XII. Gravitational waves

12.1. What are gravitational waves

Generalities

GW can provide us information about General Relativity in the high energy regime (high masses, strong gravitational forces,...).

Gravitational waves are ripples in space-time caused by accelerating masses. They can be sourced by:

1. Bynary systems (BH-BH, NS-NS,...)
2. Tensor perturbations (seeded by inflation, which affect the CMB)
3. Supernovae (core collapse)

   A massive star ($\sim 10-30 M_\odot$) developes an iron core, which collapses in $T \sim 100$ ms. After the collapse there is a bounce (given by the equations of state), and in the end a neutron star is formed. This bounce produce GW ($\sim$ outside our detectors).

From now o

> NOTE:
> Newton and Einstein had different conceptions about spacetime.
>
> • Newton's fixed space • Einstein's flexible space-time
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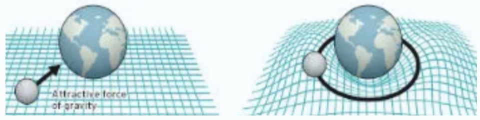
> 
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History

In 1915-16 Einstein formulated General Relativity: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

Soon after, he conjectured the existence of wave solutions, but was uncertained due to gauge artifacts. He wrote a letter to Schwarzschild in 1916:

> *"Since then [November 14] I have handled Newton's case differently, of course, according to the final theory [the theory of General Relativity]. Thus there are no gravitational waves analogous to light waves. This probably is also related to the one-sidedness of the sign of the scalar T, incidentally [this implies the nonexistence of a "gravitational dipole"] [6].*

Later Einstein found three types of waves, but Eddington showed two of them were spurius due to a choice of frame (not physical)

In 1936 he tried to publish a paper in Physical Review that GW do not exist, and the referee (Robertson) rejected it. So, Einstein sent an angry letter to the editor:

> *July 27, 1936*
> *Dear Sir.*
> *"We (Mr. Rosen and I) had sent you our manuscript for publication and had not authorized you to show it to specialists before it is printed. I see no reason to address the—in any case erroneous—comments of your anonymous expert. On the basis of this incident I prefer to publish the paper elsewhere."*
> *Respectfully*
> *Einstein*
> *P.S. Mr. Rosen, who has left for the Soviet Union, has authorized me to represent him in this matter.*

Later Einstein changed his mind again and now believed in GWs after realizing the error in his calculations. He then changed the title and published the paper as "On gravitational waves".

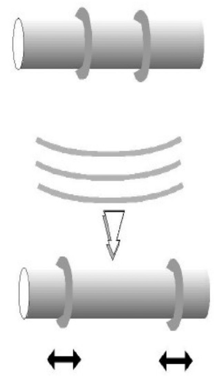
> "Note—The second part of this article was considerably altered by me after the departure to Russia of Mr. Rosen as we had misinterpreted the results of our formula. I want to thank my colleague Professor Robertson for their friendly help in clarifying the original error. I also thank Mr. Hoffmann your kind assistance in translation."

The argument was settled forever in 1957 by Feynmann:

> In a letter to Victor Weisskopf, Feynman recalls the 1957 conference in Chapel Hill and says, "I was surprised to find that a whole day of the conference was spent on this issue and that 'experts' were confused. That's what happens when one is considering energy conservation tensors, etc. instead of questioning, can waves do work?" [19].

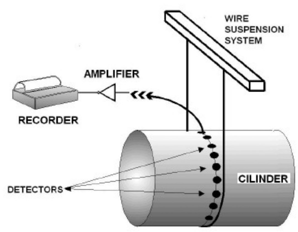
Direct detection

Feynmann's argumented that if GWs are real: they displace the beads (rings around a cilinder) thus producing heat (due to friction).



If we detect the heat, they are real.

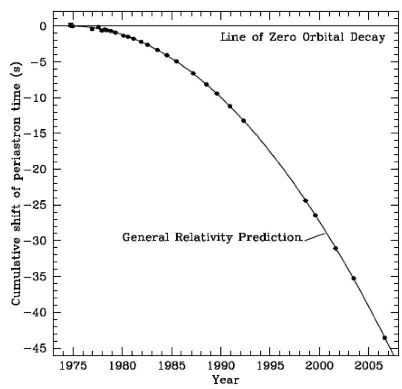
The first detector was built on 1960 by Joseph Weber: it would record small changes in current produced if a GW deformed the cilinder.



Indirect detection

A pulsar is a highly magnetized rotating neutron star that emits beams of EM radiation out of its magnetic poles.

They are very precise clocks. Eg. J0437-4715 has a period of 0.005757451936712637 secs with error of 1.7×10^{-17} secs.

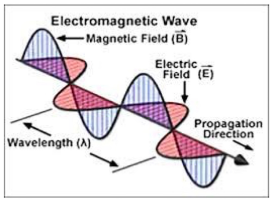


Having a pair of pulsars orbiting around each other, one can measure the properties of the system (seminajor axis, eccentricity, period, ...).

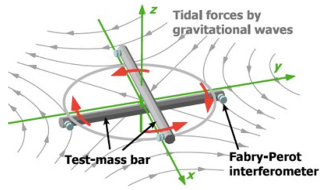
In 1974, Hulse and Taylor found that a pair of binary pulsars was inspiralling in perfect agreement with GR: pulsars are radiating energy, coming closer and closer to each other.

The better way to detect GWs is with interferometry. In 2002, LIGO started operating until 2010. Adv LIGO started in 2015.

Differences between GWs and EM waves



- EM waves travel through space, GWs are ripples in spacetime itself.
- EM waves can be absorbed, GWs cannot.
- GWs are weakly interacting, EM waves strongly interact with charges (ISM)
- GWs are produced (at minimum) by quadrupole, EM by dipole
- GWs are travelling, time-dependent tidal forces



- GW allow for a measurement of the luminosity distance $d_L(z)$, but not the redshift z (without a model)
- With EM counterpart, we can reconstruct $d_L(z)$ as for supernovae (GW + Gamma ray burst [GRB]) → GRB allows to measure z, GW gives $d_L(z)$. Taking first order expansion, one can measure the hubble constant.

12.2. Formalism in GR

Linearization

Gravity is weak and GWs interact weakly, so we need to linearize GR (perturbation theory).

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

We also know that GR is diffeomorphism invariant:

$$x^\mu \longrightarrow x'^\mu(x) \longrightarrow g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

(metric tensor transforms as a tensor)

Small perturbations around empty space can be written as:

$$\left.\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \\ |h_{\mu\nu}| &\ll 1 \\ x^\mu \rightarrow x'^\mu &= x^\mu + \xi^\mu(x) \end{aligned}\right\} \quad h_{\mu\nu}(x) \longrightarrow h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

↖ introducing x'^μ in the metric transformation rule

where $\eta_{\mu\nu}$ is the Minkowski background and $h_{\mu\nu}$ is a small perturbation and ξ^μ tells how have we set up our coordinate system.

We can also plug in the perturbation in the Riemann tensor and linearize it:

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho}\right)$$

We can also introduce "bared" h:

$$\left.\begin{aligned} h &= \eta^{\mu\nu} h_{\mu\nu} \\ \bar{h} &= h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \end{aligned}\right\} \quad \begin{aligned} \bar{h} &\equiv \eta^{\mu\nu}\bar{h}_{\mu\nu} = h - 2h = -h \\ h_{\mu\nu} &= \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \end{aligned}$$

↳ Trace

Combining everything for Einstein's equation:

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu \bar{h}_{\mu\rho} - \partial^\rho\partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

REMINDER
$\Box \rightarrow$ D'Alembert operator
$\Box = \eta_{\mu\nu}\partial^\mu\partial^\nu = \partial_\mu\partial^\mu$

GR has some residual freedom, so we can choose a gauge (usually, the Lorentz gauge).
This makes the GR equations decouple.
Lorentz gauge: $\partial^\mu \bar{h}_{\mu\nu} = 0$
This is possible because

$$\Box = \eta_{\mu\nu}\partial^\mu\partial^\nu = \partial_\mu\partial^\mu$$

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho) \Rightarrow \partial^\nu\bar{h}_{\mu\nu} \rightarrow (\partial^\nu\bar{h}_{\mu\nu})' = \partial^\nu\bar{h}_{\mu\nu} - \Box\xi_\mu$$

$$\partial^\nu\bar{h}_{\mu\nu} = f_\mu(x) \Rightarrow \Box\xi_\mu = f_\mu(x)$$

The final equations are:

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} \quad \text{(with sources)}$$
$$\Box \bar{h}_{\mu\nu} = 0 \quad \text{(in vacuum)}$$

We can use the gauge to remove spurious degrees of freedom (dof).
Let us find the number of degrees of freedom of GR. During the lecture on gauge inv. perturbations that the metric has 16 d.o.f. However, since it is simetric, only 10 or them are independent. 4 dof can be removed by choosing a coordinate system, and another 4 choosing the gauge. This leaves us with only 2 physical propagating degrees of freedom. If we go back to how the perturbations on the metric transforms (and knowing that we can choose ξ as we want):

$\Box\xi_\mu = 0 \quad h_{\mu\nu}(x) \longrightarrow h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu)$

We decide that:

$$\left.\begin{array}{ll} \xi^0 \longrightarrow \bar{h} = 0 & (\text{trace} = 0) \\ \xi^i(x) \rightarrow h^{0i}(x) = 0 & (\text{spatial part}) \end{array}\right\} \text{transverse traceless gauge (TT)}$$

Assuming that we are in vacuum, we can eliminate some of the h_{ij}:

$$\partial^\nu \bar{h}_{\mu\nu} = 0 \longrightarrow \partial^0 h_{00} + \partial^i h_{0i} = 0 \longrightarrow \partial^0 h_{00} = 0$$

Finally, the TT gauge: $\boxed{h^{0\mu} = 0 \qquad h^i{}_i = 0 \qquad \partial^j h_{ij} = 0}$

Solutions in vacuum are plane waves

$$\Box \bar{h}_{\mu\nu} = 0 \longrightarrow \boxed{h_{ij}^{TT}(x) = e_{ij}(\vec{k})e^{ikx}} \qquad k^\mu = \left(\frac{\omega}{c}, \vec{k}\right) \text{ and } \omega/c = |\vec{k}|$$

The polarizations:

$$\left.\begin{array}{l}\hat{n} = \vec{k}/|\vec{k}| \\ \partial^j h_{ij} = 0\end{array}\right\} \quad n^i h_{ij} = 0$$ (using TT gauge we find a condition on the polarization)

$$\boxed{h_{ij}^{TT} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos[\omega(t - z/c)]}$$

$\longrightarrow$ polarization tensor

$\longrightarrow$ oscillatory part + phase

Once we have this, we can write the structure of spacetime:

$$ds^2 = -c^2dt^2 + dz^2 + \left(1 + h_+ \cos[\omega(t - z/c)]\right)dx^2 + \left(1 - h_+ \cos[\omega(t - z/c)]\right)dy^2 + 2h_\times \cos[\omega(t - z/c)]\, dx\,dy$$

