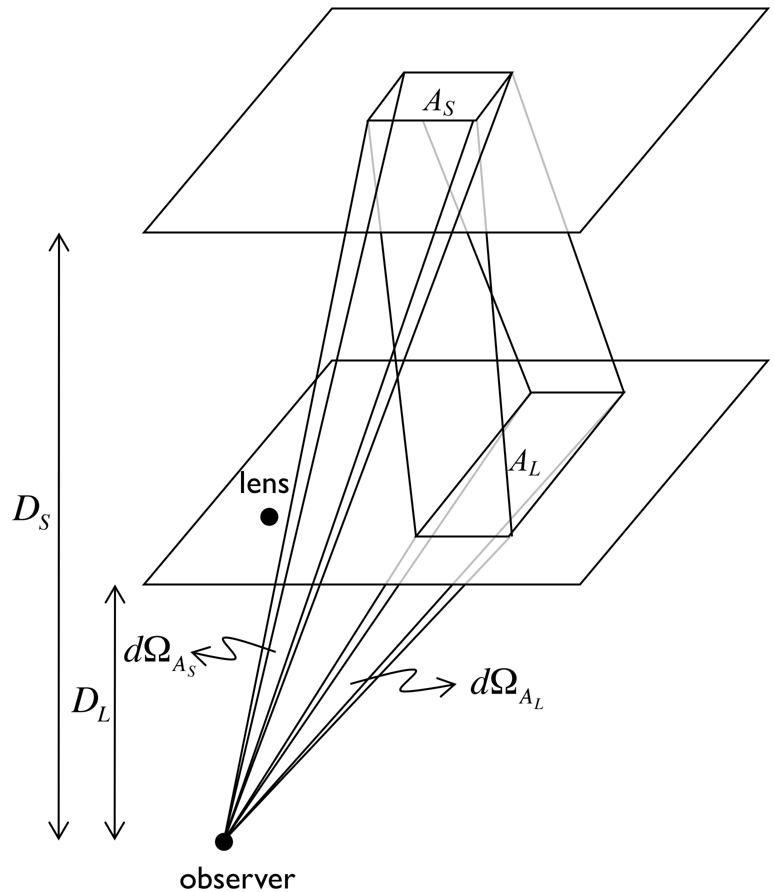
- differential deflection of light-rays

$$d\Omega_{A_S} \neq d\Omega_{A_L}$$

as the number of photons is conserved, the ratio between the two solid angles determines the magnification:

$$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}}$$

lensing preserves surface brightness



▪ magnification

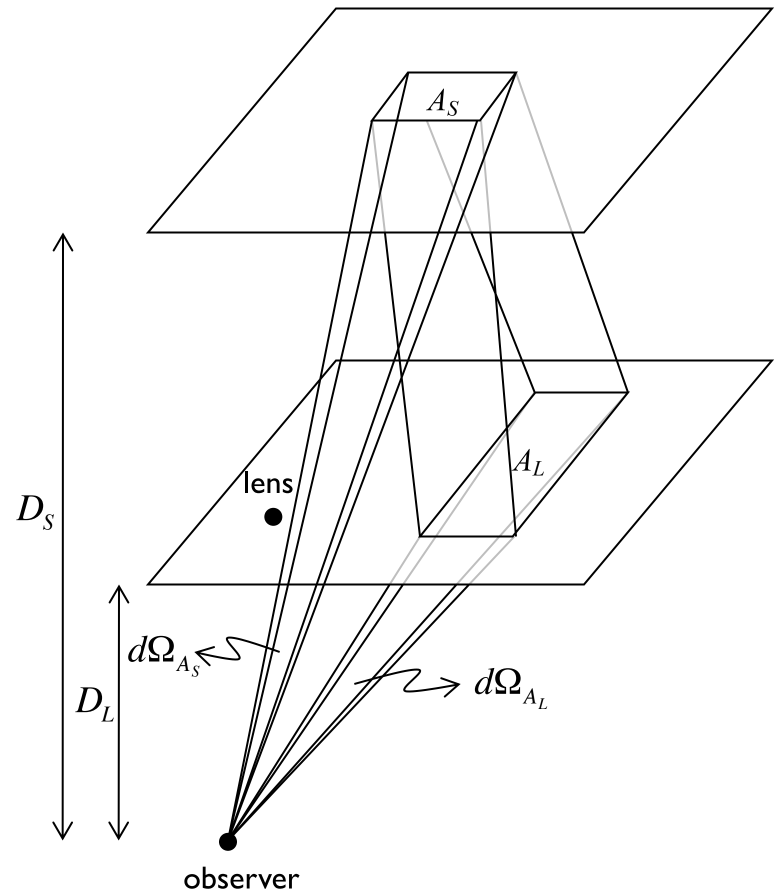
- differential deflection of light-rays

$$d\Omega_{A_S} \neq d\Omega_{A_L}$$

as the number of photons is conserved, the ratio between the two solid angles determines the magnification:

$$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}} = ?$$

lensing preserves surface brightness

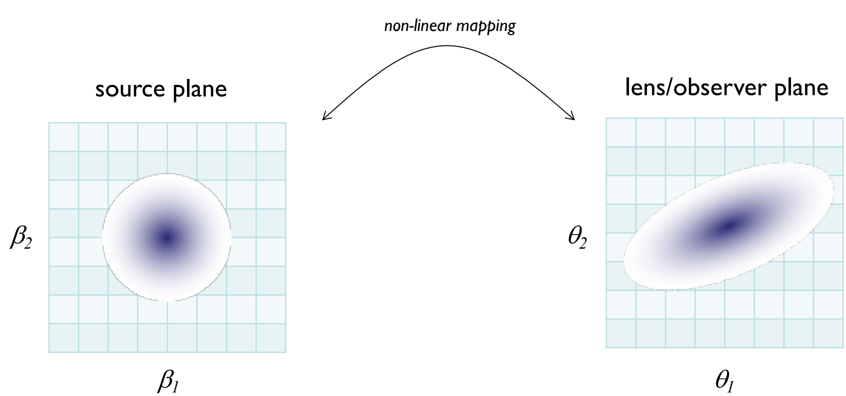


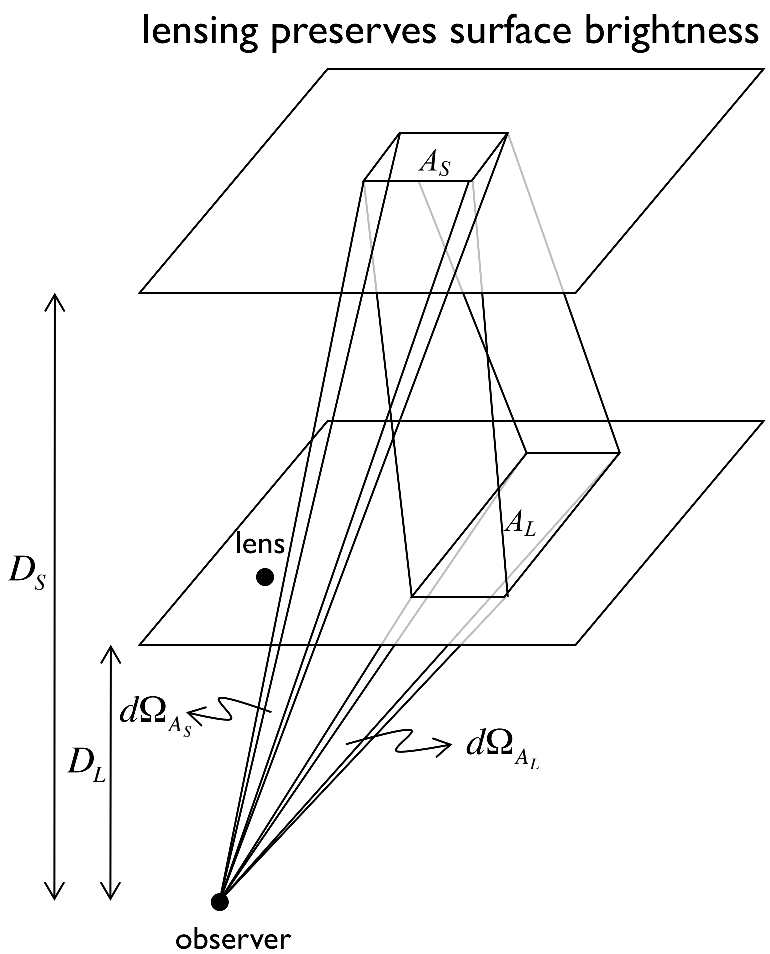
- magnification
  - differential deflection of light-rays

$$d\Omega_{A_S} \neq d\Omega_{A_L}$$

as the number of photons is conserved, the ratio between the two solid angles determines the magnification:

$$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}} = ?$$



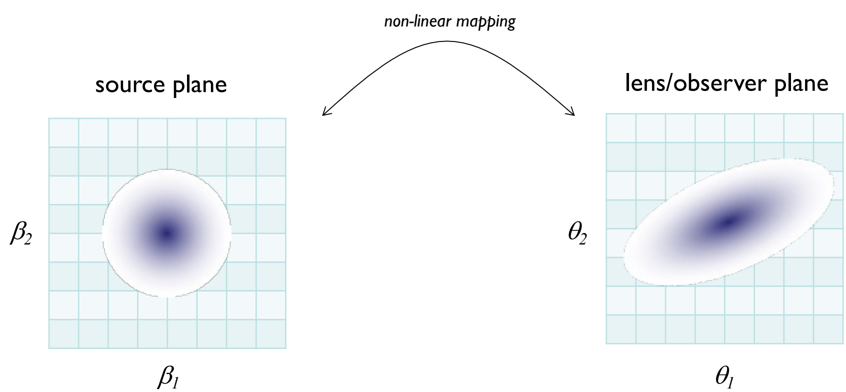


- magnification
  - differential deflection of light-rays

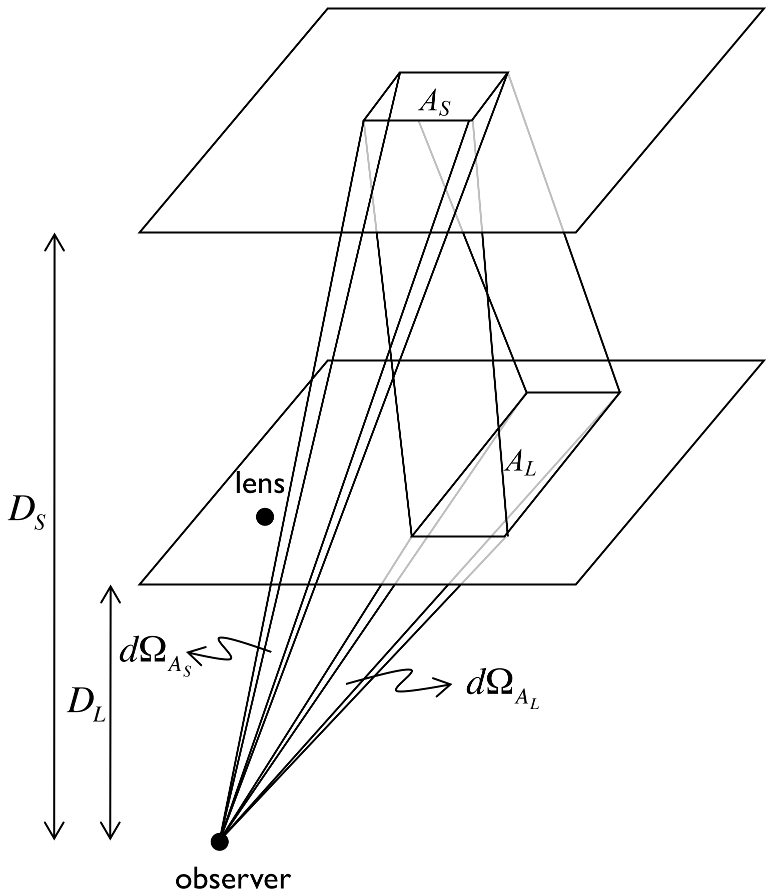
$$d\Omega_{A_S} \neq d\Omega_{A_L}$$

  - coordinate transformation β to θ

$$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}} = \left[det\left(\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\right)\right]^{-1}$$



lensing preserves surface brightness



▪ magnification

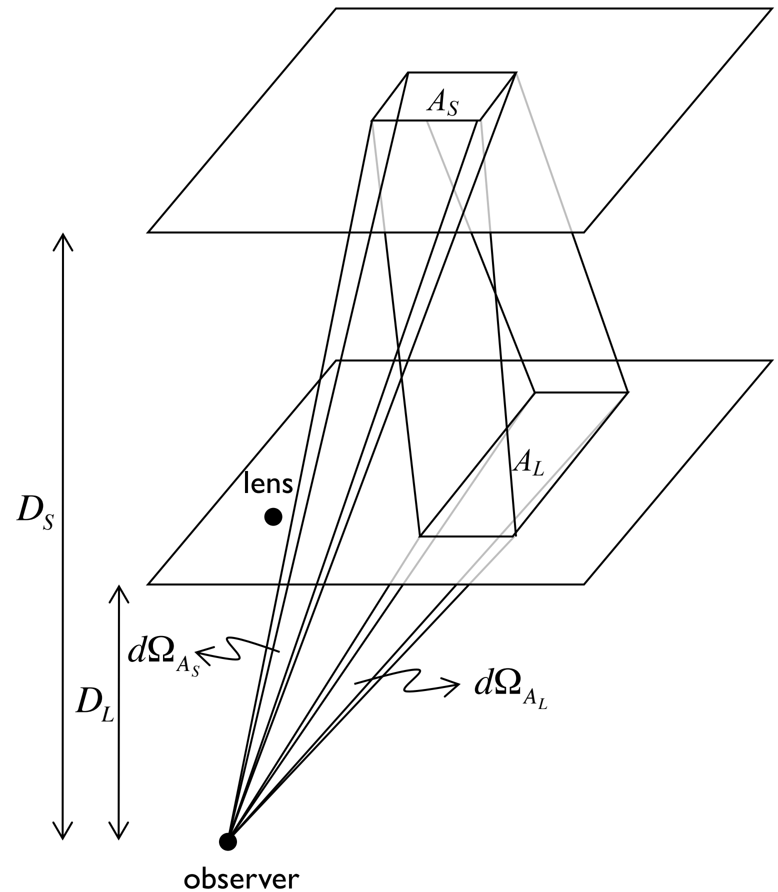
- differential deflection of light-rays

$$d\Omega_{A_S} \neq d\Omega_{A_L}$$

- coordinate transformation β to θ

$$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}} = \left[det\left(\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\right)\right]^{-1}$$

