IX. Large Scale Structure formation

Introduction

One of the goals of Cosmology is explaining how the structures that we observe were formed. We already know about the origin of matter perturbations and that dark matter was forming structures before matter decoupling, but we need to understand how this matter perturbations grew and why did they formed inhomogeneous structures. (Note: that when we talk about the homogeneity of the Universe, we refer to a certain scale. Below that scale, we have other structures). Throughout the whole lecture we only consider matter perturbations well inside the Hubble radius (i.e., the horizon).

9.1. Governing equations.

Basic newtonian equations

We start with a non-cosmological setting, taking a newtonian approach to find how matter evolves with time.

- Poisson's equation:

$$\Delta \Psi = 4\pi G\left(\rho + \frac{3p}{c^2}\right) \quad \rightarrow \text{self gravity}$$

- Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\left(\rho+\frac{p}{c^2}\right)\vec{v}\right) = 0 \quad \rightarrow \text{mass conservation}$$

- Conservation of momentum:

$$\left(\rho+\frac{p}{c^2}\right)\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\left(\rho+\frac{p}{c^2}\right)\nabla\Psi - \nabla p$$

- Equation of state

$$p = p(\rho, S) \quad \longrightarrow \text{covers D.M. and baryonic matter}$$

Cosmological setting

We can change this equations considering a non relativistic fluid in an expanding Universe characterized by small perturbations about a homogeneous and isotropic background.

Non relativistic fluids

For non-relativistic fluids, we have $p \ll \rho c^2$. Thus, P can be discarded when it is compared to the energy density. However, we cannot discard ∇P because it can be large. Baryonic matter fulfills $p \ll \rho c^2$ too. We further assume adiabatic perturbations, whose equation of state will be given by: $\nabla p = c_s^2 \nabla \rho$

We obtain:

- Poisson's equation:

$$\Delta \Psi = 4\pi G\left(\rho + \frac{3p}{c^2}\right) \longrightarrow \Delta \Psi = 4\pi G \rho$$

- Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\left(\rho + \frac{p}{c^2}\right)\vec{v}\right) = 0 \longrightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

- Conservation of momentum:

$$\left(\rho + \frac{p}{c^2}\right)\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\left(\rho + \frac{p}{c^2}\right)\nabla\Psi - \nabla p \longrightarrow \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla\Psi - \frac{\nabla p}{\rho}$$

- Equation of state

$$p = p(\rho, S) \longrightarrow \nabla p = c_s^2 \nabla \rho$$

Expanding Universe

We can change to comoving coordinates to consider the expansion of the Universe.

REMINDER
$\vec{r} = a\vec{x}$
$\vec{x} \equiv$ comoving coordinate

Transforming the velocity:

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{u} + \frac{\dot{a}}{a}\vec{r}, \quad \text{with} \quad \boxed{\vec{u} = a\dot{\vec{x}}}$$

Peculiar velocity field

Dem:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(a\vec{x}) = a\dot{\vec{x}} + \dot{a}\vec{x} = a\dot{\vec{x}} + \dot{a}\left(\frac{\vec{r}}{a}\right)$$

- The peculiar velocity $\vec{u}$ is associated to the motion of matter within an expanding Universe
- $\frac{\dot{a}}{a}\vec{r}$ is the Hubble drag term, related to the expansion. We can handle this expansion with the Friedmann equations.

The peculiar potential Φ can be defined as:

$$\boxed{\Psi = \Phi - \frac{1}{2}a\ddot{a}x^2}$$

As it will be seen later, Φ is sourced by matter perturbations. The second term of the potential is related to the expansion.

It is also necessary to change the operators. We find that:

$$\nabla = \frac{1}{a}\nabla_x$$

"Proof":

$$\frac{\partial}{\partial r} \longrightarrow \frac{1}{a}\frac{\partial}{\partial x}$$

It is harder to change the time derivative. The expression for the convective time derivative is given by:

$$\left.\frac{\partial}{\partial t}\right|_{\vec{r}} = \left.\frac{\partial}{\partial t}\right|_{\vec{x}} - \frac{\dot{a}}{a}(\vec{x}\cdot\nabla_x)$$

Proof: Calculating the total derivatives of df and dg:

1. $$df = \frac{\partial f(\vec r,t)}{\partial r}\Big|_t \cdot d\vec r + \frac{\partial f(\vec r,t)}{\partial t}\Big|_{\vec r} dt$$