Taking the expansion in Fourier space:

$$h_{ab}(t, \mathbf{x}) = \sum_{A=+,\times} \int_{-\infty}^{\infty} df \int d^2\hat{\mathbf{n}}\, \tilde{h}_A(f, \hat{\mathbf{n}}) e^{A}_{ab}(\hat{\mathbf{n}})\, e^{-2\pi i f(t - \hat{\mathbf{n}}\cdot\mathbf{x}/c)} \qquad e^+_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ab} \qquad e^\times_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ab}$$

Effect on the masses

To get a first approach to the effect on masses, we study of geodesic deviation for two geodesics:

$\nearrow$ coupled system of diff eqn

$$\left.\begin{array}{l} x^\mu(\tau) \\ x^\mu(\tau) + \xi^\mu(\tau) \end{array}\right\} \qquad \frac{D^2\xi^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma}\, \xi^\rho \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} \longrightarrow \boxed{\ddot{\xi}^i = \frac{1}{2}\ddot{h}_{ij}^{TT}\xi^j}$$

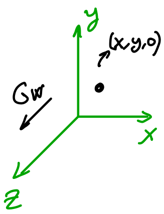
$\longrightarrow$ geodesics

$\longrightarrow$ geodesic deviation equation

Taking each polarization separatelly:

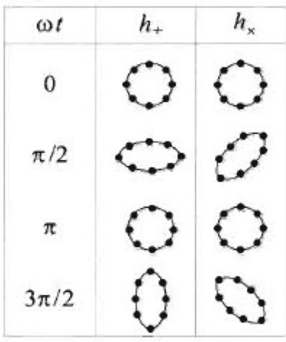
The + polarization:

$$h_{ab}^{TT} = h_+ \sin\omega t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (z = 0)$$



And substituting in the geodesic deviation equation:

$$\left.\begin{array}{l} \delta\ddot{x} = -\dfrac{h_+}{2}(x_0 + \delta x)\,\omega^2 \sin(\omega t) \\ \delta\ddot{y} = +\dfrac{h_+}{2}(y_0 + \delta y)\,\omega^2 \sin(\omega t) \end{array}\right\} \longrightarrow \left\{\begin{array}{l} \delta x(t) = +\dfrac{h_+}{2} x_0 \sin(\omega t) \\ \delta y(t) = -\dfrac{h_+}{2} y_0 \sin(\omega t) \end{array}\right.$$



The × polarization

Following the same procedure gives:

$$\left.\begin{array}{l} \delta x(t) = \dfrac{h_\times}{2} y_0 \sin(\omega t) \\ \delta y(t) = \dfrac{h_\times}{2} x_0 \sin(\omega t) \end{array}\right\}$$

GW energy

Feynman showed that GWs do work and carry energy. Energy of a wave is $E \sim h^2$, so we need to expand to second order:

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} + \dots$$

Rewritting the Einstein eqs and averaging over a wavelength:

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) \longrightarrow \tilde{R}_{\mu\nu} = -\langle R_{\mu\nu}^{(2)} \rangle + \frac{8\pi G}{c^4} \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle$$

Averaging over a wavelength, the background part will remain and the first order term will go to 0 (closed cycles). We can take it to the right hand side and define an effective energy-momentum tensor for the gravitational waves.

$$t_{\mu\nu} = -\frac{c^4}{8\pi G} \langle R_{\mu\nu}^{(2)} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \rangle \xrightarrow{\text{E. eq.}} \tilde{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = \frac{8\pi G}{c^4} (\bar{T}_{\mu\nu} + t_{\mu\nu})$$

Do the expansion:

$$R_{\mu\nu}^{(2)} = \frac{1}{2}\left[\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - h^{\alpha\beta}\partial_\nu\partial_\beta h_{\alpha\mu} - h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu} + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial^\beta h^\alpha_\nu \partial_\beta h_{\alpha\mu} - \partial^\beta h^\alpha_\nu \partial_\alpha h_{\beta\mu} - \partial_\beta h^{\alpha\beta}\partial_\nu h_{\alpha\mu}\right.$$
$$\left. + \partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu} - \partial_\beta h^{\alpha\beta}\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial^\alpha h \partial_\alpha h_{\mu\nu} + \frac{1}{2}\partial^\alpha h \partial_\nu h_{\alpha\mu} + \frac{1}{2}\partial^\alpha h \partial_\mu h_{\alpha\nu}\right]$$

The GW energy momentum tensor is:

energy density of the system

$$t_{\mu\nu} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle \longrightarrow \boxed{t^{00} = \frac{c^2}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle}$$

The energy flux and momentum by the waves are (integrating over a volume / surface)

$$E_V = \int_V d^3x \, t^{00} \xrightarrow{\partial_\mu t^{\mu\nu} = 0}$$

$$\frac{dE}{dt} = \frac{c^3 r^2}{32\pi G} \int d\Omega \, \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle \qquad \frac{dP^k}{dt} = -\frac{c^3}{32\pi G} r^2 \int d\Omega \, \langle \dot{h}_{ij}^{TT} \partial^k h_{ij}^{TT} \rangle$$

angular momentum

$$\frac{dE}{dA} = \frac{c^3}{16\pi G} \int_{-\infty}^{\infty} dt \, \left(\dot{h}_+^2 + \dot{h}_\times^2\right) \qquad J^i = \frac{c^2}{32\pi G} \int d^3x \left[-\epsilon^{ikl} \dot{h}_{ab}^{TT} x^k \partial^l h_{ab}^{TT} + 2\epsilon^{ikl} h_{ak}^{TT} \dot{h}_{al}^{TT}\right]$$

Gravitational waves carry energy, momentum and angular momentum

Solutions with sources can be obtained using retarded Green functions:

$$\left.\begin{aligned} \Box \bar{h}_{\mu\nu} &= -\frac{16\pi G}{c^4} T_{\mu\nu} \\ \Box_x G(x-x') &= \delta^4(x-x') \end{aligned}\right\} \quad \begin{aligned} \bar{h}_{\mu\nu}(x) &= -\frac{16\pi G}{c^4} \int d^4x' \, G(x-x') T_{\mu\nu}(x') \\ G(x-x') &= -\frac{1}{4\pi|\vec{x}-\vec{x}'|} \delta(x^0_{ret} - x'^0) \end{aligned} \qquad t_{ret} = t - \frac{|\vec{x}-\vec{x}'|}{c}$$

The solution can be written:

$$\bar{h}_{\mu\nu}(t,\vec{x}) = \frac{4G}{c^4} \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|} T_{\mu\nu}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right)$$

Low velocity expansion

$$|\vec{x}-\vec{x}'| = r - \vec{x}' \cdot \hat{n} + \mathcal{O}\left(\frac{d^2}{r}\right) \longrightarrow T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}'\right) \simeq T_{kl}(t - \frac{r}{c}, \mathbf{x}') + \frac{x'^i n^i}{c} \partial_0 T_{kl} + \frac{1}{2c^2} x'^i x'^j n^i n^j \partial_0^2 T_{kl} + \dots$$

GW radiated power

Quadrupole

One can define the moments of the 00 part of the energy-momentum tensor as:

$$M = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x}),$$ total mass/energy - conserved

$$M^i = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})x^i,$$ dipole (centre of mass) - removable

$$M^{ij} = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})x^i x^j, \text{ quadrupole} \longrightarrow \left[h_{ij}^{TT}(t,\vec{x})\right]_{quad} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{n})\ddot{M}^{kl}(t-r/c)$$

$$M^{ijk} = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})x^i x^j x^k,$$ octupole

Thus we can introduce the quadrupole tensor as:

$$M^{kl} = \left(M^{kl} - \frac{1}{3}\delta^{kl}M_{ii}\right) + \frac{1}{3}\delta^{kl}M_{ii}$$

$$\boxed{Q^{ij} = M^{ij} - \frac{1}{3}\delta^{ij}M_{kk} = \int d^3x\,\rho(t,\vec{x})\left(x^i x^j - \frac{1}{3}r^2\delta^{ij}\right)}$$

$\rho = \frac{1}{c^2}T^{00}$

$$\boxed{\left[h_{ij}^{TT}(t,\vec{x})\right]_{quad} = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}^{TT}\left(t-\frac{r}{c}\right)}$$

Amplitude in terms of the quadrupole tensor

If we have a distribution of matter (eg. orbits of BH) one just needs to calculate the cuadrupole tensor and derivate to obtain the amplitude, from the which one can calculate the power, energy, momentum, angular momentum, ...

Radiated power and angular momentum:

$$\left(\frac{dP}{d\Omega}\right)_{quad} = \frac{r^2c^3}{32\pi G}\left\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\right\rangle$$

$$J^i = \frac{c^2}{32\pi G}\int d^3x\left[-\varepsilon^{ijk}\dot{h}_{ab}^{TT}x^k\partial^l h_{ab}^{TT} + 2\varepsilon^{ikl}h_{ak}^{TT}\dot{h}_{al}^{TT}\right] \longrightarrow$$

$$\boxed{\begin{aligned} P_{quad} &= \frac{G}{5c^5}\left\langle\dddot{Q}_{ij}\dddot{Q}_{ij}\right\rangle \\ \left(\frac{dJ^i}{dt}\right)_{quad} &= \frac{2G}{5c^5}\varepsilon^{ikl}\left\langle\ddot{Q}_{ka}\dddot{Q}_{la}\right\rangle\end{aligned}}$$

In order to get GW, it is not only necessary to have a time-varying distribution of mass, but also to have a non-zero third derivative in order to radiate power.

Radiation from Octupole:

$$O^{klm} = M^{klm} - \frac{1}{5}\left(\delta^{kl}M^{k'k'm} + \delta^{km}M^{k'lk'} + \delta^{lm}M^{kk'k'}\right)$$

$$M^{ijk}(t) = \mu x_0^i(t)\,x_0^j(t)\,x_0^k(t)$$

$$\left(h_{ij}^{TT}\right)_{oct} = \frac{1}{r}\frac{2G}{3c^5}\Lambda_{ij,kl}(\hat{n})n_m\dddot{O}^{klm}$$

$$P = \frac{G}{c^5}\left[\frac{1}{5}\langle\dddot{Q}_{ij}\dddot{Q}_{ij}\rangle + \frac{1}{c^2}\frac{1}{189}\langle\frac{d^4O_{ijk}}{dt^4}\frac{d^4O_{ijk}}{dt^4}\rangle + O\left(\frac{v^4}{c^4}\right)\right]$$

Note:

> The power of the octupole is suppressed with respect to the octupole by a factor of $1/c^2$ $\longrightarrow$ can get a good prediction without considering it

Particular cases

Inspiral binaries in circular orbits.