lensing preserves surface brightness



- spherical symmetry

$$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}} = \frac{dA_L}{4\pi D_L^2}\frac{4\pi D_S^2}{dA_S} = \frac{D_S^2 dA_L}{D_L^2 dA_S} = \frac{D_S^2 d(D_L^2\theta^2)}{D_L^2 d(D_S^2\beta^2)} = \frac{D_S^2 D_L^2 d(\theta^2)}{D_L^2 D_S^2 d(\beta^2)} = \frac{\theta d\theta}{\beta d\beta}$$

small angle approximation

Note:
$\mu < 0$ => mirror inversion of image

▪ magnification

- differential deflection of light-rays

$$d\Omega_{A_S} \neq d\Omega_{A_L}$$

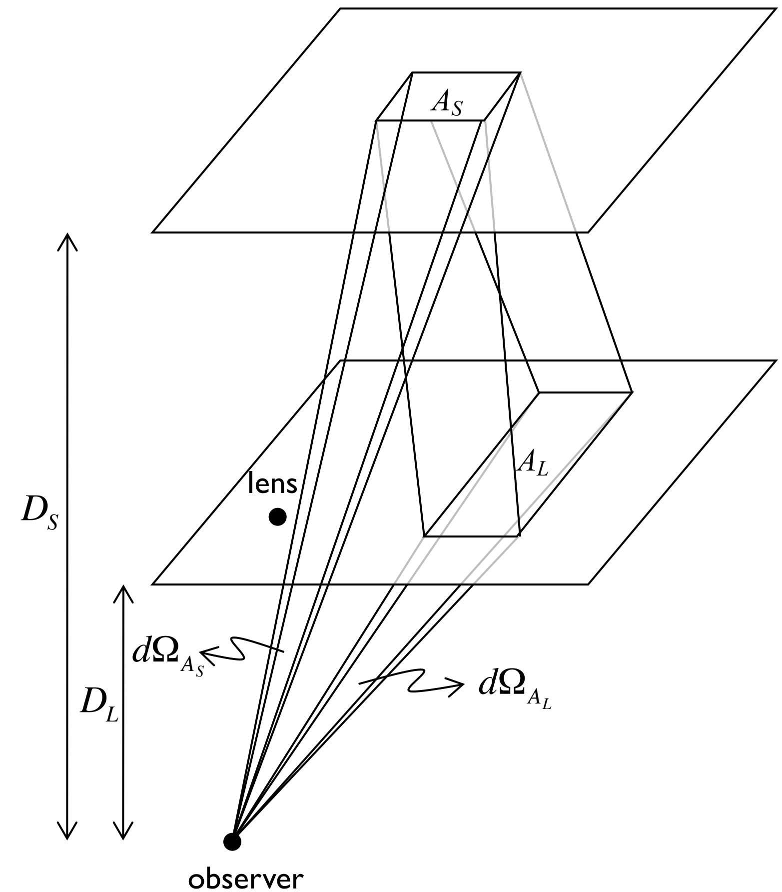
- coordinate transformation β to θ

$$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}} = \left[det \left(\frac{\partial \vec{\beta}}{\partial \vec{\theta}} \right) \right]^{-1}$$

- spherical symmetry

$$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}} = \frac{\theta d\theta}{\beta d\beta}$$

lensing preserves surface brightness



- magnification
  - differential deflection of light-rays

$$d\Omega_{A_S} \neq d\Omega_{A_L}$$

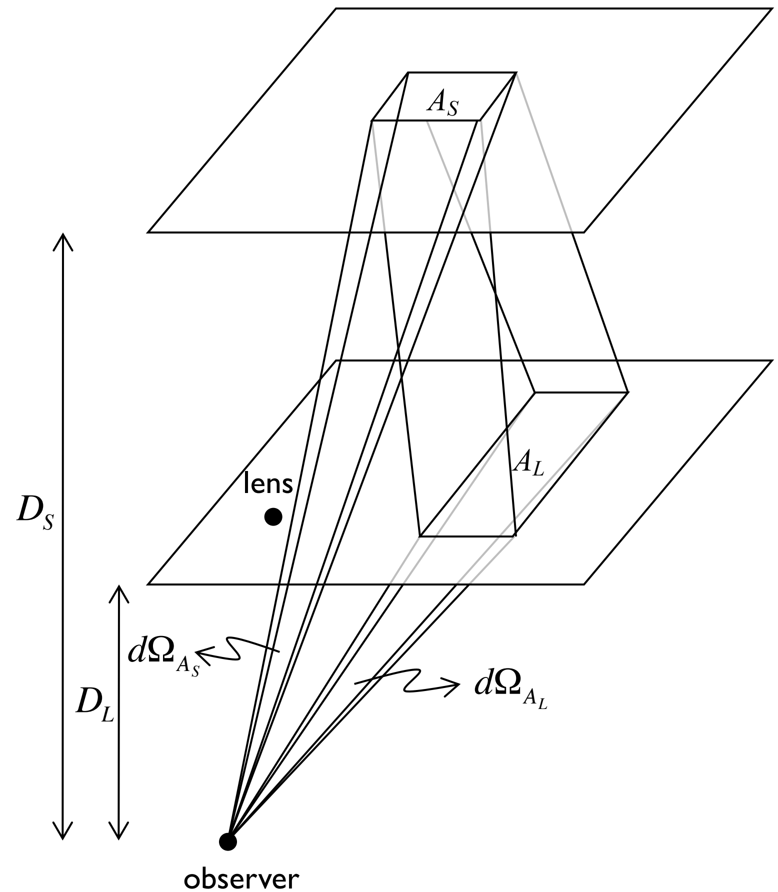
  - coordinate transformation β to θ

$$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}} = \left[det \left(\frac{\partial \vec{\beta}}{\partial \vec{\theta}} \right) \right]^{-1}$$

  - spherical symmetry

$$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}} = \frac{\theta d\theta}{\beta d\beta}$$

lensing preserves surface brightness



Note:
there are usually multiple images and
the sum of their magnifications equals unity

▪ magnification

- differential deflection of light-rays

$$d\Omega_{A_S} \neq d\Omega_{A_L}$$

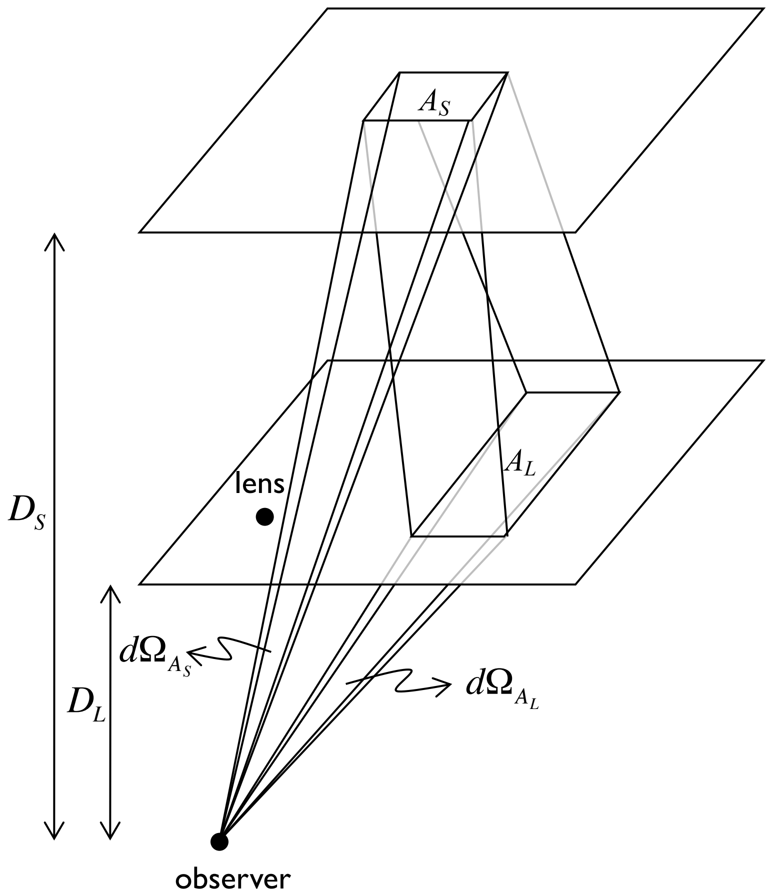
- coordinate transformation β to θ

$$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}} = \left[\underbrace{det\left(\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\right)}_{=0\,\rightarrow\,\mu=\infty\ \ !?} \right]^{-1}$$

- spherical symmetry

$$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}} = \frac{\theta d\theta}{\beta d\beta}$$

lensing preserves surface brightness



<u>Note:</u>
there are usually multiple images and
the sum of their magnifications equals unity

- theory
  - the basics of lensing…
    - the lens equation
    - the lensing potential
    - critical surface mass density
    - magnification
    - **caustics and critical curves**
    - distortion
    - mass-sheet degeneracy
  - some sample lenses…
    - point mass
    - extended mass
    - singular isothermal sphere

- caustics and critical curves
  - magnification: $$\mu = \frac{d\Omega_{A_L}}{d\Omega_{A_S}} = \left[det\left(\frac{\partial \vec{\beta}}{\partial \vec{\theta}} \right) \right]^{-1}$$

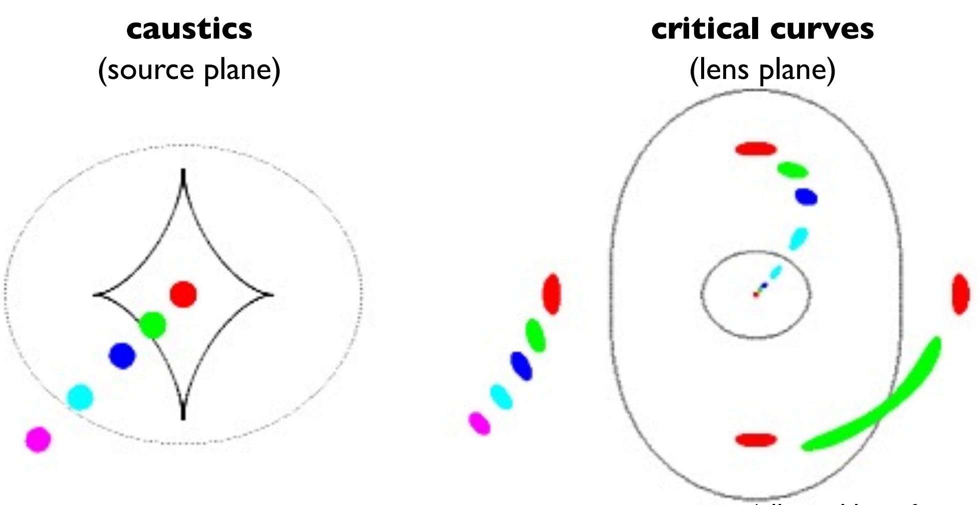
- caustics and critical curves
  - (formally) infinite magnification: $\mu = \infty \quad \Leftrightarrow \quad \det\left(\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\right) = 0$

- caustics and critical curves
  - (formally) infinite magnification: $\mu = \infty \quad \Leftrightarrow \quad \det\left(\frac{\partial \vec{\beta}}{\partial \vec{\theta}}\right) = 0$

$$\det\left(\frac{\partial \vec{\beta}}{\partial \vec{\theta}}\right) = 0$$

$\vec{\beta}_{\mu=\infty}$: **caustics** (in ***source plane***)

$\vec{\theta}_{\mu=\infty}$: **critical curves** (in ***lens plane***)



(elliptical lens, figure taken from Natarayan & Bartelmann1995)

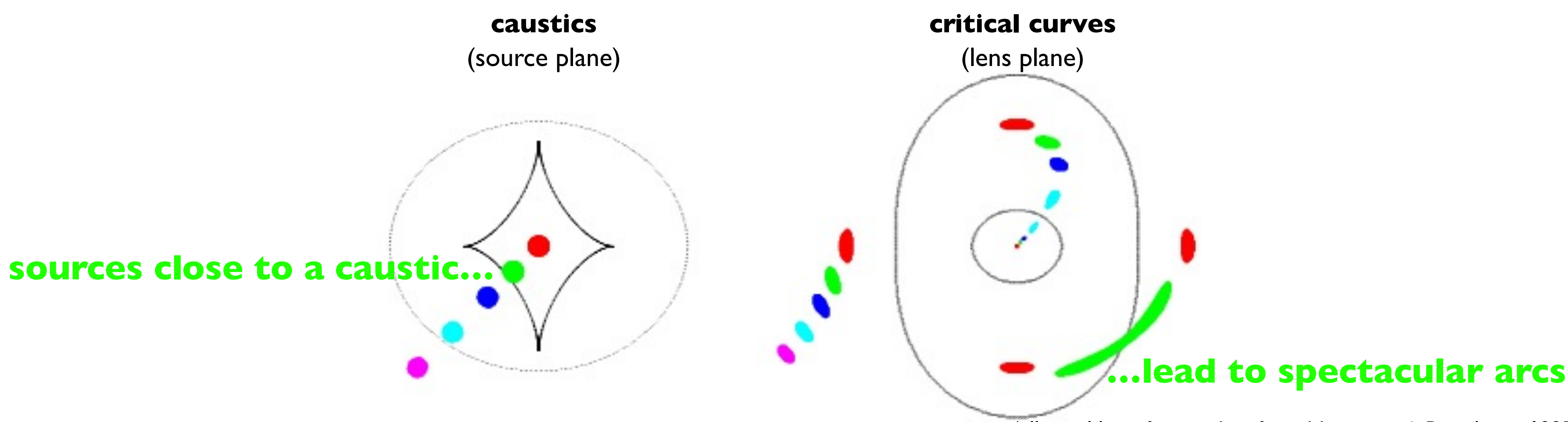
- caustics and critical curves

  - (formally) infinite magnification: $\mu = \infty \quad \Leftrightarrow \quad \det\left(\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\right) = 0$

$$\det\left(\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\right) = 0$$

$\vec{\beta}_{\mu=\infty}$: **caustics** (in ***source plane***)