$d\vec r = a\,d\vec x + \vec x\,da$
$$= \nabla f(\vec r,t)\cdot(a\,d\vec x + \vec x\,da) + \frac{\partial f(\vec r,t)}{\partial t}\Big|_{\vec r} dt$$

$da = \dot a\,dt$
$$= \left[a\nabla f(\vec r,t)\right]\cdot d\vec x + \left[\dot a\vec x\cdot(\nabla f(\vec r,t)) + \frac{\partial f(\vec r,t)}{\partial t}\Big|_{\vec r}\right] dt$$

2. $$dg = \frac{\partial g(\vec x,t)}{\partial x}\Big|_t \cdot d\vec x + \frac{\partial g(\vec x,t)}{\partial t}\Big|_{\vec x} dt$$
$$= \nabla_x g(\vec x,t)\cdot d\vec x + \frac{\partial g(\vec x,t)}{\partial t}\Big|_{\vec x} dt$$

We must have $df = dg$

3. $df = dg$
$$\underline{\left[a\nabla f(\vec r,t)\right]}\cdot d\vec x + \left[\dot a\vec x\cdot(\nabla f(\vec r,t)) + \frac{\partial f(\vec r,t)}{\partial t}\Big|_{\vec r}\right] dt = \underline{\nabla_x g(\vec x,t)}\cdot d\vec x + \frac{\partial g(\vec x,t)}{\partial t}\Big|_{\vec x} dt$$

Once we have these ingredients, we can rewrite the equations using the comoving coordinates.

- **Poisson's equation**:

Associated to perturbations, not background expansion

$$\Delta\Psi = 4\pi G\rho \longrightarrow \Delta_x\Phi = 4\pi G a^2(\rho - \bar\rho)$$

$\bar\rho$: background density

$\rho \neq cte$, will be written as a density contrast

Proof:

$$\Psi = \Phi - \frac{1}{2}a\ddot a x^2 \qquad \Delta_x\Phi = \Delta_x\left(\Psi + \frac{1}{2}a\ddot a|\vec x|^2\right)$$
$$= \Delta_x\Psi + \frac{1}{2}a\ddot a\Delta_x|\vec x|^2$$

$\Delta = \frac{1}{a^2}\Delta_x$ $\qquad \Delta_x|\vec x|^2 = 6$
$$= a^2 4\pi G\rho + \frac{1}{2}a\ddot a 6$$

2nd Friedmann equation: $\ddot a = -\frac{4\pi G}{3}\bar\rho a$
$$= 4\pi G a^2\rho + 3a\left(-\frac{4\pi G}{3}\bar\rho a\right)$$
$$= 4\pi G a^2(\rho - \bar\rho)$$ q.e.d.

NOTE

- Friedmann equations are for the "background" $\bar\rho$
- The comoving potential Φ is responsible for the growth of perturbations
- There is no solution for Poisson's equation in infinite space unless the source function averages to zero
- The inclusion of a Λ-term will not change the result (it would be compensated by the appearance in the 2nd Friedmann equation

- **[Comoving] continuity equation**

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec v) = 0 \longrightarrow \frac{\partial\rho}{\partial t} + \frac{1}{a}\nabla_x\cdot(\rho\vec u) + 3\frac{\dot a}{a}\rho = 0$$

NOTE:

- It contains an additional drag term due to the cosmic expansion.

Proof:

$$\underbrace{\frac{\partial\rho}{\partial t}}_{A}+\underbrace{\nabla\cdot(\rho\vec{v})}_{B}=0$$

A

$$\left.\frac{\partial\rho(\vec{r},t)}{\partial t}\right|_{\vec{r}}=\left.\frac{\partial\rho(\vec{x},t)}{\partial t}\right|_{\vec{x}}-\frac{\dot{a}}{a}\vec{x}\cdot[\nabla_x\rho(\vec{x},t)]$$