$\omega_s^2 = \frac{Gm}{R^3}$ → Frequency of the orbit

$$\begin{cases} x_0(t) = R\cos\left(\omega_s t + \frac{\pi}{2}\right) \\ y_0(t) = R\sin\left(\omega_s t + \frac{\pi}{2}\right) \\ z_0(t) = 0 \end{cases} \xrightarrow[\text{reduced mass}]{\mu = \frac{m_1 m_2}{m_1+m_2}}$$

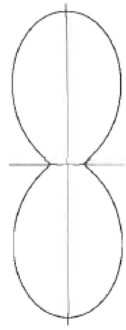
Viewing angle

$$h_+(t;\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{ret} + 2\phi),$$
$$h_\times(t;\theta,\phi)) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\cos\theta\sin(2\omega_s t_{ret} + 2\phi).$$

Power:

$$\left(\frac{dP}{d\Omega}\right)_{quad} = \frac{r^2c^3}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle \longrightarrow \left(\frac{dP}{d\Omega}\right)_{quad} = \frac{2G\mu^2R^4\omega_s^6}{\pi c^5}\, g(\theta) \qquad g(\theta) = \left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta$$

$$P_{quad} = \frac{32}{5}\frac{G\mu^2}{c^5}R^4\omega_s^6 = \frac{1}{10}\frac{G\mu^2}{c^5}R^4\omega^6 \qquad \boxed{\omega = 2\omega_s} \leftarrow \omega \text{ Frequency of the GWs}$$



Introduce the chirp mass (to simplify the expresions for the polarizatons)

$$\omega_s^2 = \frac{Gm}{R^3} \longrightarrow$$

$$\boxed{M_c = \mu^{3/5}\, m^{2/5} = \frac{(m_1m_2)^{3/5}}{(m_1+m_2)^{1/5}}}$$

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}}{c}\right)^{2/3}\frac{1+\cos^2\theta}{2}\cos(2\pi f_{gw}t_{ret} + 2\phi)$$
$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}}{c}\right)^{2/3}\cos\theta\sin(2\pi f_{gw}t_{ret} + 2\phi),$$

$$P = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{gw}}{2c^3}\right)^{10/3}$$

The system is losing energy, thus the frecuency changes:

they get closer

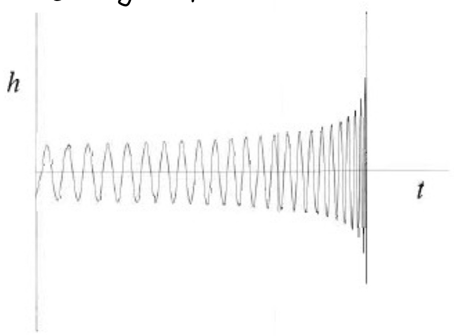
$$\omega_s^2 = \frac{Gm}{R^3}$$
$$E_{orbit} = E_{kin} + E_{pot} = -\frac{Gm_1m_2}{2R}$$
$$\longrightarrow \left.\begin{aligned} \dot{R} &= -\frac{2}{3}R\frac{\dot{\omega}_s}{\omega_s} = -\frac{2}{3}R\frac{\dot{\omega}_s}{\omega_s} \\ E_{orbit} &= -\left(\frac{G^2M_c^5\omega_{gw}^2}{32}\right)^{1/3} \end{aligned}\right\} \quad \dot{\omega}_{gw} = \frac{12}{5}2^{1/3}\left(\frac{GM_c}{c^3}\right)^{5/3}\omega_{gw}^{11/3}$$

One can solve the differential equation to get the time to coalescence (when they merge):

$$\dot{f}_{gw} = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3}f_{gw}^{11/3} \longrightarrow f_{gw}(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8} \simeq 134\,\text{Hz}\left(\frac{1.21M_\odot}{M_c}\right)^{5/8}\left(\frac{1\text{s}}{\tau}\right)^{3/8}$$

$\tau \equiv t_{coal} - t$

Change of amplitude with time: (solving numerically and substituting).



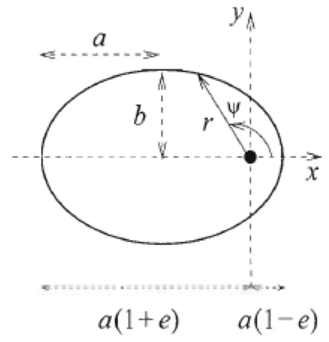
After the merger $h \to 0$

Elliptical orbits

Taking again a semi Keplerian/newtonian approximation:

One can calculate the second moment, obtaining:

$$M_{ab} = \mu r^2 \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{pmatrix}_{ab}$$



$$e^2 = 1 + \frac{2EL^2}{G^2m^2\mu^3}$$

$$\dot{\psi} = \frac{(GmR)^{1/2}}{r^2}$$

$$r = \frac{a(1-e^2)}{1+e\cos\psi}$$

Radiated power:

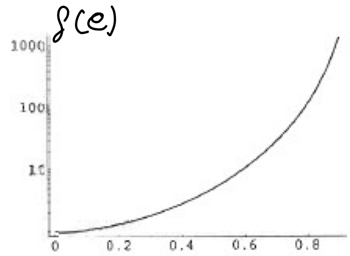
$$P(\psi) = \frac{G}{5c^5}\left[\dddot{M}_{11}^2 + \dddot{M}_{22}^2 + 2\dddot{M}_{12}^2 - \frac{1}{3}(\dddot{M}_{11} + \dddot{M}_{22})^2\right] = \frac{2G}{15c^5}\left[\dddot{M}_{11}^2 + \dddot{M}_{22}^2 + 3\dddot{M}_{12}^2 - \dddot{M}_{11}\dddot{M}_{22}\right]$$

$$= \frac{8G^4}{15c^5}\frac{\mu^2 m^3}{a^5(1-e^2)^5}(1+e\cos\psi)^4\left[12(1+e\cos\psi)^2 + e^2\sin^2\psi\right]$$

Average over orbit ($T \equiv$ period)

$$P \equiv \frac{1}{T}\int_0^T dt\, P(\psi) = \frac{\omega_0}{2\pi}\int_0^{2\pi}\frac{d\psi}{\dot{\psi}}P(\psi) = (1-e^2)^{3/2}\int_0^{2\pi}\frac{d\psi}{2\pi}(1+e\cos\psi)^{-2}P(\psi)$$

$$= \frac{8G^4\mu^2m^3}{15c^5a^5}(1-e^2)^{-7/2}$$

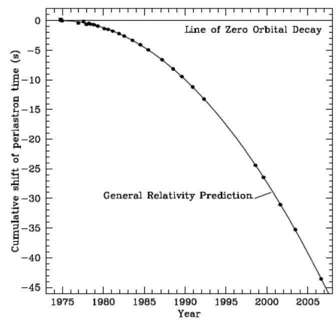


$$P = \frac{32G^4\mu^2m^3}{5c^5a^5}f(e) \qquad f(e) = \frac{1}{(1-e^2)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right)$$

One can find the change in period:

$$\left.\begin{aligned} a &= \frac{Gm\mu}{2|E|} \\ \omega_0^2 &= \frac{Gm}{a^3} \\ T &= \text{const} \times (-E)^{-3/2} \end{aligned}\right\} \quad \frac{\dot{T}}{T} = -\frac{96}{5}\frac{G^{5/3}\mu m^{2/3}}{c^5}\left(\frac{T}{2\pi}\right)^{-8/3}f(e)$$

Measured $\longrightarrow$ for pulsar



Change in orbital elements: the system is radiating energy and angular momentum.
(E → related to semimajor axis. J → related to the eccentricity)

$$\frac{dE}{dt} = -\frac{32}{5}\frac{G^4\mu^2m^3}{c^5a^5}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

$$\frac{dL}{dt} = -\frac{32}{5}\frac{G^{7/2}\mu^2m^{5/2}}{c^5a^{7/2}}\frac{1}{(1-e^2)^2}\left(1+\frac{7}{8}e^2\right)$$

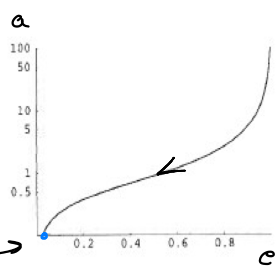
$\xrightarrow[\text{diff equation}]{\text{translate to}}$

$$\frac{da}{dt} = -\frac{64}{5}\frac{G^3\mu m^2}{c^5a^3}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

$$\frac{de}{dt} = -\frac{304}{15}\frac{G^3\mu m^2}{c^5a^4}\frac{e}{(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right)$$

Orbit circularization (dividing both equations):

$$\frac{da}{de} = \frac{12}{19}a\frac{1+(73/24)e^2+(37/96)e^4}{e(1-e^2)[1+(121/304)e^2]} \longrightarrow a(e) = C_0\frac{e^{12/19}}{1-e^2}\left(1+\frac{121}{304}e^2\right)^{870/2299}$$



The orbits tend to e = 1: they circularize $\longrightarrow$

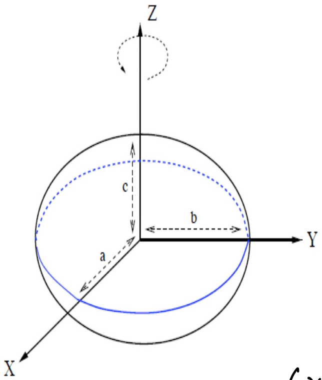
Time to coalescence (e.g. Hulse-Taylor pulsar)

$$\tau(a_0, e_0) = \frac{15}{304}\frac{c^5}{G^3 m^2 \mu}\int_0^{e_0} de\, \frac{a^4(e)(1-e^2)^{5/2}}{e\left(1+\frac{121}{304}e^2\right)}$$

$$\simeq 9.829\,\mathrm{Myr}\left(\frac{T_0}{1\,\mathrm{hr}}\right)^{8/3}\left(\frac{M_\odot}{m}\right)^{2/3}\left(\frac{M_\odot}{\mu}\right)F(e_0)$$

$\longrightarrow$ $m_1 = m_2 \simeq 1.4\,M_\odot$ (neutron stars), $T_0 \simeq 7.75$ hr $\Big\}\ \tau(a_0, e_0) \simeq 300$ Myr

Rotating spherically symmetric objects



A spherically symmetric rotating matter distribution does NOT emit GW. Let us see why. (SPOILER: It has to do with the third derivative of the quadrupole moment.)