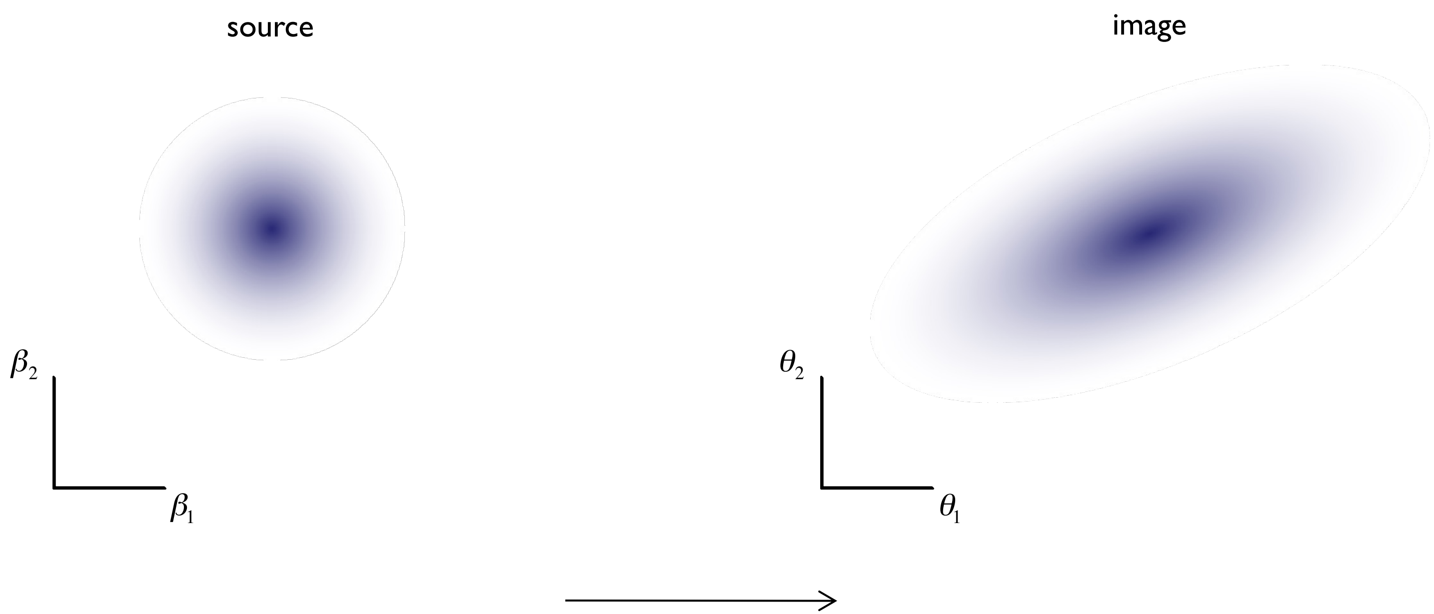
$\vec{\theta}_{\mu=\infty}$: **critical curves** (in ***lens plane***)



(elliptical lens, figure taken from Natarayan & Bartelmann1995)

- theory
  - the basics of lensing…
    - the lens equation
    - the lensing potential
    - critical surface mass density
    - magnification
    - caustics and critical curves
    - **distortion**
    - mass-sheet degeneracy
  - some sample lenses…
    - point mass
    - extended mass
    - singular isothermal sphere

- the distortion matrix

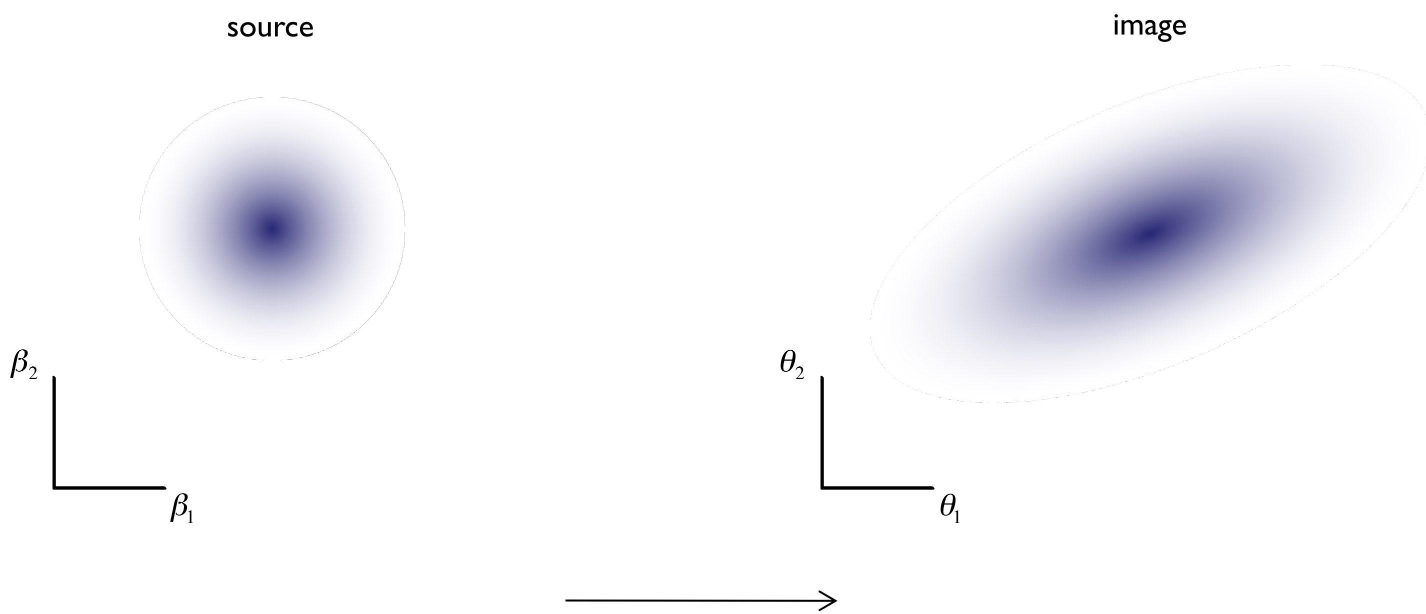


coordinate transformation from $\boldsymbol{\beta}$ to $\boldsymbol{\theta}$:

$$\beta = \theta - \alpha(\theta)$$

(the lens equation)

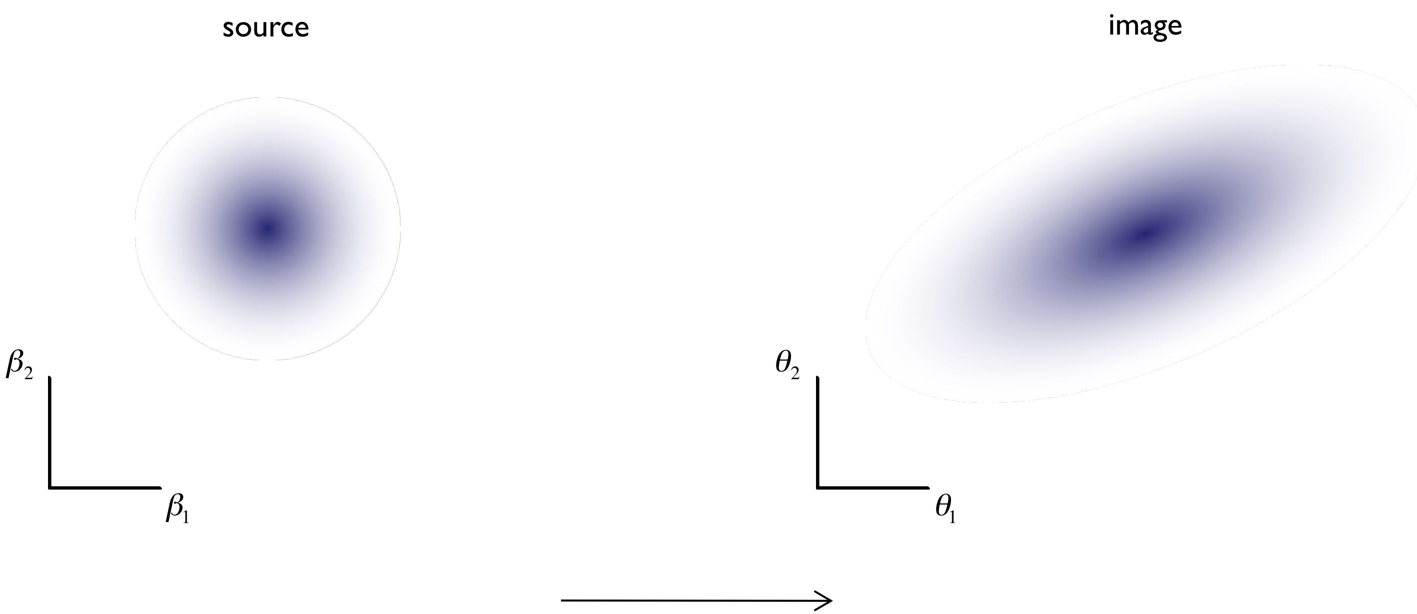
- the distortion matrix



coordinate transformation from $\boldsymbol{\beta}$ to $\boldsymbol{\theta}$: $\beta = \theta - \alpha(\theta)$

$$A_{ij} = \frac{\partial \beta_i}{\partial \theta_j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\partial \alpha_i}{\partial \theta_j}$$

- the distortion matrix



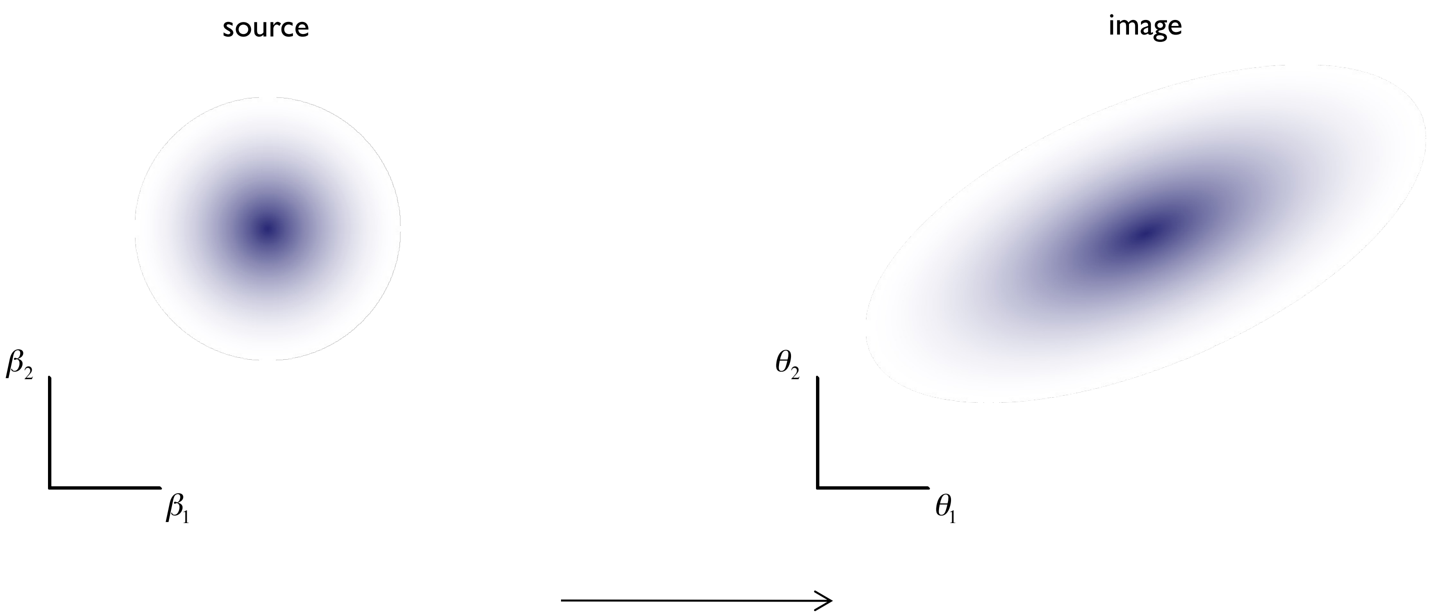
coordinate transformation from $\boldsymbol{\beta}$ to $\boldsymbol{\theta}$: $\beta = \theta - \alpha(\theta)$

$$A_{ij} = \frac{\partial \beta_i}{\partial \theta_j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\partial \alpha_i}{\partial \theta_j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1 \end{pmatrix}$$

decomposition of a symmetric matrix* into a diagonal and a trace-free part...

*why symmetric?