B

$$\vec{v}=\vec{u}+\frac{\dot{a}}{a}\vec{r} \quad \text{with} \quad \vec{u}=a\dot{\vec{x}}$$

$$\begin{aligned}\nabla\cdot(\rho\vec{v})&=\rho[\nabla\cdot\vec{v}]+\vec{v}[\nabla\rho]\\&=\rho[\nabla\cdot(\frac{\dot{a}}{a}\vec{r}+\vec{u})]+(\frac{\dot{a}}{a}\vec{r}+\vec{u})\cdot[\nabla\rho]\\&=\rho[\frac{\dot{a}}{a}3+\nabla\cdot\vec{u}]+\frac{\dot{a}}{a}\vec{r}\cdot\nabla\rho+\vec{u}\cdot\nabla\rho\\&=\frac{\dot{a}}{a}3\rho+\frac{1}{a}\rho\nabla_x\cdot\vec{u}+\frac{\dot{a}}{a}\vec{x}\nabla_x\rho+\frac{1}{a}\vec{u}\cdot\nabla_x\rho\\&=\frac{\dot{a}}{a}(3\rho+\vec{x}\cdot\nabla_x\rho)+\frac{1}{a}\nabla_x\cdot(\rho\vec{u})\end{aligned}$$

$$\frac{\partial\rho}{\partial t}+\frac{1}{a}\nabla_x\cdot(\rho\vec{u})+3\frac{\dot{a}}{a}\rho=0$$

q.e.d.

- Comoving conservation of momentum

$$\left(\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right)=-\nabla\Psi-\frac{\nabla p}{\rho}\longrightarrow\frac{\partial\vec{u}}{\partial t}+\frac{1}{a}(\vec{u}\cdot\nabla_x)\vec{u}+\frac{\dot{a}}{a}\vec{u}=-\frac{1}{a}\nabla_x\Phi-\frac{1}{a}\frac{\nabla_x p}{\rho}$$

Proof:

$$\underbrace{\frac{\partial\vec{v}}{\partial t}}_{A}+\underbrace{(\vec{v}\cdot\nabla)\vec{v}}_{B}=-\underbrace{\nabla\Psi}_{C}-\underbrace{\frac{\nabla p}{\rho}}_{D}$$

A

$$\begin{aligned}\frac{\partial\vec{v}}{\partial t}&=\frac{\ddot{a}}{a}\vec{r}-\frac{\dot{a}^2}{a^2}\vec{r}+\left.\frac{\partial\vec{u}}{\partial t}\right|_{\vec{x}}-\frac{\dot{a}}{a}(\vec{x}\cdot\nabla_x)\vec{u}\\&=\ddot{a}\vec{x}-\frac{\dot{a}^2}{a}\vec{x}+\left.\frac{\partial\vec{u}}{\partial t}\right|_{\vec{x}}-\frac{\dot{a}}{a}(\vec{x}\cdot\nabla_x)\vec{u}\end{aligned}$$

C

$$\begin{aligned}\nabla\Psi&=\frac{1}{a}\nabla_x(\Phi-\frac{1}{2}a\ddot{a}\vec{x}^2)\\&=\frac{1}{a}\nabla_x\Phi-\frac{1}{2}\ddot{a}\nabla_x\vec{x}^2\\&=\frac{1}{a}\nabla_x\Phi-\ddot{a}\vec{x}\end{aligned}$$

D

$$\nabla p=\frac{1}{a}\nabla_x p$$

B

$$(\vec{r}\cdot\nabla)\vec{r}=\vec{r} \qquad (\vec{u}\cdot\nabla)\vec{r}=\vec{u}$$

$$\begin{aligned}(\vec{v}\cdot\nabla)\vec{v}&=[(\frac{\dot{a}}{a}\vec{r}+\vec{u})\cdot\nabla](\frac{\dot{a}}{a}\vec{r}+\vec{u})\\&=(\frac{\dot{a}}{a}\vec{r}\cdot\nabla+\vec{u}\cdot\nabla)(\frac{\dot{a}}{a}\vec{r}+\vec{u})\\&=\frac{\dot{a}}{a}(\vec{r}\cdot\nabla)(\frac{\dot{a}}{a}\vec{r}+\vec{u})+(\vec{u}\cdot\nabla)(\frac{\dot{a}}{a}\vec{r}+\vec{u})\\&=\frac{\dot{a}^2}{a^2}\vec{r}+\frac{\dot{a}}{a}(\vec{r}\cdot\nabla)\vec{u}+\frac{\dot{a}}{a}(\vec{u}\cdot\nabla)\vec{r}+(\vec{u}\cdot\nabla)\vec{u}\\&=\frac{\dot{a}^2}{a^2}\vec{r}+\frac{\dot{a}}{a}(\vec{r}\cdot\nabla)\vec{u}+\frac{\dot{a}}{a}\vec{u}+(\vec{u}\cdot\nabla)\vec{u}\\&=\frac{\dot{a}^2}{a}\vec{x}+\frac{\dot{a}}{a}(\vec{x}\cdot\nabla_x)\vec{u}+\frac{\dot{a}}{a}\vec{u}+(\vec{u}\cdot\nabla)\vec{u}\end{aligned}$$

A+B=-C-D

$\Rightarrow$ q.e.d.