The moment of inertia tensor is given by:

$$I_{ij} = \int_V \rho\left(r^2\delta_{ij} - x_i x_j\right)\, dx^3$$

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x^2}{b}\right)^2 + \left(\frac{x^3}{c}\right)^2 = 1$$

$$\Bigg\}\ I_{ij} = \frac{M}{5}\begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & c^2+a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Going to a rotating frame:

$$x_i = R_{ij}x'_j,$$

$$R_{ij} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi = \Omega t \quad \longrightarrow$$

$$I_{ij} = R_{ik}R_{jl}I'_{kl} = (RI'R^T)_{ij}$$

$$= \begin{pmatrix} I_1\cos^2\varphi + I_2\sin^2\varphi & -\sin\varphi\cos\varphi(I_2-I_1) & 0 \\ -\sin\varphi\cos\varphi(I_2-I_1) & I_1\sin^2\varphi + I_2\cos^2\varphi & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

The quadrupole tensor is given by:

$$Q_{ij} = -\left(I_{ij} - \frac{1}{3}\delta_{ij}\,\mathrm{Tr}\, I\right) = -I_{ij} + \text{constant}$$

$$\mathrm{Tr}\, I = I_1 + I_2 + I_3 = \text{constant}$$

$$\Bigg\}\ \to\ Q_{ij} = \frac{I_2 - I_1}{2}\begin{pmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} + constant$$

For spherically symmetric objects, $a = b = c$:

$$I_1 = \frac{M}{5}(b^2+c^2)$$

$$I_2 = \frac{M}{5}(c^2+a^2)$$

$$\Bigg\}\ \boxed{Q_{ij} = 0}$$

In general (if we have a small deviation):

$$\epsilon \equiv \frac{a-b}{(a+b)/2} \longrightarrow \frac{I_2 - I_1}{I_3} = \frac{1}{2}\epsilon\,\frac{a^2+b^2+2ab}{a^2+b^2} = \epsilon + \mathcal{O}(\epsilon^3)$$

$$Q_{ij} = \frac{\epsilon\, I_3}{2}\begin{pmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} + constant \longrightarrow$$

Radiated power: $\mathcal{L}_{GW} = \frac{32G}{5c^2}\,\Omega^6\epsilon^2 I^2$ (rot. frequency)

$\hookrightarrow$ small, but existent

The Post-Newtonian expansion (PN)

Decomposing the metric and $T_{\mu\nu}$ in terms of v/c:

$$
\begin{aligned}
g_{00} &= -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + {}^{(6)}g_{00} + \dots , \\
g_{0i} &= {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + \dots , \\
g_{ij} &= \delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + \dots ,
\end{aligned}
\qquad
\begin{aligned}
T^{00} &= {}^{(0)}T^{00} + {}^{(2)}T^{00} + \dots , \\
T^{0i} &= {}^{(1)}T^{0i} + {}^{(3)}T^{0i} + \dots , \\
T^{ij} &= {}^{(2)}T^{ij} + {}^{(4)}T^{ij} + \dots .
\end{aligned}
$$

And expanding the geodesic equation:

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \longrightarrow \frac{d^2x^i}{dt^2} \simeq -c^2\Gamma^i_{00} = c^2\left(\frac{1}{2}\partial^i h_{00} - \partial_0 h_0{}^i\right) = \frac{c^2}{2}\partial^i h_{00}$$

The expansion becomes:

$$\frac{dy_1^i}{dt} = v_1^i \qquad \frac{dv_1^i}{dt} = A_1^i + \frac{1}{c^2}B_1^i + \frac{1}{c^4}C_1^i + \frac{1}{c^5}D_1^i + \mathcal{O}(6)$$

Where:

$$A_1^i = -\frac{Gm_2}{r^2}n^i ,$$

0PN (Newton's term)

$$
\begin{aligned}
B_1^i = \frac{Gm_2}{r^2}\Bigg\{ & n^i\left[-v_1^2 - 2v_2^2 + 4(v_1v_2) + \frac{3}{2}(nv_2)^2 + 5\frac{Gm_1}{r} + 4\frac{Gm_2}{r}\right] \\
& + (v_1^i - v_2^i)\Big[4(nv_1) - 3(nv_2)\Big]\Bigg\} ,
\end{aligned}
$$

1PN

$$
\begin{aligned}
C_1^i = \frac{Gm_2}{r^2}\Bigg\{ & n^i\Bigg[-2v_2^4 + 4v_2^2(v_1v_2) - 2(v_1v_2)^2 + \frac{3}{2}v_1^2(nv_2)^2 \\
& + \frac{9}{2}v_2^2(nv_2)^2 - 6(v_1v_2)(nv_2)^2 - \frac{15}{8}(nv_2)^4 \\
& + \frac{Gm_1}{r}\left(-\frac{15}{4}v_1^2 + \frac{5}{4}v_2^2 - \frac{5}{2}(v_1v_2) + \frac{39}{2}(nv_1)^2 - 39(nv_1)(nv_2) + \frac{17}{2}(nv_2)^2\right) \\
& + \frac{Gm_2}{r}\left(4v_2^2 - 8(v_1v_2) + 2(nv_1)^2 - 4(nv_1)(nv_2) - 6(nv_2)^2\right)\Bigg] \\
& + (v_1^i - v_2^i)\Bigg[v_1^2(nv_2) + 4v_2^2(nv_1) - 5v_2^2(nv_2) - 4(v_1v_2)(nv_1) \\
& \quad + 4(v_1v_2)(nv_2) - 6(nv_1)(nv_2)^2 + \frac{9}{2}(nv_2)^3 \\
& + \frac{Gm_1}{r}\left(-\frac{63}{4}(nv_1) + \frac{55}{4}(nv_2)\right) + \frac{Gm_2}{r}\Big(-2(nv_1) - 2(nv_2)\Big)\Bigg]\Bigg\} \\
& + \frac{G^3m_2}{r^4}n^i\left\{-\frac{57}{4}m_1^2 - 9m_2^2 - \frac{69}{2}m_1m_2\right\} ,
\end{aligned}
$$

2PN

$$D_1^i = \frac{4}{5}\frac{G^2m_1m_2}{r^3}\left\{v^i\left[-v^2 + 2\frac{Gm_1}{r} - 8\frac{Gm_2}{r}\right] + n^i(nv)\left[3v^2 - 6\frac{Gm_1}{r} + \frac{52}{3}\frac{Gm_2}{r}\right]\right\}$$

2.5PN

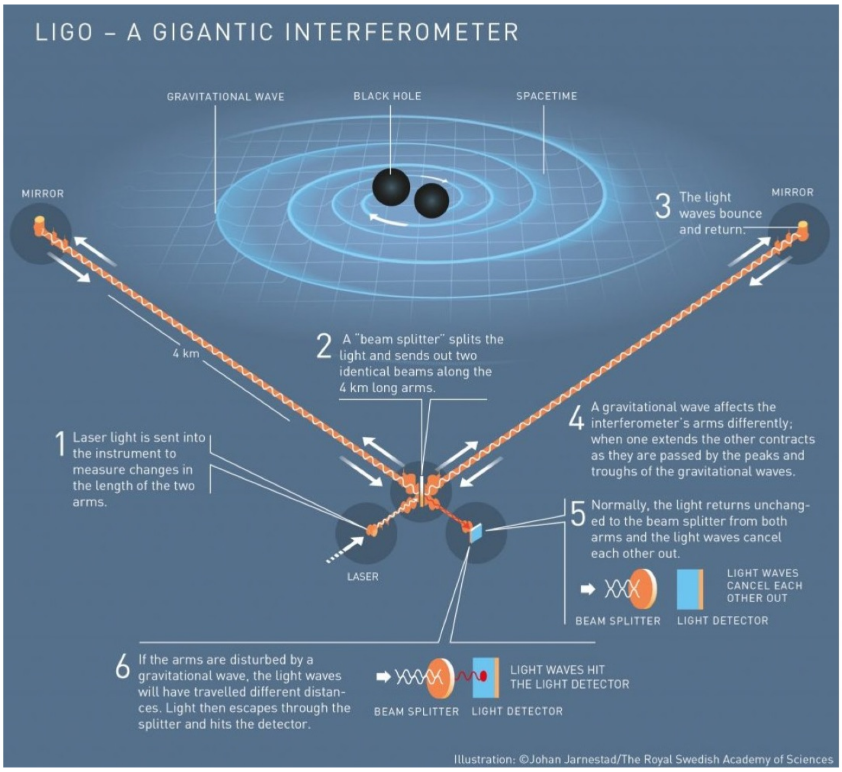
Not conservative!!!
Reason for GWs

(Breaks time reversal)

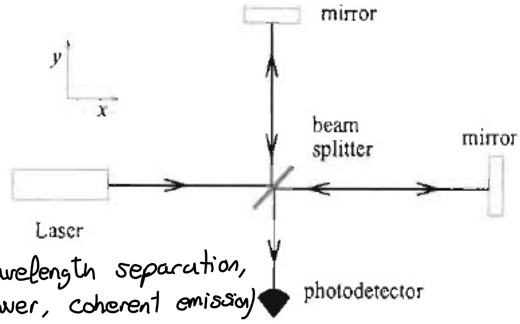
12.3. Detection techniques

Interferometry

The expected amplitude of gravitational waves is small, so we need a technique to measure very small displacements. One well developed technique to do so is interferometry. GW require photon-based distance measurements to be detected. We need something that travels with the speed of light (which is constant). Hence:



Laser interferometry, Michelson (1887)



(wavelength separation, power, coherent emission)

Some of the photons go through the mirror, others are reflected.
Differences in flight distance (e.g. due to gravitational waves) creates different interference patterns.

Electric field measured:

$$E_1 = -\frac{1}{2} E_0 e^{-i\omega_L t + 2ik_L L_x}$$
$$E_2 = +\frac{1}{2} E_0 e^{-i\omega_L t + 2ik_L L_y}$$
$$E_{out} = E_1 + E_2$$

on the detector

Flight path difference

$$E_{out} = -iE_0 e^{-i\omega_L t + ik_L(L_x + L_y)} \sin[k_L(L_y - L_x)]$$

$$\boxed{|E_{out}|^2 = E_0^2 \sin^2[k_L(L_y - L_x)]}$$

Connection with GW (effect on distances):

We have seen that GW have an effect on the distance. Since photons move along null geodesics, we can calculate this:

$$h_+(t) = h_0 \cos \omega_{gw} t$$
$$ds^2 = -c^2 dt^2 + [1 + h_+(t)]\, dx^2 + [1 - h_+(t)]\, dy^2 + dz^2$$

$$ds^2 = 0 \longrightarrow dx = \pm c\,dt \left[1 - \frac{1}{2} h_+(t)\right]$$

$$L_x = c(t_1 - t_0) - \frac{c}{2}\int_{t_0}^{t_1} dt'\, h_+(t')$$

Similarly (on the way back):

$$L_x = c(t_2 - t_1) - \frac{c}{2}\int_{t_1}^{t_2} dt'\, h_+(t')$$

Total time and difference in phase:

$$\boxed{t_2 - t_0 = \frac{2L_x}{c} + \frac{1}{2}\int_{t_0}^{t_2} dt'\, h_+(t') = \frac{2L_x}{c} + \frac{L_x}{c} h(t_0 + L_x/c)\, \frac{\sin(\omega_{gw} L_x/c)}{(\omega_{gw} L_x/c)}}$$

$$\boxed{\Delta\phi_x(t) = h_0 \frac{\omega_L L_x}{c} \mathrm{sinc}\,(\omega_{gw} L_x/c)\, \cos[\omega_{gw}(t - L_x/c)]}$$

We can also calculate the power detected:

$$P = P_0 \sin^2[\phi_0 + \Delta\phi_x(t)] = \frac{P_0}{2}\{1 - \cos[2\phi_0 + 2\Delta\phi_x(t)]\}$$

$$\Delta\phi_{\text{Mich}} \equiv \Delta\phi_x - \Delta\phi_y = 2\Delta\phi_x$$

$$(\Delta P)_{\text{GW}} = \frac{P_0}{2}|\sin 2\phi_0|\,(\Delta\phi)_{\text{Mich}}$$

Noise and sensitivity.