- the distortion matrix



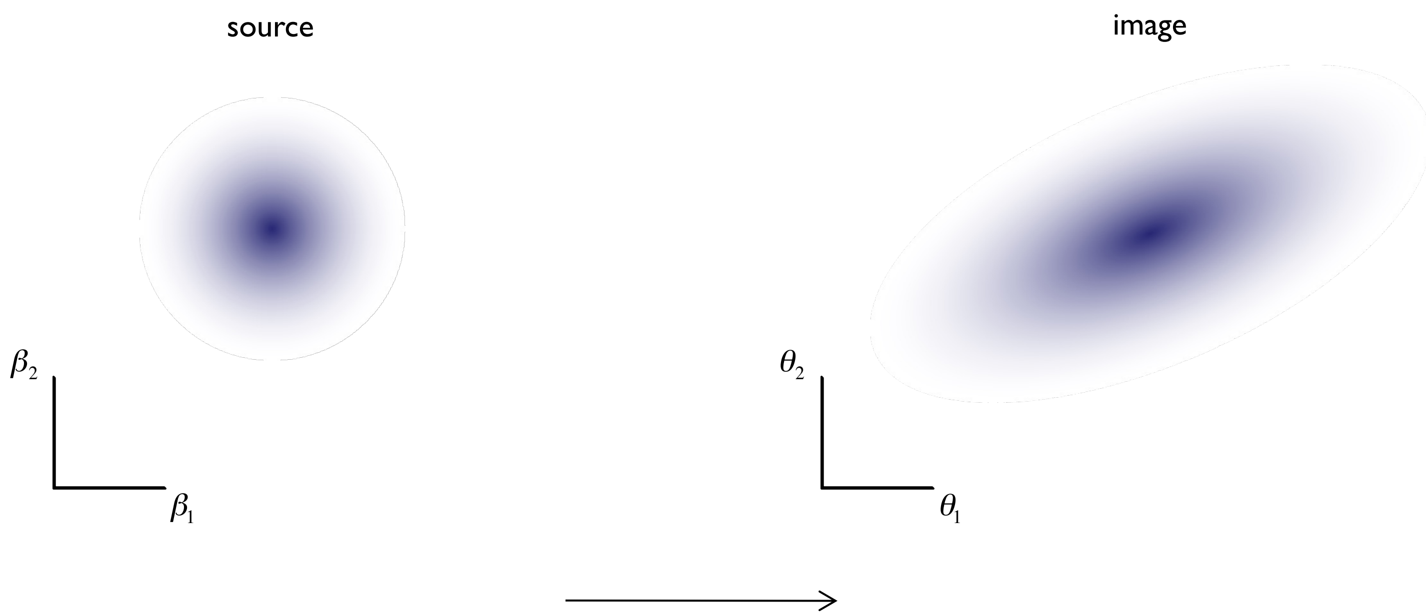
coordinate transformation from $\boldsymbol{\beta}$ to $\boldsymbol{\theta}$: $\beta = \theta - \alpha(\theta)$

$$A_{ij} = \frac{\partial \beta_i}{\partial \theta_j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\partial \alpha_i}{\partial \theta_j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1 \end{pmatrix}$$

decomposition of a symmetric matrix* into a diagonal and a trace-free part...

*A_{ij} is symmetric, because $\alpha = \nabla \varphi$ and hence $\partial\alpha_i/\partial\theta_j = \partial\alpha_j/\partial\theta_i$

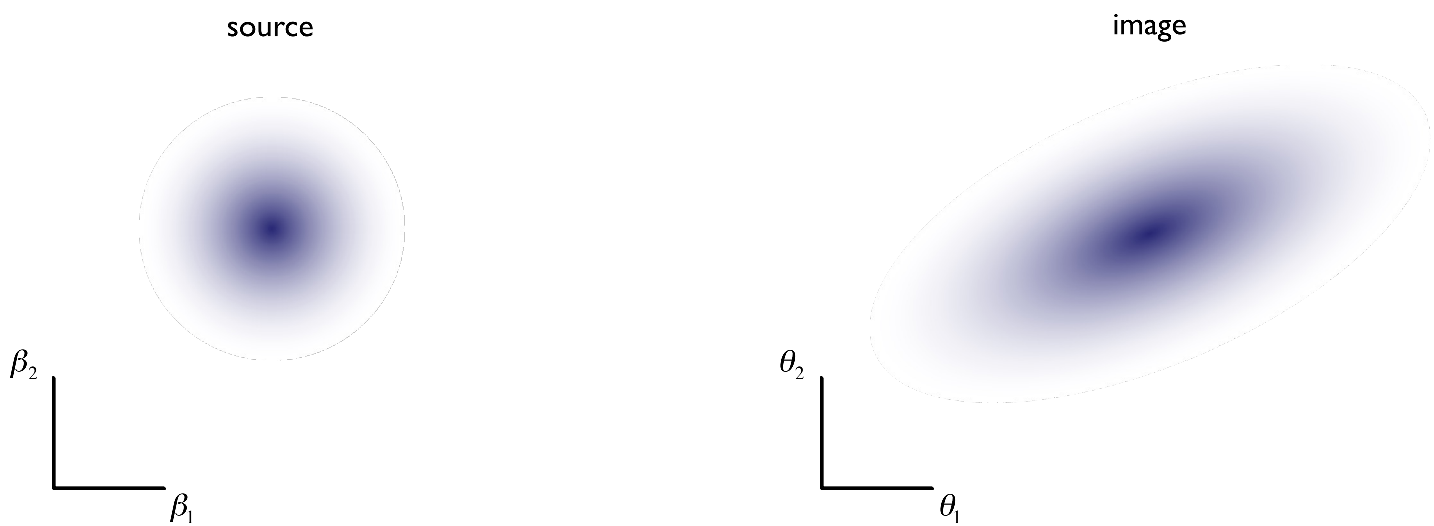
▪ the distortion matrix



coordinate transformation from $\boldsymbol{\beta}$ to $\boldsymbol{\theta}$: $\beta = \theta - \alpha(\theta)$

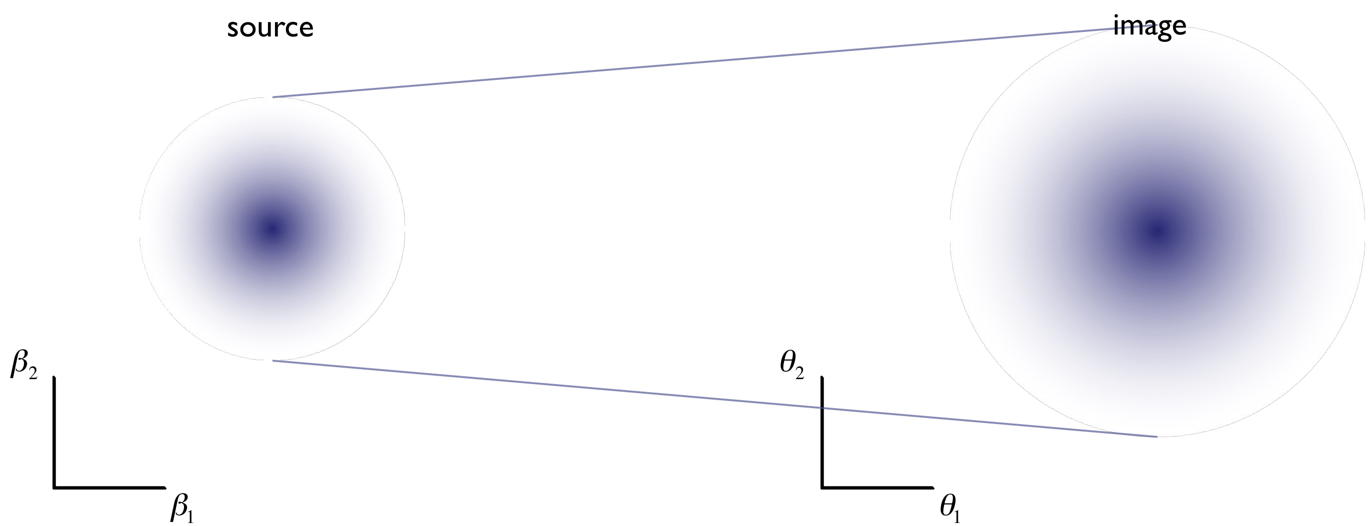
$$A_{ij} = \frac{\partial \beta_i}{\partial \theta_j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\partial \alpha_i}{\partial \theta_j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix} + \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

- the distortion matrix



$$A_{ij} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

- the distortion matrix

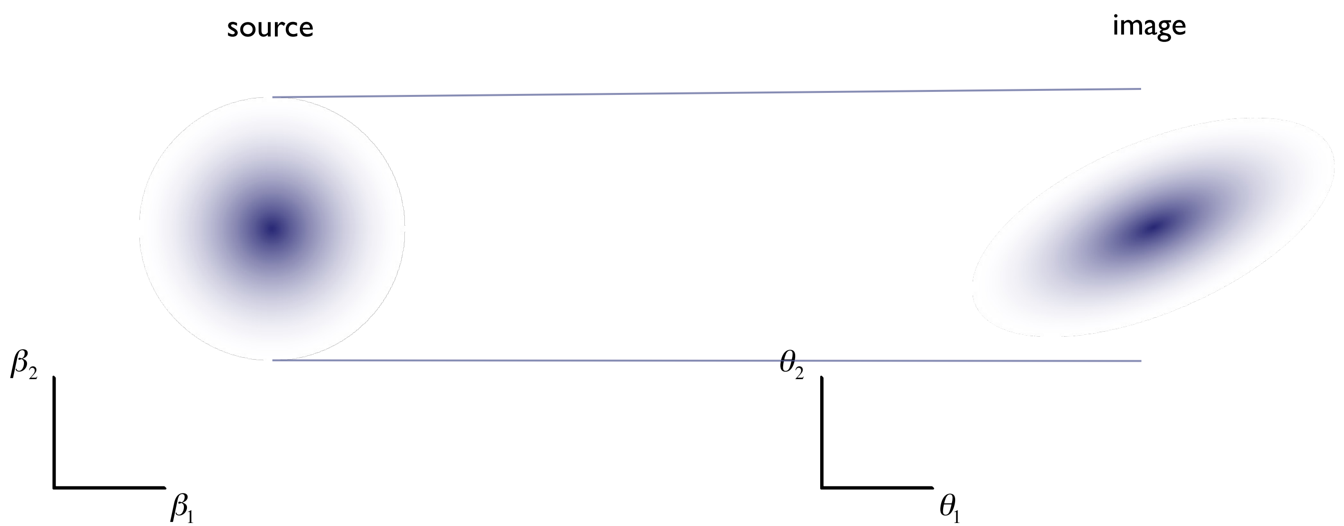


$$\kappa = \frac{\Sigma(\theta)}{\Sigma_{crit}} = \frac{1}{2}\nabla_\theta^2 \varphi(\theta)$$ ***(exercise)***

$$A_{ij} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

magnification

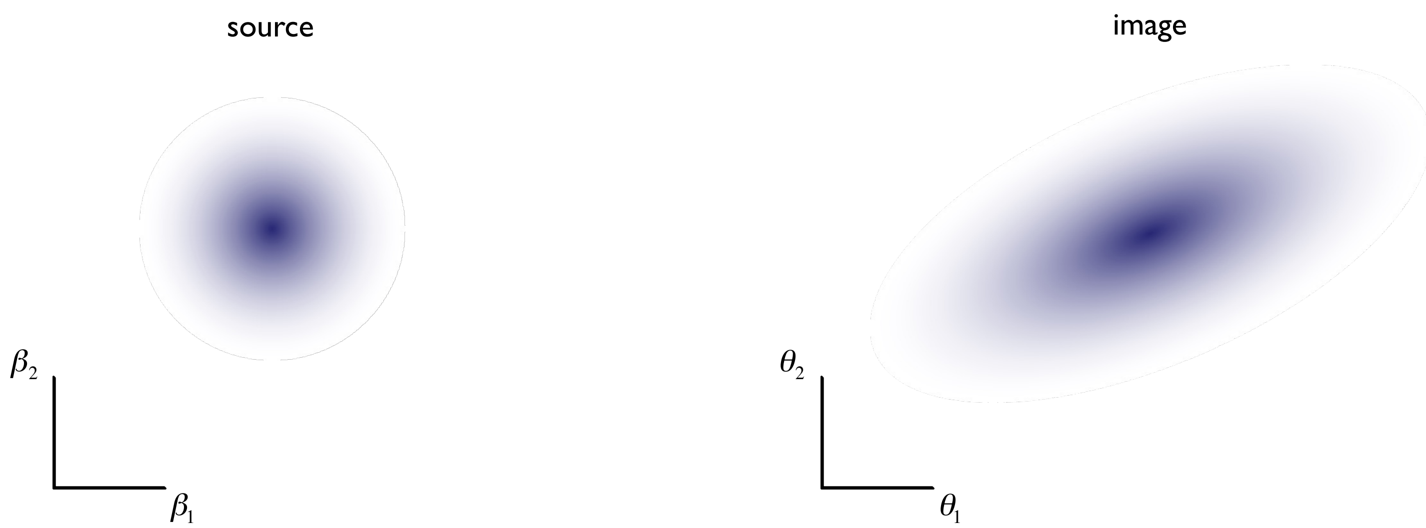
- the distortion matrix



$$A_{ij} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

shear

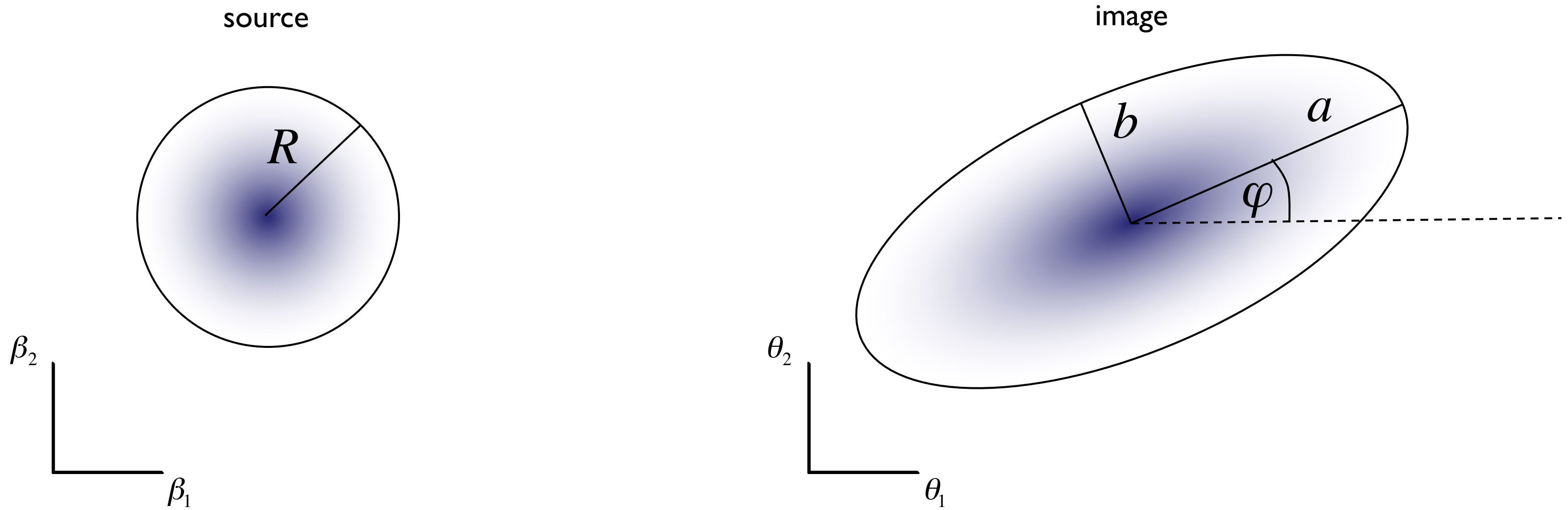
- the distortion matrix



$$A_{ij} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

magnification ***shear***

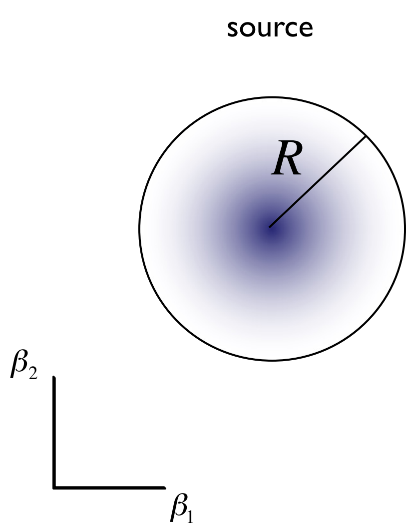
- the distortion matrix



how are κ and γ related to φ, a, b?