The detector measures the total strain but measurements given in terms of signal to noise

$h(t) = D^{ij} h_{ij}(t)$

$\longrightarrow$ Depends on detector geometry

Final measurement depends on the transfer function $T(f)$ $\leftarrow$ sensitivity in terms of the frequency

$\tilde{h}_{out}(f) = T(f)\,\tilde{h}(f)$

The output also includes the noise (more later):

$$\left.\begin{array}{l} S_{out}(t) = h_{out}(t) + n_{out}(t) \\ \langle \tilde{n}^*(f)\,\tilde{n}(f')\rangle = \delta(f-f')\,\frac{1}{2}\,S_n(f) \\ \qquad \text{and} \quad \langle n(t)\rangle = 0 \end{array}\right\} \longrightarrow \langle |\tilde{n}(f)|^2\rangle = \frac{1}{2} S_n(f)\, T$$

$$\delta(f=0) \rightarrow \left[\int_{-T/2}^{T/2} dt\, e^{i2\pi f t}\right]\Bigg|_{f=0} = T$$

$\downarrow$

noise profile in terms of the frequency

Spectral noise density $S_n(f)$ is variance of the noise

$$\left.\begin{array}{l} \Delta f = 1/T \\ \frac{1}{2} S_n(f) = \langle |\tilde{n}(f)^2|\rangle \Delta f \end{array}\right\} \langle n^2(t)\rangle = \langle n^2(t=0)\rangle = \int_{-\infty}^{+\infty} df\, df' \langle n^*(f)\, n(f')\rangle = \frac{1}{2}\int_{-\infty}^{+\infty} df\, S_n(f) = \int_0^{\infty} df\, S_n(f)$$

spectral noise density

Signal to noise ratio

We can calculate the signal using the filter function K, which provides the sensitivity for each frequency:

$$S = \int_{-\infty}^{\infty} dt\, \langle s(t)\rangle K(t) = \int_{-\infty}^{\infty} dt\, h(t) K(t) = \int_{-\infty}^{\infty} df\, \tilde{h}(f)\tilde{K}^*(f)$$

$$\langle n(t)\rangle = 0$$

Noise:

$$N^2 = \left[\langle \hat{s}^2(t)\rangle - \langle \hat{s}(t)\rangle^2\right]_{h=0} = \langle \hat{s}^2(t)\rangle_{h=0} = \int_{-\infty}^{\infty} dt dt'\, K(t)K(t')\,\langle n(t) n(t')\rangle$$

$$= \int_{-\infty}^{\infty} dt dt' K(t)K(t') \int_{-\infty}^{\infty} df\, df'\, e^{2\pi i f t - 2\pi i f' t'} \langle \tilde{n}^*(f)\tilde{n}(f')\rangle = \int_{-\infty}^{\infty} df\, \frac{1}{2} S_n(f) |\tilde{K}(f)|^2$$

Final expresion for the signal to noise

$$\frac{S}{N} = \frac{\int_{-\infty}^{\infty} df\, \tilde{h}(f)\tilde{K}^*(f)}{\left[\int_{-\infty}^{\infty} df\, (1/2) S_n(f) |\tilde{K}(f)|^2\right]^{1/2}} \qquad \tilde{K}(f) = \text{const.}\, \frac{\tilde{h}(f)}{S_n(f)}$$

Then:

$$\boxed{\left(\frac{S}{N}\right)^2 = 4\int_0^{\infty} df\, \frac{|\tilde{h}(f)|^2}{S_n(f)}}$$

Example 1: stochastic backgrounds (white noise background of unresolved sources)

$$\langle h_{ij}(t)h^{ij}(t)\rangle = 4\int_0^{\infty} df\, S_h(f) \qquad \rho_{\rm gw} = \frac{c^2}{32\pi G}\langle \dot{h}_{ij}\dot{h}^{ij}\rangle$$

$$\Longrightarrow \quad \rho_{\rm gw} \equiv \int_{f=0}^{f=\infty} d(\log f)\, \frac{d\rho_{\rm gw}}{d\log f} \quad \Longrightarrow \quad \Omega_{\rm gw}(f) \equiv \frac{1}{\rho_c}\frac{d\rho_{\rm gw}}{d\log f} = \frac{4\pi^2}{3H_0^2} f^3 S_h(f)$$

Example 2: Distance to coalescing binaries

$$\tilde{h}(f) = \left(\frac{5}{6}\right)^{1/2} \frac{1}{2\,\pi^{2/3}}\frac{c}{r}\left(\frac{GM_c}{c^3}\right)^{5/6} f^{-7/6} e^{i\Psi}\, Q(\theta,\phi;\iota)$$

Function that depends on geometry of the system, inclinatio, etc.

$$\left(\frac{S}{N}\right)^2 = \frac{5}{6}\frac{1}{\pi^{4/3}}\frac{c^2}{r^2}\left(\frac{GM_c}{c^3}\right)^{5/3} |Q(\theta,\phi;\iota)|^2 \int_0^{f_{\max}} df\, \frac{f^{-7/3}}{S_n(f)}$$

Averaging over inclination (etc) we can solve for the distance:

$$d_{\rm sight} = \frac{2}{5}\left(\frac{5}{6}\right)^{1/2}\frac{c}{\pi^{2/3}}\left(\frac{GM_c}{c^3}\right)^{5/6}\left[\int_0^{f_{\max}} df\, \frac{f^{-7/3}}{S_n(f)}\right]^{1/2} (S/N)^{-1}$$

Average amplitude on Earth and length of detector:

$$\Delta L = \frac{1}{2} h_0 L \xrightarrow{h_0 \sim 10^{-21}} \Delta L \sim 2\times 10^{-18}\,m$$

Sources of noise

- Shot noise: photons are discrete, they follow Poisson distribution

$$P = \frac{1}{T} N_\gamma \hbar\omega_L \qquad p(N;\bar{N}) = \frac{1}{N!}\bar{N}^N e^{-\bar{N}}$$

$$\Longrightarrow \quad \Delta N_\gamma = \sqrt{N_\gamma}. \quad \Longrightarrow \quad (\Delta P)_{\rm shot} = \frac{1}{T} N_\gamma^{1/2}\hbar\omega_L = \left(\frac{\hbar\omega_L}{T}P\right)^{1/2} = \left(\frac{\hbar\omega_L}{T}P_0\right)^{1/2} |\sin\phi_0|$$

$$P = P_0 \sin^2\phi_0$$

Large N limit of Poisson → Gaussian!
Can you prove it?

Total signal to noise:

$$P = P_0\sin^2[\phi_0 + \Delta\phi_x(t)] = \frac{P_0}{2}\{1-\cos[2\phi_0 + 2\Delta\phi_x(t)]\}$$

$$\Delta\phi_{\rm Mich} \equiv \Delta\phi_x - \Delta\phi_y = 2\Delta\phi_x$$

$$(\Delta P)_{\rm GW} = \frac{P_0}{2}|\sin 2\phi_0|\,(\Delta\phi)_{\rm Mich}$$

$$\Longrightarrow \quad (\Delta P)_{\rm GW} = \frac{P_0}{2}|\sin 2\phi_0|\frac{4\pi L}{\lambda_L}h_0$$

$$\frac{S}{N} = \frac{(\Delta P)_{\rm GW}}{(\Delta P)_{\rm shot}} = \left(\frac{P_0 T}{\hbar\omega_L}\right)^{1/2}\frac{4\pi L}{\lambda_L} h_0 |\cos\phi_0|$$

- Radiation pressure

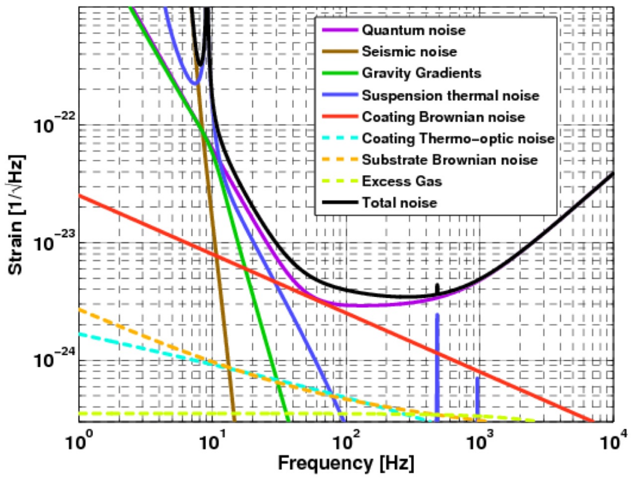
$$\Delta F = 2\Delta P/c = 2\sqrt{\frac{\hbar\omega_L P}{c^2 T}} \quad \Longrightarrow \quad S_F^{1/2} = 2\sqrt{\frac{2\hbar\omega_L P}{c^2}}$$

$$\langle A^2(t)\rangle = \frac{1}{2T}S_A$$

- The quantum limit

  (shot noise + radiation pressure)

  $$S_n(f)|_{opt} = S_n(f)|_{shot} + S_n(f)|_{rad}$$



- Seismic noise

  $$x(f) \simeq A\left(\frac{1\,Hz}{f_\omega}\right) m\,Hz^{-1/2}$$

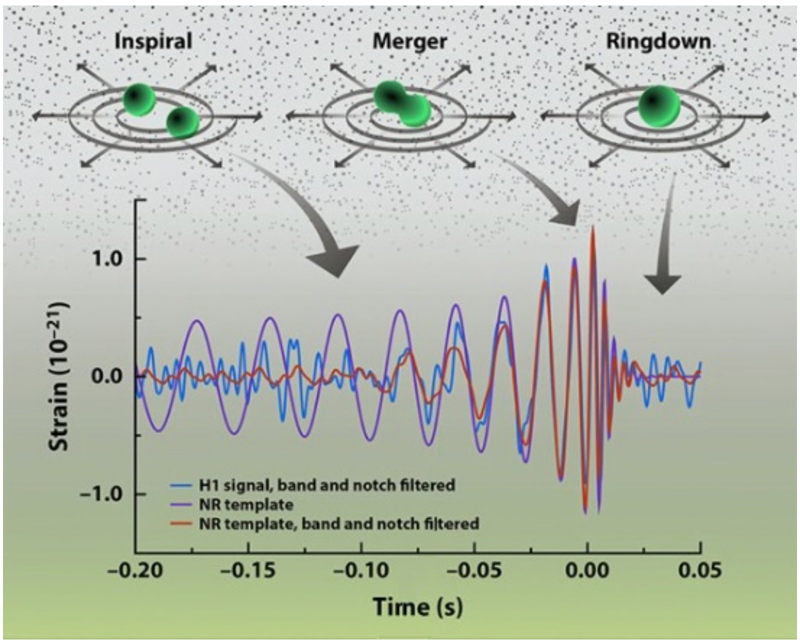
Quantum noise domains for larger freq.