$$A_{ij} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

- the distortion matrix

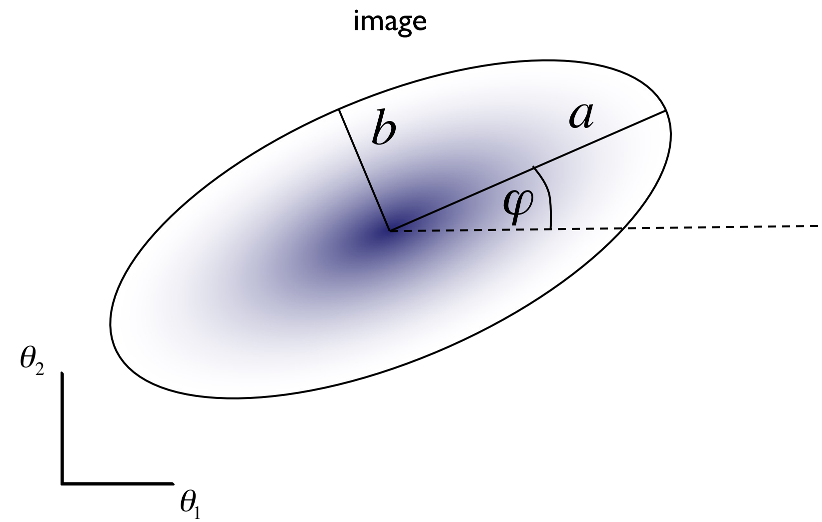


$$\gamma = |\gamma| e^{i2\varphi}$$

$$a = \frac{R}{1-\kappa-|\gamma|}$$

$$b = \frac{R}{1-\kappa+|\gamma|}$$

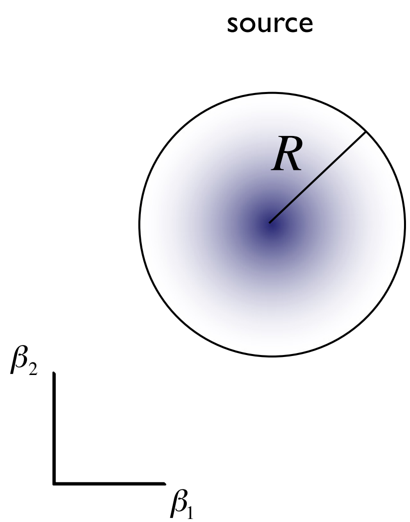
$$|\gamma| = \sqrt{\gamma_1^2 + \gamma_2^2}$$



$$A_{ij} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

- the distortion matrix

2φ because rotation about 180º maps ellipse onto itself

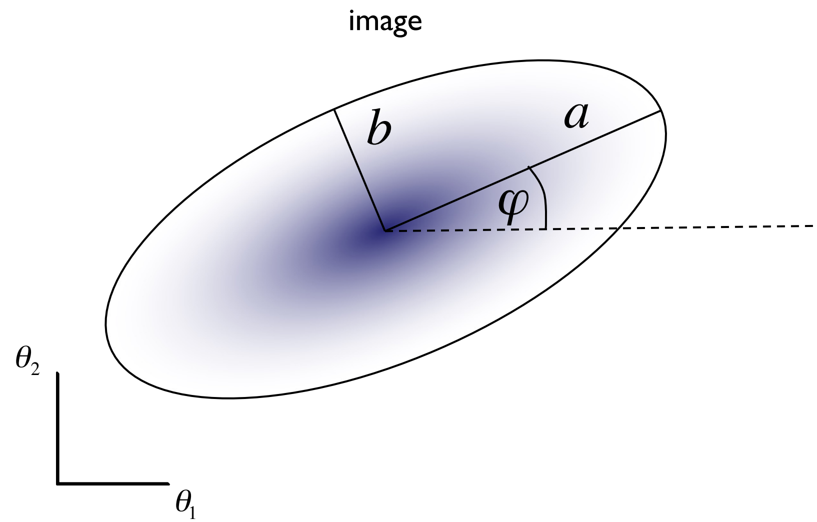


$$\gamma = |\gamma| e^{i2\varphi}$$

$$a = \frac{R}{1-\kappa-|\gamma|}$$

$$b = \frac{R}{1-\kappa+|\gamma|}$$

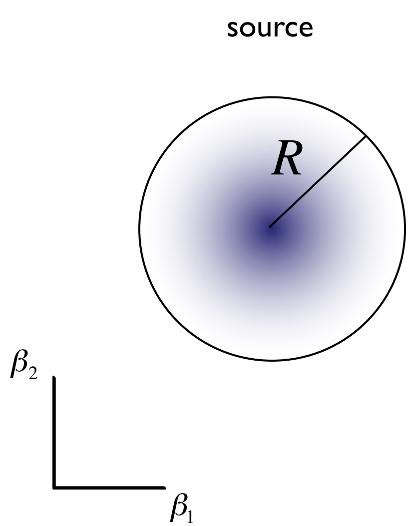
$$|\gamma| = \sqrt{\gamma_1^2 + \gamma_2^2}$$



$$A_{ij} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

- the distortion matrix

2φ because rotation about 180º maps ellipse onto itself

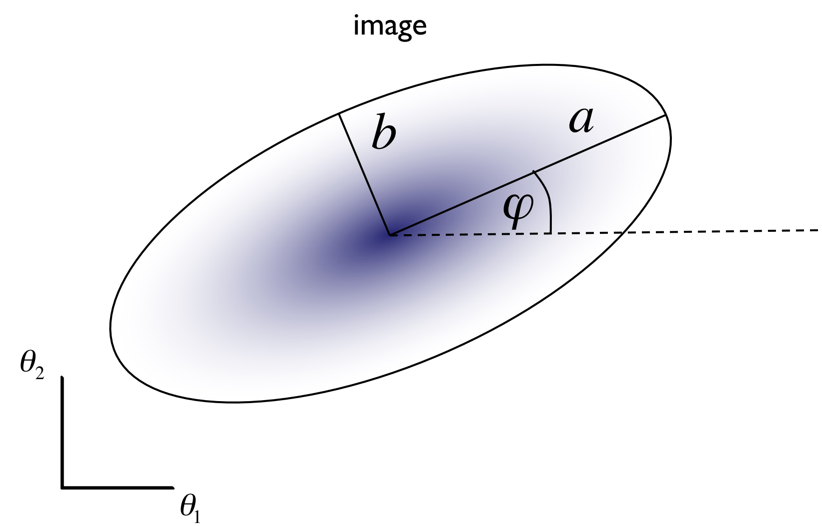


$$\gamma = |\gamma| e^{i2\varphi}$$

$$a = \frac{R}{1-\kappa-|\gamma|}$$

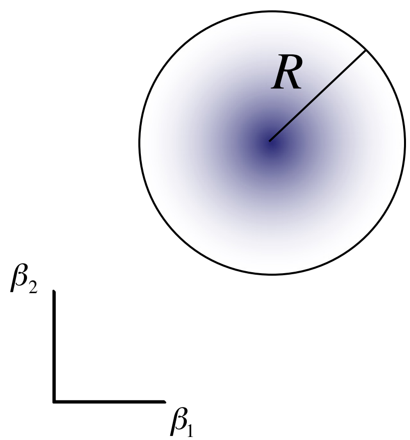
$$b = \frac{R}{1-\kappa+|\gamma|}$$

$$|\gamma| = \sqrt{\gamma_1^2 + \gamma_2^2}$$



$$A_{ij} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \gamma \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix}$$

- the distortion matrix

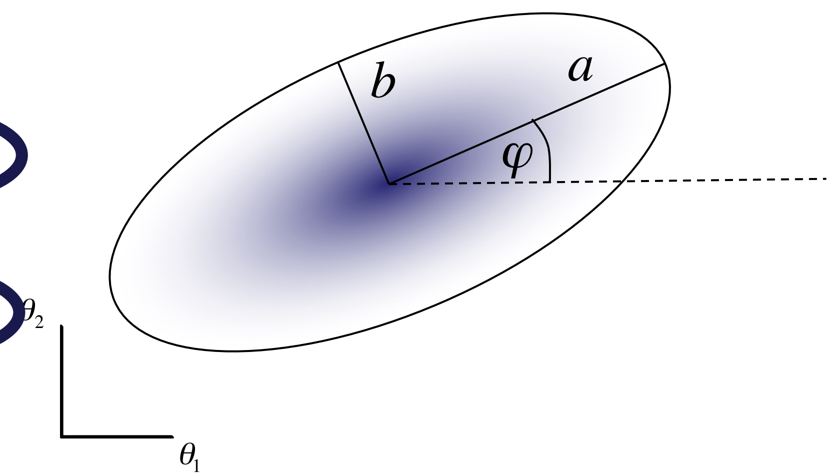


$$\gamma = |\gamma| e^{i2\varphi}$$

$$a = \frac{R}{1-\kappa-|\gamma|}$$

$$b = \frac{R}{1-\kappa+|\gamma|}$$

$$|\gamma| = \sqrt{\gamma_1^2 + \gamma_2^2}$$

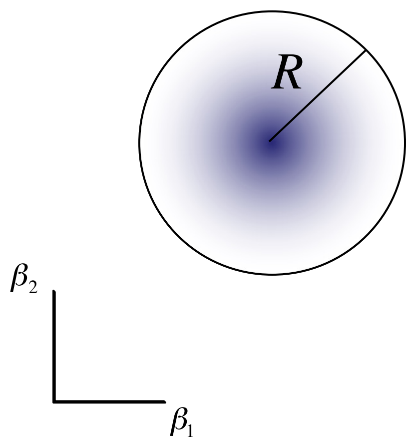


eigenvalues of the distortion matrix ***(exercise)***

$$A_{ij} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \gamma \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix}$$

- the distortion matrix

source



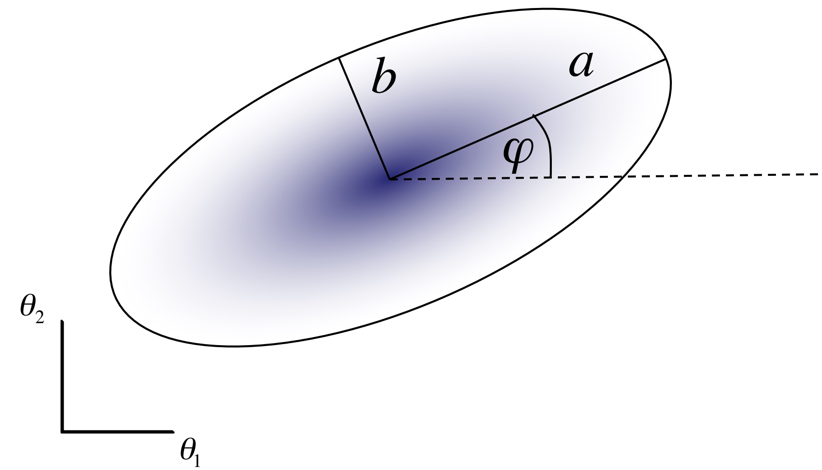
$$\gamma = |\gamma| e^{i2\varphi}$$

$$a = \frac{R}{1-\kappa-|\gamma|}$$

$$b = \frac{R}{1-\kappa+|\gamma|}$$

$$|\gamma| = \sqrt{\gamma_1^2 + \gamma_2^2}$$

image

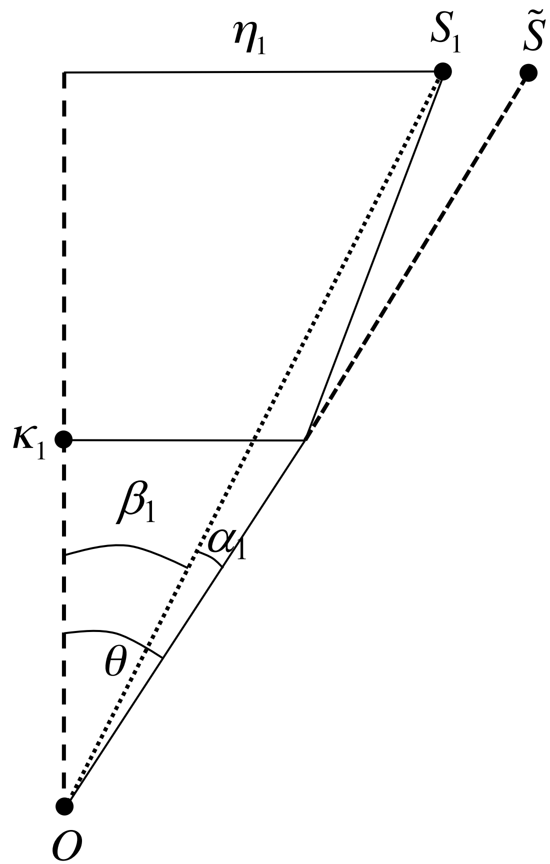


circular source => measuring a and b gives reduced shear $g=|\gamma|/(1-\kappa)$ ***(exercise)***

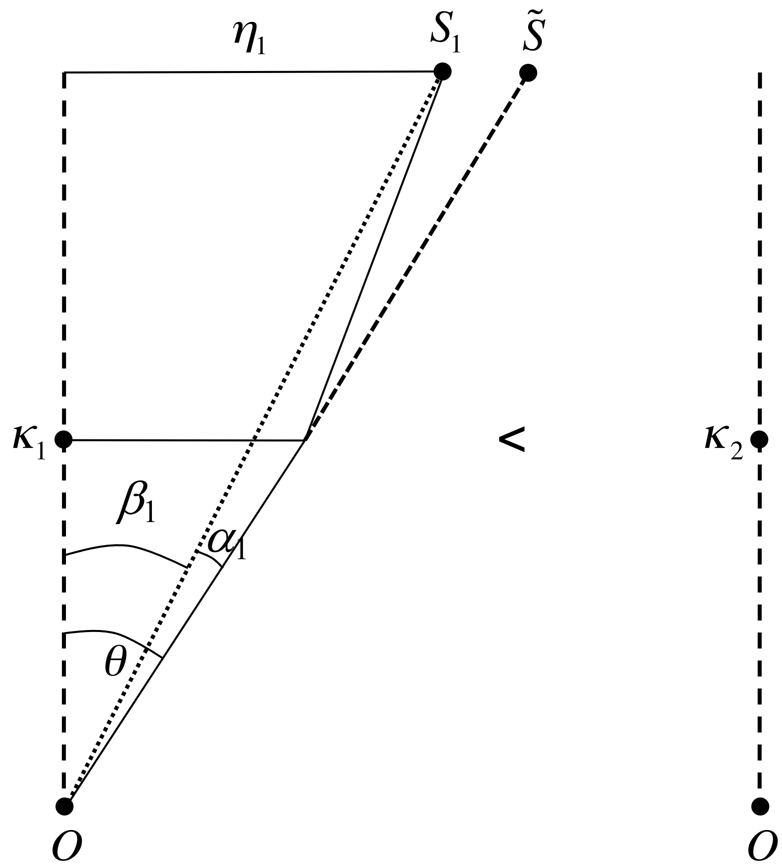
$$A_{ij} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \gamma \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix}$$

- theory
  - the basics of lensing…
    - the lens equation
    - the lensing potential
    - critical surface mass density
    - magnification
    - caustics and critical curves
    - distortion
    - **mass-sheet degeneracy**
  - some sample lenses…
    - point mass
    - extended mass
    - singular isothermal sphere

- mass-sheet degeneracy – visualisation

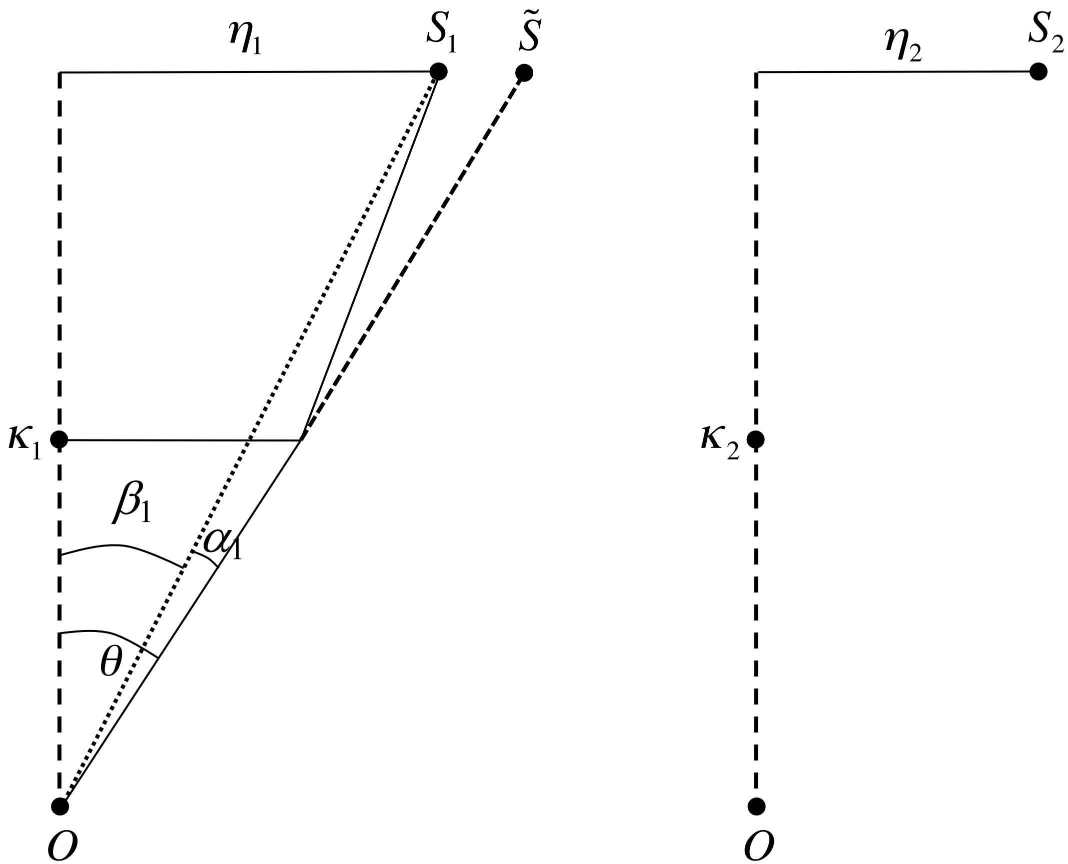


- mass-sheet degeneracy – visualisation



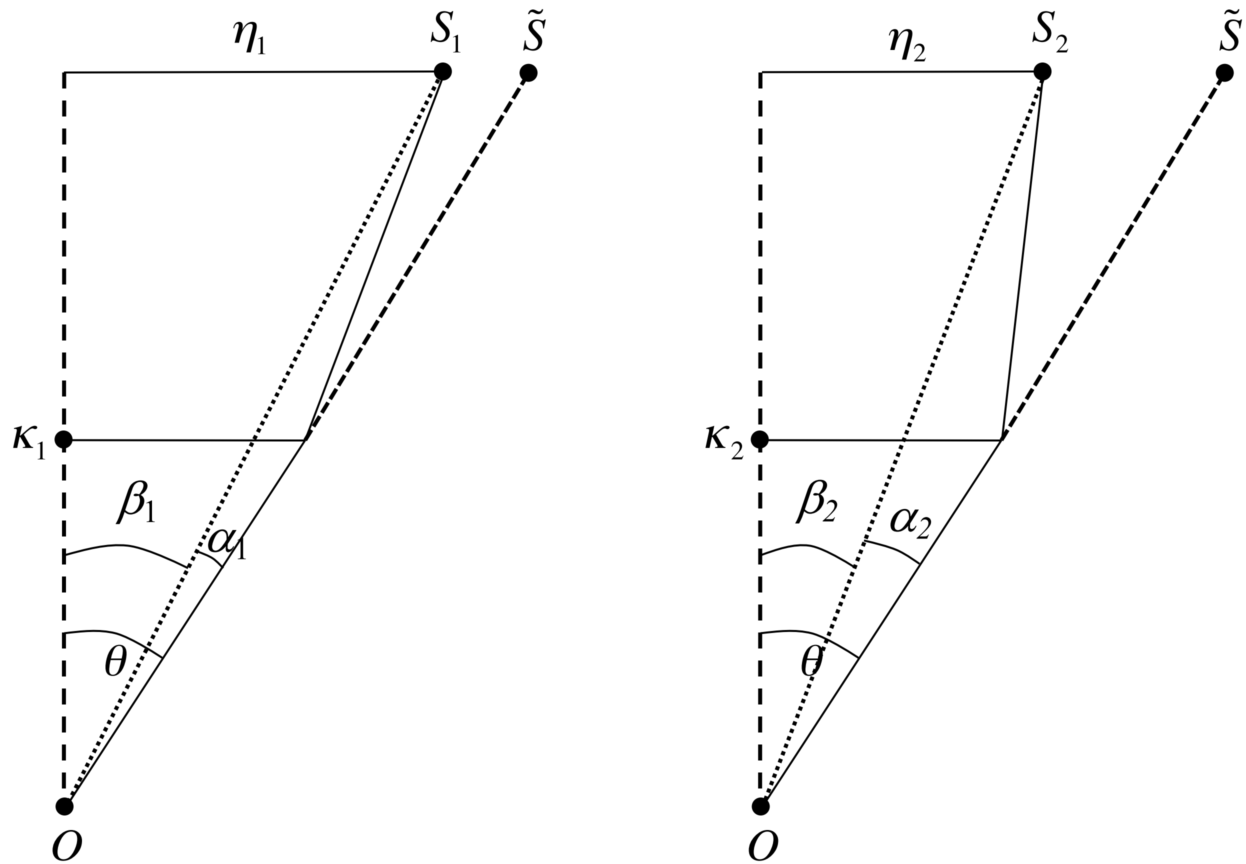
a larger $\kappa_2 > \kappa_1$ leads to stronger deflection

- mass-sheet degeneracy – visualisation



a larger $\kappa_2 > \kappa_1$ leads to stronger deflection,
but for $\eta_2 < \eta_1$ we might get the same θ in the end

- mass-sheet degeneracy – visualisation



a larger $\kappa_2 > \kappa_1$ leads to stronger deflection,
but for $\eta_2 < \eta_1$ we might get the same θ in the end

- mass-sheet degeneracy

$$\beta = \theta - \alpha\left(\theta\right) \qquad \alpha(\theta) = \nabla_\theta \varphi(\theta)$$

$$\nabla_\theta \alpha(\theta) = \Delta_\theta \varphi(\theta) = 2\kappa(\theta)$$

transformation of projected surface mass...

$$\kappa_\lambda(\theta) = (1-\lambda) + \lambda\kappa(\theta)$$

...corresponds to transformation of deflection angle...

$$\alpha_\lambda(\theta) = \left[(1-\lambda)\theta + \lambda\alpha(\theta)\right]$$

...which leads to an effective transformation of coordinates in source plane

$$\beta = \theta - \alpha_\lambda(\theta) = \theta - \left[(1-\lambda)\theta + \lambda\alpha(\theta)\right] = \lambda\theta - \lambda\alpha(\theta)$$

$$\frac{\beta}{\lambda} = \theta - \alpha(\theta)$$ => such a shift is not observable!

▪ summary

- deflection angle

$$\vec{\alpha}(\vec{\theta}) = \nabla_\theta \varphi(\vec{\theta}) \quad \text{with} \quad \nabla_\theta^2 \varphi(\vec{\theta}) = 2\kappa(\vec{\theta})$$

- lens (ray-tracing) equation

$$\kappa(\vec{\theta}) = \frac{\Sigma(\vec{\theta})}{\Sigma_{crit}}$$

$$\vec{\beta}\left(\vec{\theta}\right) = \vec{\theta} - \vec{\alpha}\left(\vec{\theta}\right)$$

$$\Sigma(\theta) = \int \rho(\theta, z)dz$$

- magnification

$$\mu = \left|\det\left(\frac{\partial\vec{\beta}}{\partial\vec{\theta}}\right)\right|^{-1}$$

$$\Sigma_{crit} = \frac{c^2}{4\pi G}\frac{D_S}{D_{LS}D_L}$$

- distortion

$$\frac{\partial\vec{\beta}}{\partial\vec{\theta}} = (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}$$

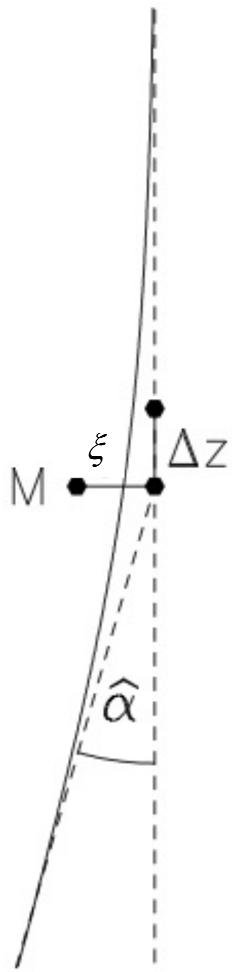
and now for some examples…

- theory
  - the basics of lensing…
    - the lens equation
    - the lensing potential
    - critical surface mass density
    - magnification
    - caustics and critical curves
    - distortion
    - mass-sheet degeneracy
  - some sample lenses…
    - **point mass**
    - extended mass
    - singular isothermal sphere

▪ lensing by point masses

- deflection angle

$$\hat{\alpha} = \frac{2}{c^2} \int \nabla_{\xi} \Phi(\xi) \ dz$$



…

=> $$\boxed{\hat{\alpha} = \frac{4GM}{c^2} \frac{1}{\xi}}$$ **(exercise)**

▪ lensing by point masses

- deflection angle

$$\hat{\alpha} = \frac{4GM}{c^2\xi}$$

- examples

object	mass M	impact parameter ξ	deflection angle α
sun	$1\ M_\odot$	$7\text{x}10^5$ km	1.75”
star	$1\ M_\odot$	10^{-2}pc	$3\text{x}10^{-6}$”
galaxy	$10^{11}\ M_\odot$	10^4pc	0.4”
galaxy cluster	$10^{14}\ M_\odot$	$2\text{x}10^5$pc	20”

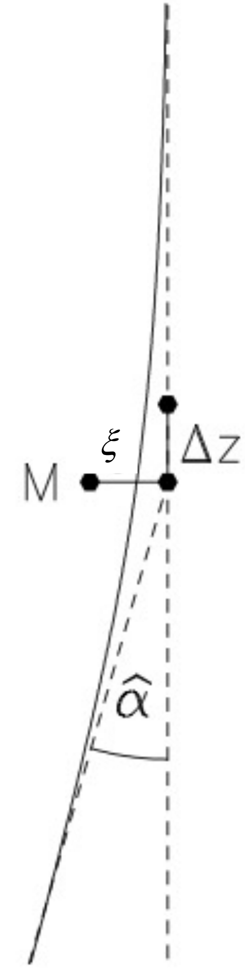


figure taken from Narayan & Bartelmann (1995)

- lensing by point masses
  - lens (ray-tracing) equation

$$\beta(\theta) = \theta - \alpha(\theta)$$

- lensing by point masses

  - lens (ray-tracing) equation

$$\beta(\theta) = \theta - \alpha(\theta)$$

point mass:

$$\alpha = \frac{D_{LS}}{D_S}\hat{\alpha} = \frac{D_{LS}}{D_S}\frac{4GM}{c^2\xi}$$

- lensing by point masses

  - lens (ray-tracing) equation

$$\beta(\theta) = \theta - \alpha(\theta)$$

point mass:

$$\alpha = \frac{D_{LS}}{D_S}\hat{\alpha} = \frac{D_{LS}}{D_S}\frac{4GM}{c^2\xi} \qquad \xi = D_L\theta$$

- lensing by point masses
  - lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{D_{LS}}{D_S D_L} \frac{4GM}{c^2} \frac{1}{\theta}$$

- lensing by point masses
  - lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}\frac{1}{\theta}$$

$$\theta_E = \sqrt{\frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}}$$

θ_E: Einstein radius

- lensing by point masses

  - lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{\theta_E^2}{\theta}$$

$$\theta_E = \sqrt{\frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}}$$

θ_E: *Einstein radius*

▪ lensing by point masses

- lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{\theta_E^2}{\theta}$$

$$\theta_E = \sqrt{\frac{D_{LS}}{D_S D_L} \frac{4GM}{c^2}}$$

θ_E*: Einstein radius*

what are the possible images θ for a given source β?