Low frequencies are quenched by seismic noise.

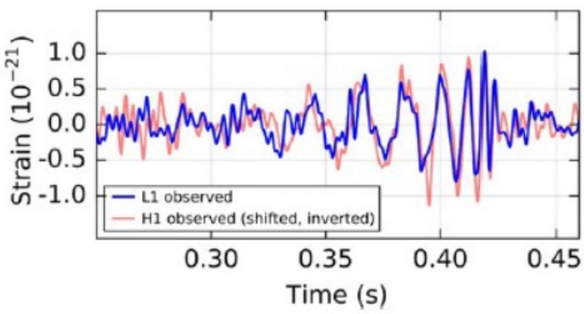
12.4. Recent observations (BH - BH, NS - NS)

BH - BH

First detected by LIGO in 2015



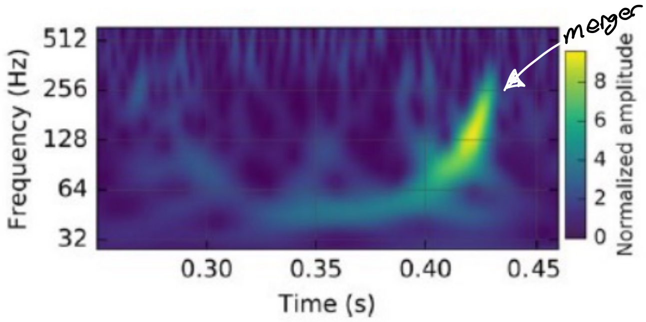
The two LIGO detectors. The signals have to appear in both of them!

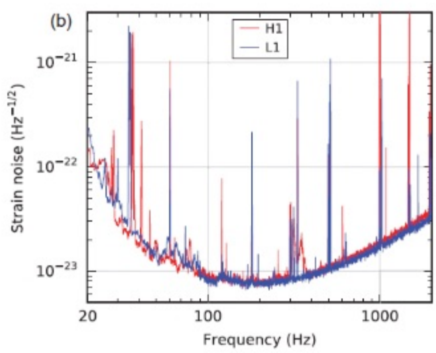


The signal (strain) from both LIGO detectors (S/N ~24) ⟶ A true signal must be detected by both detectors with a 10 ms delay (given by the distance between detectors.

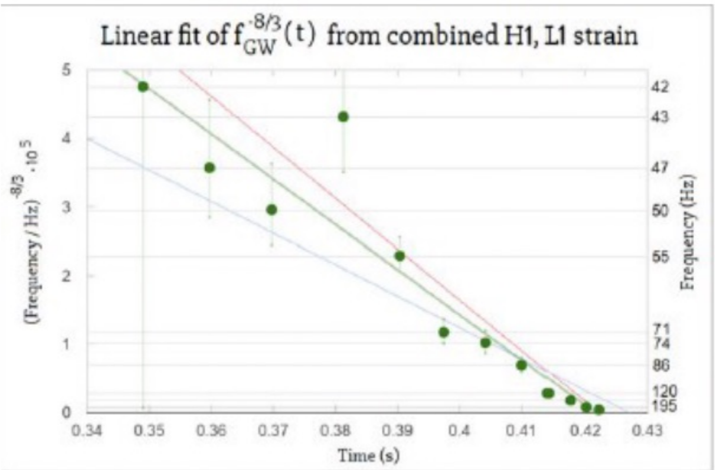
The signal in time-frequency and strain-frequency domain:

(FFT to the signal)





Fit to the data (determination of the chirp mass):



$$\mathcal{M} = \frac{c^3}{G}\left(\left(\frac{5}{96}\right)^3 \pi^{-8} (f_{GW})^{-11} (\dot{f}_{GW})^3\right)^{1/4}$$

$$f_{GW}^{-8/3}(t) = \frac{(8\pi)^{8/3}}{5}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}(t_c - t)$$

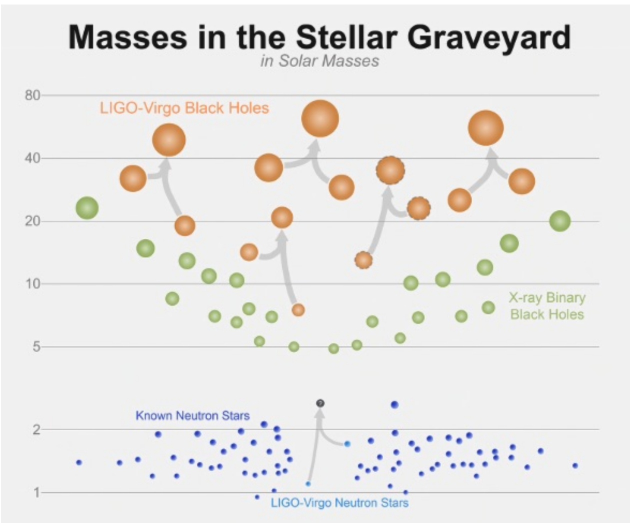
(time of coalescence)

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} \longrightarrow \mathcal{M} = 30\, M_\odot$$

Primary black hole mass	$36^{+5}_{-4} M_\odot$
Secondary black hole mass	$29^{+4}_{-4} M_\odot$
Final black hole mass	$62^{+4}_{-4} M_\odot$
Final black hole spin	$0.67^{+0.05}_{-0.07}$
Luminosity distance	410^{+160}_{-180} Mpc
Source redshift z (+ cosmological model)	$0.09^{+0.03}_{-0.04}$

$$d_L \sim 45\ \mathrm{Gpc}\left(\frac{\mathrm{Hz}}{\dot{f}_{GW}|_{max}}\right)\left(\frac{10^{-21}}{h|_{max}}\right)$$

Overview and comparison of all GW observations



Spin ~ 0 : primordial BH?