- lensing by point masses

  - lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{\theta_E^2}{\theta}$$

$$\theta_E = \sqrt{\frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}}$$

θ_E: Einstein radius

$$\Longrightarrow \quad 0 = \theta^2 - \theta\beta - \theta_E^2$$

$$= \theta^2 - \theta\beta + \left(\frac{1}{2}\beta\right)^2 - \left(\frac{1}{2}\beta\right)^2 - \theta_E^2$$

$$= \theta^2 - \theta\beta + \left(\frac{1}{2}\beta\right)^2 - \left[\left(\frac{1}{2}\beta\right)^2 + \theta_E^2\right]$$

$$= \left(\theta - \frac{\beta}{2}\right)^2 - \left[\left(\frac{1}{2}\beta\right)^2 + \theta_E^2\right]$$

$$\left(\theta_\pm - \frac{\beta}{2}\right) = \sqrt{\left(\frac{1}{2}\beta\right)^2 + \theta_E^2}$$

▪ lensing by point masses

- lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{\theta_E^2}{\theta}$$

$$\theta_E = \sqrt{\frac{D_{LS}}{D_S D_L} \frac{4GM}{c^2}}$$

θ_E: Einstein radius

=> $$\theta_{\pm} = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

▪ lensing by point masses

- lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{\theta_E^2}{\theta}$$

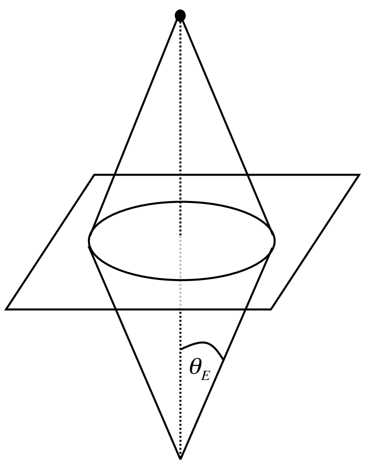
$$\theta_E = \sqrt{\frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}}$$

θ_E: Einstein radius

=> $$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

$\underline{\beta = 0:}$ $$\theta_\pm = \theta_E = \sqrt{\frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}}$$

Einstein Ring



- lensing by point masses

  - lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{\theta_E^2}{\theta}$$

$$\theta_E = \sqrt{\frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}}$$

θ_E: Einstein radius

=> $$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

$\underline{\beta = 0:}$ $$\theta_\pm = \theta_E = \sqrt{\frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}}$$

$\underline{\beta \neq 0:}$ $$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

$\theta_+ > \theta_E$ image outside Einstein ring

$\theta_- < \theta_E$ image inside Einstein ring

▪ lensing by point masses

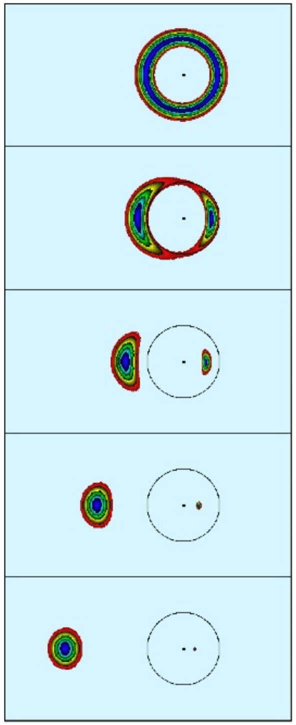
- lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{\theta_E^2}{\theta}$$

$$\theta_E = \sqrt{\frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}}$$

θ_E: Einstein radius

=> $$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$



Wambsganss (1998)

$\underline{\beta = 0:}$ $$\theta_\pm = \theta_E = \sqrt{\frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}}$$

$\underline{\beta \neq 0:}$ $$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

$\theta_+ > \theta_E$ image outside Einstein ring

$\theta_- < \theta_E$ image inside Einstein ring

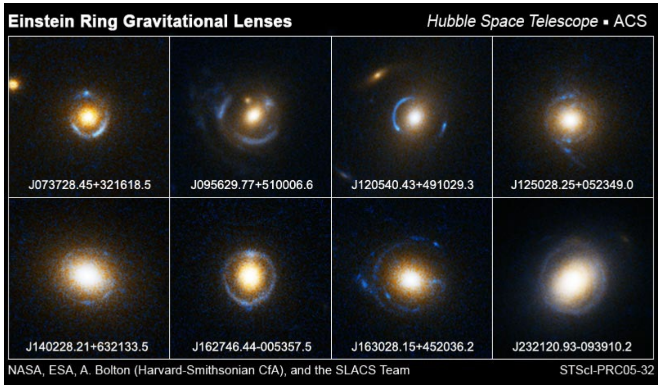
- lensing by point masses
  - lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{\theta_E^2}{\theta}$$

$$\theta_E = \sqrt{\frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}}$$

θ_E: Einstein radius

=> $$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

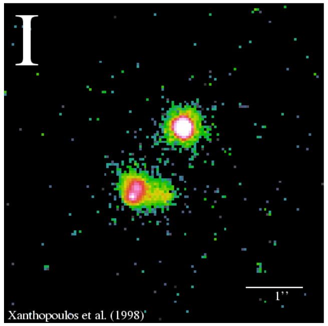


$\underline{\beta = 0:}$ $$\theta_\pm = \theta_E = \sqrt{\frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}}$$

$\underline{\beta \neq 0:}$ $$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

$\theta_+ > \theta_E$ image outside Einstein ring

$\theta_- < \theta_E$ image inside Einstein ring



B1030+074

- lensing by point masses
  - lens (ray-tracing) equation ... now in full 2D

$$\begin{pmatrix}\theta_{1,\pm}\\ \theta_{2,\pm}\end{pmatrix} = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)\begin{pmatrix}\frac{\beta_1}{\beta}\\ \frac{\beta_2}{\beta}\end{pmatrix}$$

$$\theta_E = \sqrt{\frac{D_{LS}}{D_S D_L}\frac{4GM}{c^2}}$$

θ_E: *Einstein radius*

$$\beta = \sqrt{\beta_1^2 + \beta_2^2}$$

- lensing by point masses

  - magnification

$$\mu_\pm = \frac{\theta_\pm}{\beta}\frac{d\theta_\pm}{d\beta} = \left(1 - \left(\frac{\theta_E}{\theta_\pm}\right)^4\right)^{-1} \qquad \theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

- lensing by point masses

  - magnification

$$\mu_\pm = \frac{\theta_\pm}{\beta}\frac{d\theta_\pm}{d\beta} = \left(1 - \left(\frac{\theta_E}{\theta_\pm}\right)^4\right)^{-1} \qquad \theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

$$\theta_- < \theta_E \Rightarrow$$ the image inside the Einstein radius has negative magnification, meaning it is mirror-inverted

- lensing by point masses

  - magnification

$$\mu_\pm = \frac{\theta_\pm}{\beta}\frac{d\theta_\pm}{d\beta} = \left(1 - \left(\frac{\theta_E}{\theta_\pm}\right)^4\right)^{-1} \qquad \theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right) \qquad u = \frac{\beta}{\theta_E}$$

- lensing by point masses

  - magnification

$$\mu_\pm = \frac{\theta_\pm}{\beta}\frac{d\theta_\pm}{d\beta} = \left(1 - \left(\frac{\theta_E}{\theta_\pm}\right)^4\right)^{-1} = \frac{u^2+2}{2u\sqrt{u^2+4}} \pm \frac{1}{2}$$

$$u = \frac{\beta}{\theta_E}$$

=> $$\mu = |\mu_+| + |\mu_-| = \frac{u^2+2}{u\sqrt{u^2+4}}$$

$$\mu = \mu_+ + \mu_- = 1$$

▪ lensing by point masses

- deflection angle

$$\alpha = \frac{D_{LS}}{D_S D_L} \frac{4GM}{c^2 \theta}$$

- lens (ray-tracing) equation

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

$$\theta_E = \sqrt{\frac{D_{LS}}{D_S D_L} \frac{4GM}{c^2}}$$

θ_E: Einstein radius

- magnification

$$\mu = |\mu_+| + |\mu_-| = \frac{u^2 + 2}{u\sqrt{u^2 + 4}}$$

$$u = \frac{\beta}{\theta_E}$$

- theory
  - the basics of lensing…
    - the lens equation
    - the lensing potential
    - critical surface mass density
    - magnification
    - caustics and critical curves
    - distortion
    - mass-sheet degeneracy
  - some sample lenses…
    - point mass
    - **extended mass**
    - singular isothermal sphere

- lensing by extended masses
  - surface mass density:

$$\Sigma\left(\vec{\xi}\right) = \int \rho\left(\vec{\xi}, z\right) dz$$

- lensing by extended masses

  - surface mass density:

$$\Sigma\left(\vec{\xi}\right) = \int \rho\left(\vec{\xi}, z\right) dz$$

$$\kappa(\vec{\theta}) = \frac{\Sigma(\vec{\theta})}{\Sigma_{crit}}$$

$$\nabla_\theta^2 \varphi(\vec{\theta}) = 2\kappa(\vec{\theta})$$

$$\vec{\alpha}(\vec{\theta}) = \nabla_\theta \varphi(\vec{\theta})$$

- lensing by extended masses

  - surface mass density:

$$\Sigma\left(\vec{\xi}\right) = \int \rho\left(\vec{\xi}, z\right) dz$$

$$\kappa(\vec{\theta}) = \frac{\Sigma(\vec{\theta})}{\Sigma_{crit}}$$

$$\nabla_\theta^2 \varphi(\vec{\theta}) = 2\kappa(\vec{\theta})$$

$$\vec{\alpha}(\vec{\theta}) = \nabla_\theta \varphi(\vec{\theta})$$

- lensing by extended masses

  - surface mass density:

$$\Sigma\left(\vec{\xi}\right) = \int \rho\left(\vec{\xi}, z\right) dz$$

$$\kappa(\vec{\theta}) = \frac{\Sigma(\vec{\theta})}{\Sigma_{crit}}$$

$$\nabla_\theta^2 \varphi(\vec{\theta}) = 2\kappa(\vec{\theta})$$

$$\vec{\alpha}(\vec{\theta}) = \nabla_\theta \varphi(\vec{\theta})$$

  - deflection angles are additive!

- lensing by extended masses

  - surface mass density:

$$\Sigma\left(\vec{\xi}\right) = \int \rho\left(\vec{\xi}, z\right) dz$$

$$\kappa(\vec{\theta}) = \frac{\Sigma(\vec{\theta})}{\Sigma_{crit}}$$

$$\nabla_\theta^2 \varphi(\vec{\theta}) = 2\kappa(\vec{\theta})$$

$$\vec{\alpha}(\vec{\theta}) = \nabla_\theta \varphi(\vec{\theta})$$

  - deflection angles are additive:

integrate over mass distribution…

- lensing by extended masses

  - surface mass density:

$$\Sigma(\vec{\xi}) = \int \rho(\vec{\xi}, z)\, dz$$

$$dM(\vec{\xi}) = \Sigma(\vec{\xi})\, d^2\xi$$

  - deflection angles are additive:

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2}\int \frac{\vec{b}}{b^2}\, dM = \frac{4G}{c^2}\int \frac{(\vec{\xi} - \vec{\xi}')}{(\vec{\xi} - \vec{\xi}')^2}\Sigma(\vec{\xi}')\, d^2\xi'$$

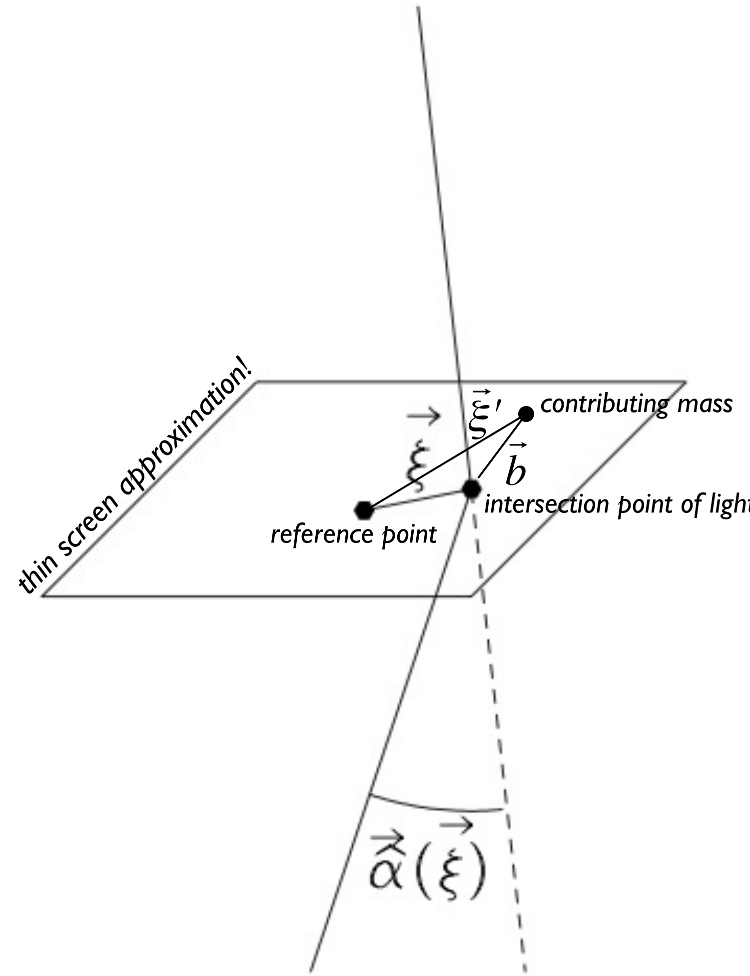


figure adapted from Narayan & Bartelmann (1995)

- lensing by extended masses:

circular lens

$$M(<\xi) = 2\pi \int \Sigma(\xi')\xi' d\xi'$$

$$\vec{\hat{\alpha}}(\vec{\xi}) = \frac{4G}{c^2}\int \frac{\vec{b}}{b^2} dM = \frac{4G}{c^2}\int \frac{\left(\vec{\xi} - \vec{\xi}'\right)}{\left(\vec{\xi} - \vec{\xi}'\right)^2}\Sigma\left(\vec{\xi}'\right) d^2\xi'$$

▪ lensing by extended masses: *circular lens*

- deflection angle

$$\hat{\alpha} = \frac{4GM(<\xi)}{c^2\xi} \qquad \text{with } M(<\xi) = 2\pi \int \Sigma(\xi')\xi' d\xi'$$

- lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{D_{LS}}{D_S D_L} \frac{4GM(<\theta)}{c^2\theta}$$

- magnification

$$\mu = \frac{\theta}{\beta} \frac{d\theta}{d\beta}$$

- lensing by extended masses: *lens with constant surface mass density*
  - deflection angle

$$\hat{\alpha} = \frac{4GM(<\xi)}{c^2\xi} \qquad \text{with } M(<\xi) = \pi\Sigma\xi^2$$

$$\alpha = \frac{D_{LS}}{D_S}\frac{4\pi G\Sigma\xi}{c^2} = \frac{4\pi G\Sigma}{c^2}\frac{D_{LS}D_L}{D_S}\theta = \frac{\Sigma}{\Sigma_{crit}}\theta$$

- lensing by extended masses: *lens with constant surface mass density*
  - deflection angle

$$\hat{\alpha} = \frac{4GM(<\xi)}{c^2\xi} \qquad \text{with } M(<\xi) = \pi\Sigma\xi^2$$

$$\alpha = \frac{\Sigma}{\Sigma_{crit}}\theta$$

- lensing by extended masses: *lens with constant surface mass density*

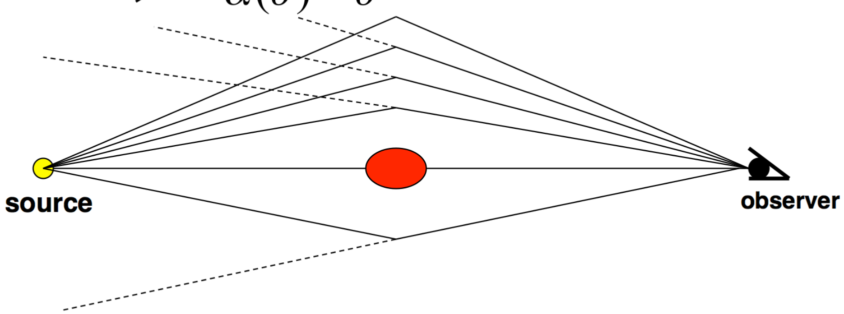
  - deflection angle

$$\hat{\alpha} = \frac{4GM(<\xi)}{c^2\xi} \qquad \text{with } M(<\xi) = \pi\Sigma\xi^2$$

$$\alpha = \frac{\Sigma}{\Sigma_{crit}}\theta$$

  - lens with critical surface mass density → perfectly focusing lens

$$\Rightarrow \quad \alpha(\theta) = \theta$$



- theory
  - the basics of lensing…
    - the lens equation
    - the lensing potential
    - critical surface mass density
    - magnification
    - caustics and critical curves
    - distortion
    - mass-sheet degeneracy
  - some sample lenses…
    - point mass
    - extended mass
    - **singular isothermal sphere**

- lensing by extended masses:

singular isothermal sphere

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$$

- lensing by extended masses:

singular isothermal sphere

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$$

$$\Sigma(\xi) = \int_{-\infty}^{+\infty} \rho(\xi,z)\,dz$$

$$\rho(r(\xi,z)) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2} = \frac{\sigma_v^2}{2\pi G}\frac{1}{\xi^2+z^2}$$

$$= \frac{\sigma_v^2}{2\pi G}\int_{-\infty}^{+\infty}\frac{1}{\xi^2+z^2}\,dz$$

$$= \frac{\sigma_v^2}{2\pi G}\left[\frac{1}{\xi}\tan^{-1}\left(\frac{z}{\xi}\right)\right]_{-\infty}^{+\infty}$$

$$= \frac{\sigma_v^2}{2\pi G}\frac{1}{\xi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]$$

$$= \frac{\sigma_v^2}{2G}\frac{1}{\xi}$$

- lensing by extended masses:

singular isothermal sphere

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$$

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G}\frac{1}{\xi}$$

$$M(<\xi) = 2\pi\int_0^{\xi}\Sigma(\xi')\xi' d\xi'$$

$$= 2\pi\int_0^{\xi}\frac{\sigma_v^2}{2G}\frac{1}{\xi'}\xi' d\xi'$$

$$= \frac{\pi\sigma_v^2}{G}\xi$$

- lensing by extended masses: *singular isothermal sphere*

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$$

  - deflection angle

$$\hat{\alpha} = \frac{4GM(<\xi)}{c^2\xi} \quad \text{with} \quad M(<\xi) = \frac{\pi\sigma_v^2}{G}\xi$$

- lensing by extended masses:

singular isothermal sphere

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$$

  - deflection angle

$$\hat{\alpha} = \frac{4\pi\sigma_v^2}{c^2}$$

- lensing by extended masses:

singular isothermal sphere

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$$

  - deflection angle

$$\hat{\alpha} = \frac{4\pi\sigma_v^2}{c^2}$$

every light ray experiences the same deflection!
(independent of sphere size!)

- lensing by extended masses:

singular isothermal sphere

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$$

- deflection angle

$$\hat{\alpha} = \frac{4\pi\sigma_v^2}{c^2}$$

- lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{D_{LS}}{D_S}\hat{\alpha}$$

- lensing by extended masses: *singular isothermal sphere*

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$$

  - deflection angle

$$\hat{\alpha} = \frac{4\pi\sigma_v^2}{c^2}$$

  - lens (ray-tracing) equation

$$\beta(\theta) = \theta - \frac{D_{LS}}{D_S}\hat{\alpha} = \theta \pm \theta_E$$

$$\theta_E^2 = \frac{D_{LS}}{D_S D_L}\frac{4G}{c^2}M(\theta_E) = \frac{D_{LS}}{D_S D_L}\frac{4G}{c^2}\frac{\pi\sigma_v^2}{G}D_L\theta_E$$

$$\theta_E = \pm\frac{D_{LS}}{D_S}\hat{\alpha}$$

$$M(<\xi) = \frac{\pi\sigma_v^2}{G}\xi$$

$$\xi = D_L\theta$$

formally speaking there is a third solution θ_E=0

- lensing by extended masses:

singular isothermal sphere

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$$

  - deflection angle

$$\hat{\alpha} = \frac{4\pi\sigma_v^2}{c^2}$$

  - lens (ray-tracing) equation

$$\theta_\pm = \beta \pm \theta_E = \beta \pm \frac{D_{LS}}{D_S}\hat{\alpha}$$

- lensing by extended masses:

singular isothermal sphere

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$$

- deflection angle

$$\hat{\alpha} = \frac{4\pi\sigma_v^2}{c^2}$$

constant offset

- lens (ray-tracing) equation

$$\theta_{\pm} = \beta \pm \theta_E = \beta \pm \frac{D_{LS}}{D_S}\hat{\alpha}$$

- lensing by extended masses:

singular isothermal sphere

$$\rho(r) = \frac{\sigma_v^2}{2\pi G}\frac{1}{r^2}$$

- deflection angle

$$\hat{\alpha} = \frac{4\pi\sigma_v^2}{c^2}$$

constant offset

- lens (ray-tracing) equation

$$\theta_\pm = \beta \pm \theta_E = \beta \pm \frac{D_{LS}}{D_S}\hat{\alpha}$$

- magnification

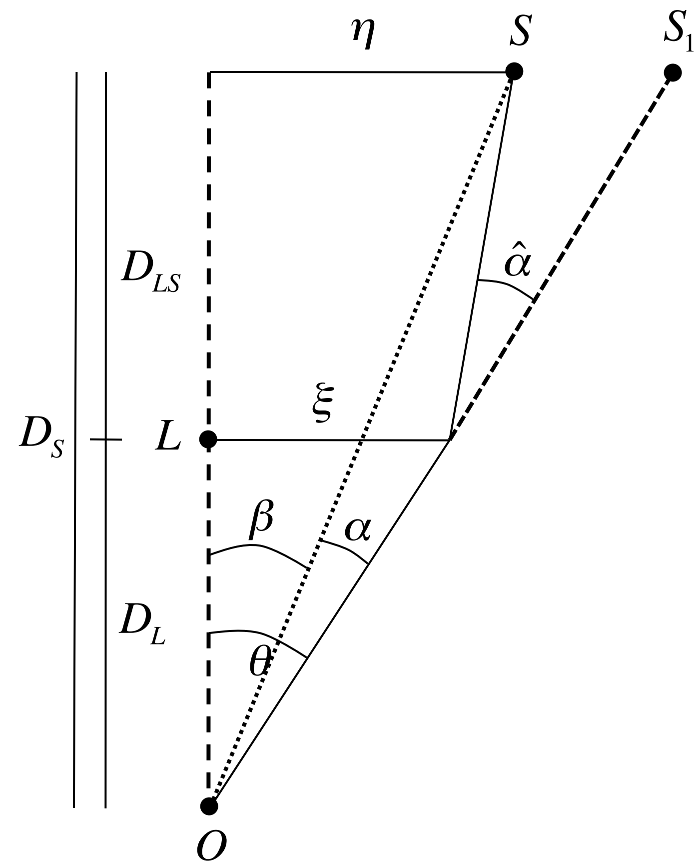
$$\mu = \frac{\theta}{\beta}\frac{d\theta}{d\beta} = \frac{\theta}{\beta} = \frac{\theta}{\theta \pm \theta_E} = \frac{1}{1 \pm \frac{\theta_E}{\theta}}$$

$$\nabla_\theta^2 \varphi(\theta) = 2\frac{\Sigma(\theta)}{\Sigma_{crit}}$$

$$\Sigma_{crit} = \frac{c^2}{4\pi G}\frac{D_S}{D_{LS} D_L}$$

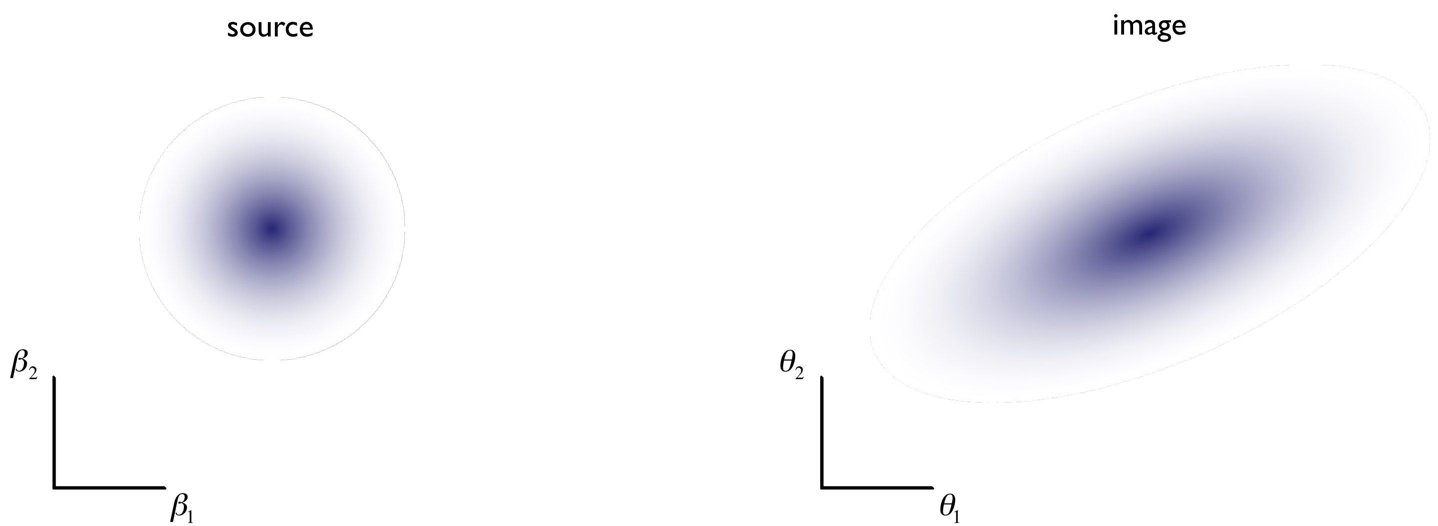
$$\alpha(\theta) = \nabla_\theta \varphi(\theta)$$

$$\beta = \theta - \alpha(\theta)$$



$\Sigma(\theta) = \int \rho(\theta, z) dz$ ≙ projected surface mass density

- the distortion matrix



$$A_{ij} = (1-\kappa)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}\gamma_1 & \gamma_2\\ \gamma_2 & -\gamma_1\end{pmatrix}$$

magnification ***shear***

$$\kappa = \frac{1}{2}(\partial_{11}\varphi + \partial_{22}\varphi) = \frac{\Sigma(\theta)}{\Sigma_{crit}}$$

$$\gamma_1 = \frac{1}{2}(\partial_{11}\varphi - \partial_{22}\varphi)$$

$$\gamma_2 = \partial_{12}\varphi = \partial_{21}\varphi$$

The Basics of Gravitational Lensing