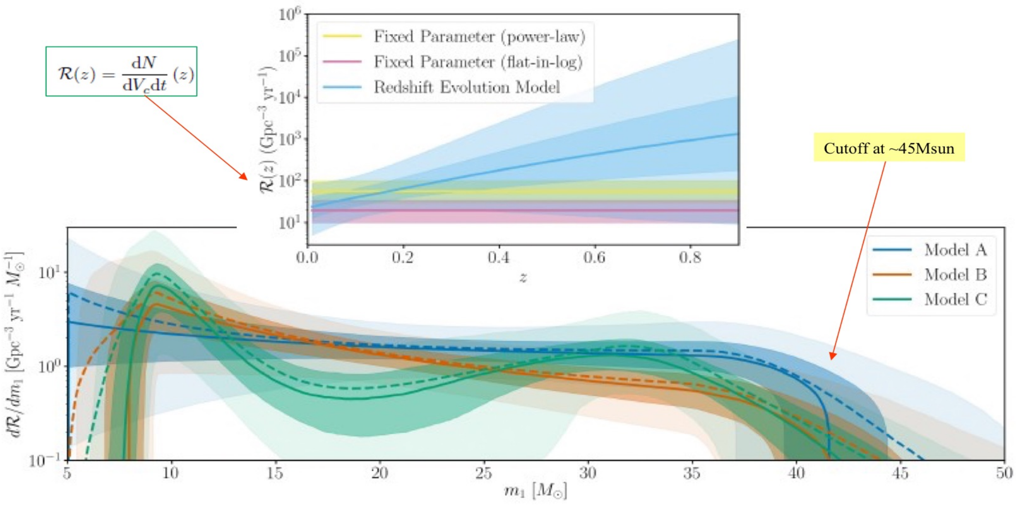
ArXiv: 1811.12907

Event	$m_1/M_\odot$	$m_2/M_\odot$	$\mathcal{M}/M_\odot$	χ_{eff}	$M_f/M_\odot$	a_f	$E_{rad}/(M_\odot c^2)$	$\ell_{peak}/(\mathrm{erg\,s^{-1}})$	d_L/Mpc	z	$\Delta\Omega/\mathrm{deg}^2$
GW150914	$35.6^{+4.8}_{-3.0}$	$30.6^{+3.0}_{-4.4}$	$28.6^{+1.6}_{-1.5}$	$-0.01^{+0.12}_{-0.13}$	$63.1^{+3.3}_{-3.0}$	$0.69^{+0.05}_{-0.04}$	$3.1^{+0.4}_{-0.4}$	$3.6^{+0.4}_{-0.4}\times 10^{56}$	430^{+150}_{-170}	$0.09^{+0.03}_{-0.03}$	179
GW151012	$23.3^{+14.0}_{-5.5}$	$13.6^{+4.1}_{-4.8}$	$15.2^{+2.0}_{-1.1}$	$0.04^{+0.28}_{-0.19}$	$35.7^{+9.9}_{-3.8}$	$0.67^{+0.13}_{-0.11}$	$1.5^{+0.5}_{-0.5}$	$3.2^{+0.8}_{-1.7}\times 10^{56}$	1060^{+540}_{-480}	$0.21^{+0.09}_{-0.09}$	1555
GW151226	$13.7^{+8.8}_{-3.2}$	$7.7^{+2.2}_{-2.6}$	$8.9^{+0.3}_{-0.3}$	$0.18^{+0.20}_{-0.12}$	$20.5^{+6.4}_{-1.5}$	$0.74^{+0.07}_{-0.05}$	$1.0^{+0.1}_{-0.2}$	$3.4^{+0.7}_{-1.7}\times 10^{56}$	440^{+180}_{-190}	$0.09^{+0.04}_{-0.04}$	1033
GW170104	$31.0^{+7.2}_{-5.6}$	$20.1^{+4.9}_{-4.5}$	$21.5^{+2.1}_{-1.7}$	$-0.04^{+0.17}_{-0.20}$	$49.1^{+5.2}_{-3.9}$	$0.66^{+0.08}_{-0.10}$	$2.2^{+0.5}_{-0.5}$	$3.3^{+0.6}_{-0.9}\times 10^{56}$	960^{+430}_{-410}	$0.19^{+0.07}_{-0.08}$	924
GW170608	$10.9^{+5.3}_{-1.7}$	$7.6^{+1.3}_{-2.1}$	$7.9^{+0.2}_{-0.2}$	$0.03^{+0.19}_{-0.07}$	$17.8^{+3.2}_{-0.7}$	$0.69^{+0.04}_{-0.04}$	$0.9^{+0.0}_{-0.1}$	$3.5^{+0.4}_{-1.3}\times 10^{56}$	320^{+120}_{-110}	$0.07^{+0.02}_{-0.02}$	396
GW170729	$50.6^{+16.6}_{-10.2}$	$34.3^{+9.1}_{-10.1}$	$35.7^{+6.5}_{-4.7}$	$0.36^{+0.21}_{-0.25}$	$80.3^{+14.6}_{-10.2}$	$0.81^{+0.07}_{-0.13}$	$4.8^{+1.7}_{-1.7}$	$4.2^{+0.9}_{-1.5}\times 10^{56}$	2750^{+1350}_{-1320}	$0.48^{+0.19}_{-0.20}$	1033
GW170809	$35.2^{+8.3}_{-6.0}$	$23.8^{+5.2}_{-5.1}$	$25.0^{+2.1}_{-1.6}$	$0.07^{+0.16}_{-0.16}$	$56.4^{+5.2}_{-3.7}$	$0.70^{+0.08}_{-0.09}$	$2.7^{+0.6}_{-0.6}$	$3.5^{+0.6}_{-0.9}\times 10^{56}$	990^{+320}_{-380}	$0.20^{+0.05}_{-0.07}$	340
GW170814	$30.7^{+5.7}_{-3.0}$	$25.3^{+2.9}_{-4.1}$	$24.2^{+1.4}_{-1.1}$	$0.07^{+0.12}_{-0.11}$	$53.4^{+3.2}_{-2.4}$	$0.72^{+0.07}_{-0.05}$	$2.7^{+0.4}_{-0.3}$	$3.7^{+0.4}_{-0.5}\times 10^{56}$	580^{+160}_{-210}	$0.12^{+0.03}_{-0.04}$	87
GW170817	$1.46^{+0.12}_{-0.10}$	$1.27^{+0.09}_{-0.09}$	$1.186^{+0.001}_{-0.001}$	$0.00^{+0.02}_{-0.01}$	≤ 2.8	≤ 0.89	≥ 0.04	$\geq 0.1\times 10^{56}$	40^{+10}_{-10}	$0.01^{+0.00}_{-0.00}$	16
GW170818	$35.5^{+7.5}_{-4.7}$	$26.8^{+4.3}_{-5.2}$	$26.7^{+2.1}_{-1.7}$	$-0.09^{+0.18}_{-0.21}$	$59.8^{+4.8}_{-3.8}$	$0.67^{+0.07}_{-0.08}$	$2.7^{+0.5}_{-0.5}$	$3.4^{+0.5}_{-0.7}\times 10^{56}$	1020^{+430}_{-360}	$0.20^{+0.07}_{-0.07}$	39
GW170823	$39.6^{+10.0}_{-6.6}$	$29.4^{+6.3}_{-7.1}$	$29.3^{+4.2}_{-3.2}$	$0.08^{+0.20}_{-0.22}$	$65.6^{+9.4}_{-6.6}$	$0.71^{+0.08}_{-0.10}$	$3.3^{+0.9}_{-0.8}$	$3.6^{+0.6}_{-0.9}\times 10^{56}$	1850^{+840}_{-840}	$0.34^{+0.13}_{-0.14}$	1651

TABLE III. Selected source parameters of the eleven confident detections. We report median values with 90% credible intervals that include statistical errors, and systematic errors from averaging the results of two waveform models for BBHs. For GW170817 credible intervals and statistical errors are shown for IMRPhenomPv2NRT with low spin prior, while the sky area was computed from TaylorF2 samples. The redshift for NGC 4993 from [87] and its associated uncertainties were used to calculate source frame masses for GW170817. For BBH events the redshift was calculated from the luminosity distance and assumed cosmology as discussed in Appendix B. The columns show source frame component masses m_i and chirp mass $\mathcal{M}$, dimensionless effective aligned spin χ_{eff}, final source frame mass M_f, final spin a_f, radiated energy E_{rad}, peak luminosity l_{peak}, luminosity distance d_L, redshift z and sky localization $\Delta\Omega$. The sky localization is the area of the 90% credible region. For GW170817 we give conservative bounds on parameters of the final remnant discussed in Sec. V E.

Merger rate of events (up to Dec 2018) as a function of redshift and mass.

ArXiv: 1811.12940



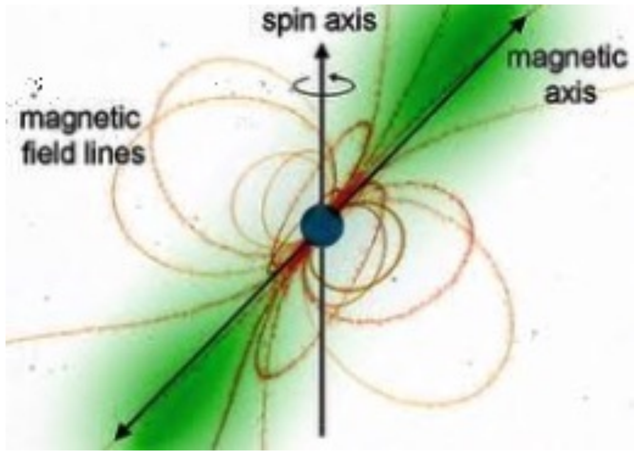
NS - NS binary

(there is an optical counterpart)

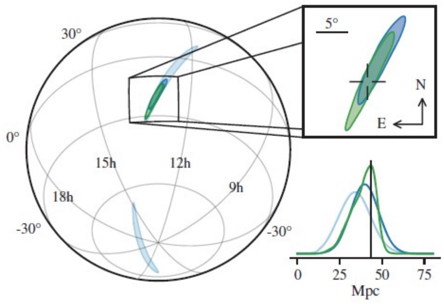
Neutron stars are collapsed stars, supported by neutron degeneracy pressure. $M < 1.4 M_\odot$

Usually emit radiation in pulses (pulsars)

LIGO saw event GW170817 linked to GRB170817A, detected by Fermi.



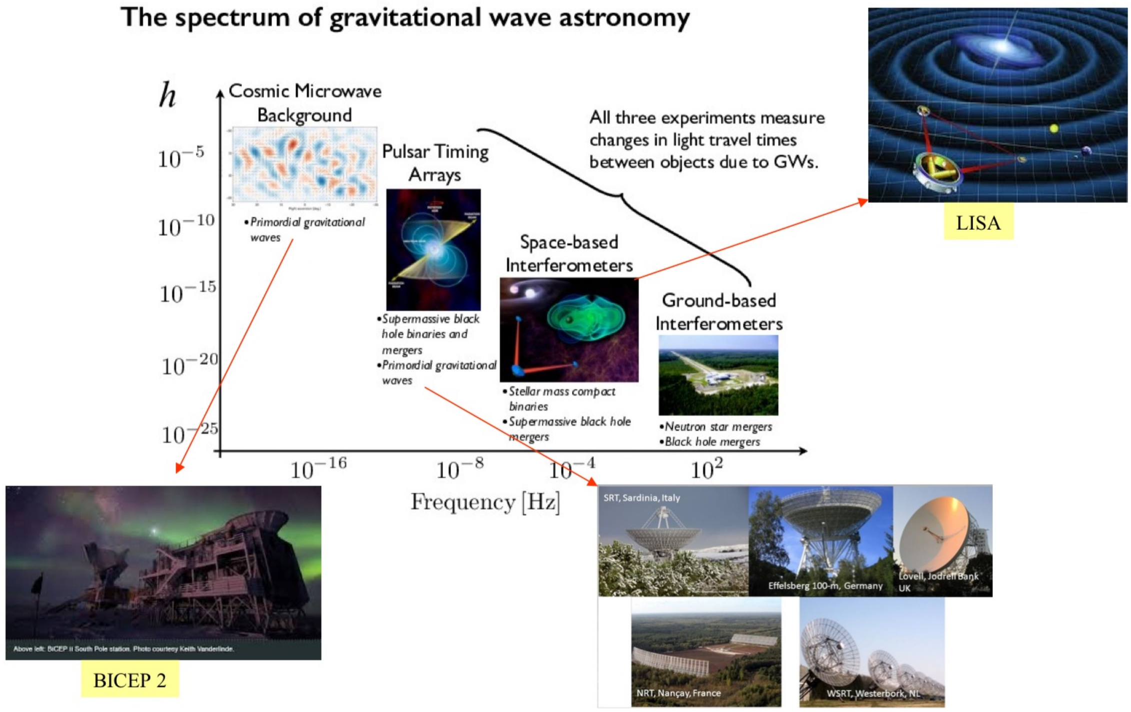
Detected by two LIGOs and Virgo → triangulation



Spin of objects is important in this case

	Low-spin priors ($\lvert\chi\rvert \leq 0.05$)	High-spin priors ($\lvert\chi\rvert \leq 0.89$)
Primary mass m_1	1.36–$1.60\ M_\odot$	1.36–$2.26\ M_\odot$
Secondary mass m_2	1.17–$1.36\ M_\odot$	0.86–$1.36\ M_\odot$
Chirp mass $\mathcal{M}$	$1.188^{+0.004}_{-0.002} M_\odot$	$1.188^{+0.004}_{-0.002} M_\odot$
Mass ratio m_2/m_1	0.7–1.0	0.4–1.0
Total mass $m_{\rm tot}$	$2.74^{+0.04}_{-0.01} M_\odot$	$2.82^{+0.47}_{-0.09} M_\odot$
Radiated energy $E_{\rm rad}$	$> 0.025 M_\odot c^2$	$> 0.025 M_\odot c^2$
Luminosity distance $D_{\rm L}$	40^{+8}_{-14} Mpc	40^{+8}_{-14} Mpc
Viewing angle Θ	$\leq 55°$	$\leq 56°$
Using NGC 4993 location	$\leq 28°$	$\leq 28°$
Combined dimensionless tidal deformability $\tilde{\Lambda}$	≤ 800	≤ 700
Dimensionless tidal deformability $\Lambda(1.4 M_\odot)$	≤ 800	≤ 1400

Other GW experimets/detectors

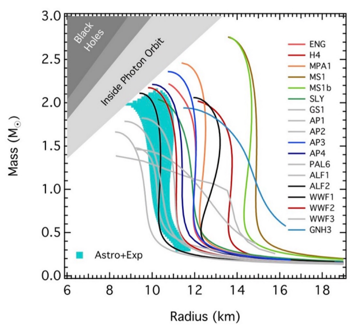


12.5. Other issues

Speed of GW

GRB 170817A was observed ~1.7s after GW170817, which provides constraints on the speed of GW and modifications of gravity:

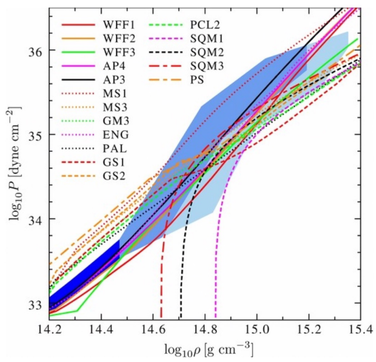
$$\left.\begin{aligned} -3\times10^{-15} \le c_g/c - 1 \le 7\times10^{-16} \\ c_g^2 = 1 + \alpha_T \end{aligned}\right\} \quad \ddot{h}_{ij} + (3+\alpha_M) H \dot{h}_{ij} + (1+\cancel{\alpha_T}) k^2 h_{ij} = 0$$



Optical counterpart → redshift → cosmological constraints

$$H_0 = 70^{+12}_{-8} \ \mathrm{Km/s/Mpc}$$

Possible constraints on the equation of state of neutron stars



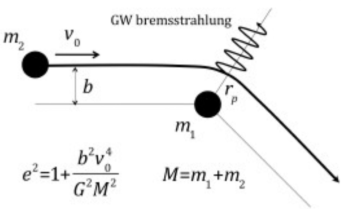
Hyperbolic encounters

Primordial black holes may scatter in clusters (a.k.a. hyperbolic encounters)

$$r(\varphi) = \frac{b\sin\varphi_0}{\cos(\varphi-\varphi_0)-\cos\varphi_0} = \frac{a(e^2-1)}{1+e\cos(\varphi-\varphi_0)}$$

$$\varphi_0 = \arccos\left(-\frac{1}{e}\right)$$

$$r_{\min} = a(e-1) = b\sqrt{\frac{e-1}{e+1}} > R_s \equiv \frac{2GM}{c^2}$$



The amplitude and the power emited are given by:

$$Q_{ij} = \mu\, r^2(\varphi) \begin{pmatrix} 3\cos^2\varphi - 1 & 3\cos\varphi\sin\varphi & 0 \\ 3\cos\varphi\sin\varphi & 3\sin^2\varphi - 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$P = \frac{dE}{dt} = -\frac{G}{45c^5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle = \frac{32G\mu^2 v_0^6}{45c^5\, b^2}\, f(\varphi,\, e)$$

$$f(\varphi,\, e) \;=\; \frac{3\,(1+e\,\cos(\varphi-\varphi_0))^4}{8(e^2-1)^4}\left[24+13\,e^2\,+48\,e\,\cos(\varphi-\varphi_0)+11\,e^2\cos 2(\varphi-\varphi_0)\right]$$

$$h_c = \frac{2G}{R\,c^4}\langle \ddot{Q}_{ij}\ddot{Q}^{ij}\rangle^{1/2}_{i,j=1,2} = \frac{2G\mu\, v_0^2}{R\,c^4}\, g(\varphi,\, e)$$

$$g(\varphi,\, e) \;=\; \frac{\sqrt{2}}{e^2-1}\left[36+59\,e^2+10\,e^4\,+(108+47\,e^2)\,e\,\cos(\varphi-\varphi_0)\,+59\,e^2\,\cos 2(\varphi-\varphi_0)+9\,e^3\cos 3(\varphi-\varphi_0)\right]^{1/2}$$

Frecuency domain and power spectrum:

$$\Delta E = \int_{-\infty}^{\infty} P(t)\,dt = \frac{1}{\pi}\int_0^{\infty} P(\omega)\,d\omega = -\frac{8}{15}\,\frac{G^{7/2}\,M^{1/2}\,m_1^2 m_2^2}{c^5\quad r_{min}^{7/2}}\, f(e)$$

$$P(\omega) = \frac{G}{45c^5}\sum_{ij}|\widehat{\dddot{Q}_{ij}}|^2 = \frac{G}{45c^5}\,\omega^6\sum_{i,j}|\widehat{Q_{ij}}|^2$$

The quadrupole tensor is given by:

$$Q_{ij} = \frac{1}{2}a^2\mu\begin{pmatrix} (3-e^2)\cosh 2\xi - 8e\cosh\xi & 3\sqrt{e^2-1}(2e\sinh\xi-\sinh 2\xi) & 0 \\ 3\sqrt{e^2-1}(2e\sinh\xi-\sinh 2\xi) & (2e^2-3)\cosh 2\xi + 4e\cosh\xi & 0 \\ 0 & 0 & 4e\cosh\xi - e^2\cosh 2\xi \end{pmatrix}$$

$$t(\xi) \;=\; \nu_0(e\sinh\xi-\xi)\,,$$
$$r(\xi) \;=\; a(e\cosh\xi-1)\,.$$
$$\nu_0 = \sqrt{a^3/GM}.$$

The power spectrum

$$P(\omega) \;=\; \frac{G^3\mu^2M^2}{a^2c^5}\left(\frac{\pi^2}{180}\,\nu^4\sum_{i,j}|\widehat{C_{ij}}|^2\right)$$
$$\;=\; \frac{G^3\mu^2M^2}{a^2c^5}\,\frac{16\pi^2}{180}\,\nu^4 F_e(\nu),$$

$$\begin{aligned} F_e(\nu) \;=\;& \left|\frac{3(e^2-1)}{e}\,H^{(1)\prime}_{i\nu}(i\nu e)+\frac{e^2-3}{e^2}\,\frac{i}{\nu}\,H^{(1)}_{i\nu}(i\nu e)\right|^2 \\ &+\left|\frac{3(e^2-1)}{e}\,H^{(1)\prime}_{i\nu}(i\nu e)+\frac{2e^2-3}{e^2}\,\frac{i}{\nu}\,H^{(1)}_{i\nu}(i\nu e)\right|^2 \\ &+\left|\frac{i}{\nu}\,H^{(1)}_{i\nu}(i\nu e)\right|^2+\frac{18\,(e^2-1)}{e^2}\,\times \\ &\times\left|\frac{(e^2-1)}{e}\,i\,H^{(1)}_{i\nu}(i\nu e)+\frac{1}{\nu}\,H^{(1)\prime}_{i\nu}(i\nu e)\right|^2 \end{aligned}$$

Hankel function

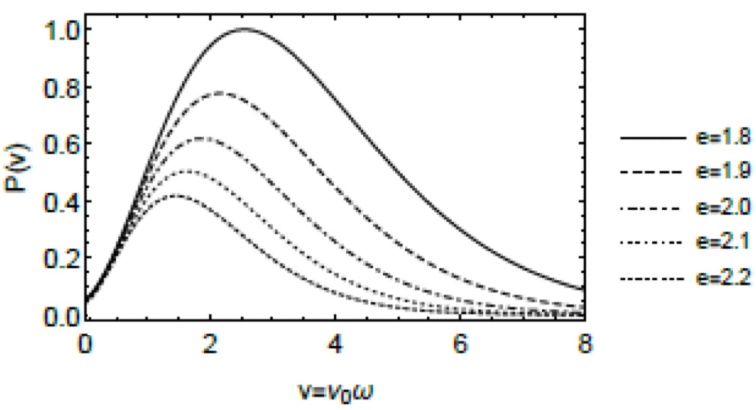
Total power and peak frequency

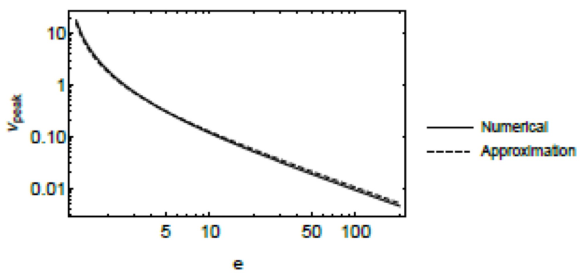
$$\Delta E \;=\; \int_{-\infty}^{+\infty} P(t)dt = \int_0^{+\infty}\frac{P(\omega)}{\pi}d\omega$$
$$\;=\; \left(\frac{G^{7/2}\mu^2M^{5/2}}{c^5a^{7/2}}\right)\frac{16\,\pi}{180}\int_0^{+\infty}\nu^4F_e(\nu)d\nu$$

$$\nu_{\max}(e) = \sqrt{\frac{e+1}{(e-1)^3}}\,,\quad \omega_{\max}(e) = \frac{v_0}{b}\left(\frac{e+1}{e-1}\right)$$

$$\nu^4F_e(\nu) \;\simeq\; \frac{12\,F_y(\nu)}{\pi\,y\,(y^2+1)^2}\,e^{-2\nu z(y)}\,,$$
$$F_y(\nu) \;=\; \nu\,(1-y^2-3\nu y^3+4y^4+9\nu y^5+6\nu^2y^6)$$
$$z(y) \;=\; y-\arctan y\,,\qquad y\equiv\sqrt{e^2-1}$$

The peak frequency is important, since it is detectable by LIGO





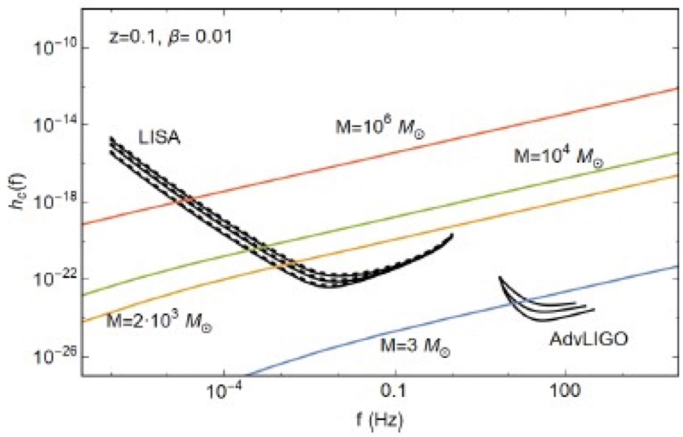
GW memory effect: After scattering $(\omega \to 0)$ spacetime remembers event

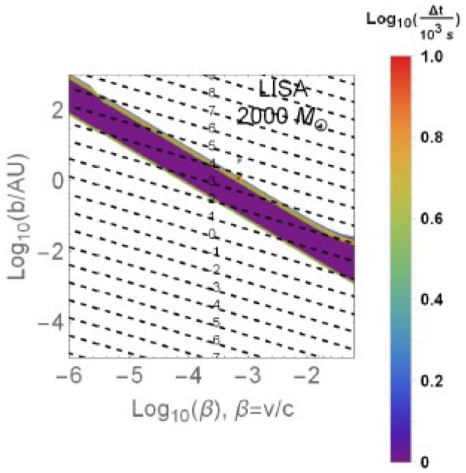
$$P(\omega=0) = \frac{G^3 \mu^2 M^2}{a^2 c^5} \frac{32(e^2-1)}{5e^4}$$

Possibility of detection by LiSA-LiGO

LiSA and LiGO are sensitive in specific frequenies - stains

These are Known as sensitivity curves.





PBH by hyperbolic encounters gives unique predictions for strain + frequency. There is also a unique stain for detector.

The scattering will be seen as a unique event (not periodic even, like in the binaries), aka a glitch.