NOTE

- It also contains a drag term due to the cosmic expansion: $\frac{\dot{a}}{a}\vec{u}$

Wrapping all the equations together:

- Poisson's equation: $\Delta_x\Phi=4\pi G a^2(\rho-\bar{\rho})$
- [Comoving] continuity equation: $\frac{\partial\rho}{\partial t}+\frac{1}{a}\nabla_x\cdot(\rho\vec{u})+3\frac{\dot{a}}{a}\rho=0$
- Comoving conservation of momentum: $\frac{\partial\vec{u}}{\partial t}+\frac{1}{a}(\vec{u}\cdot\nabla_x)\vec{u}+\frac{\dot{a}}{a}\vec{u}=-\frac{1}{a}\nabla_x\Phi-\frac{1}{a}\frac{\nabla_x p}{\rho}$
- Equation of state: $\nabla p=c_s^2\nabla\rho$

NOTE:

It is possible to define a set of coordinates known as "super comoving coordinates" that leave invariant the newtonian fluid equations. Thus, it is possible to use non-cosmological hydrodynamical codes and "translate" the result.

Small perturbations

We want to solve these equations for small perturbations about a homogeneous and isotropic background. These perturbations are the source of the potential. We can rewrite all the equations in terms of the density contrast:

$$\boxed{\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}} \longrightarrow \rho = \bar{\rho}(1+\delta)$$

- Poisson's equation:

$$\Delta_x \Phi - 4\pi G a^2 (\rho - \bar{\rho}) \xrightarrow{\rho - \bar{\rho} = \bar{\rho}\delta} \Delta_x \Phi = 4\pi G a^2 \bar{\rho}\delta$$

- Continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{a}\nabla_x \cdot (\rho \vec{u}) + 3\frac{\dot{a}}{a}\rho = 0 \longrightarrow \frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot [(1+\delta)\vec{u}] = 0$$

$$\underbrace{\frac{\partial \rho}{\partial t}}_{A} + \underbrace{\frac{1}{a}\nabla_x \cdot (\rho \vec{u})}_{B} + \underbrace{3\frac{\dot{a}}{a}\rho}_{C} = 0$$

A

$$\frac{\partial \rho}{\partial t} = \frac{\partial(\bar{\rho}(1+\delta))}{\partial t}$$
$$= \dot{\bar{\rho}}(1+\delta) + \bar{\rho}\dot{\delta}$$
$$= -3\frac{\dot{a}}{a}\bar{\rho}(1+\delta) + \bar{\rho}\dot{\delta}$$

$\bar{\rho}a^3 = const.$

we are interested in matter!
(cf. FRW lecture)

B

$$\frac{1}{a}\nabla \cdot (\rho \vec{u}) = \frac{1}{a}\bar{\rho}\nabla \cdot [(1+\delta)\vec{u}]$$

C

$$3\frac{\dot{a}}{a}\rho = 3\frac{\dot{a}}{a}\bar{\rho}(1+\delta)$$

NOTATION

From now on:

$\nabla_x \rightarrow \nabla$

- Conservation of momentum

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{a}(\vec{u}\cdot\nabla_x)\vec{u} + \frac{\dot{a}}{a}\vec{u} = -\frac{1}{a}\nabla_x \Phi - \frac{1}{a}\frac{\nabla_x p}{\rho} \longrightarrow \frac{\partial \vec{u}}{\partial t} + \frac{1}{a}(\vec{u}\cdot\nabla)\vec{u} + \frac{\dot{a}}{a}u = -\frac{1}{a}\nabla\Phi - \frac{c_s^2}{a}\frac{\nabla\delta}{1+\delta}$$

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{a}(\vec{u}\cdot\nabla_x)\vec{u} + \frac{\dot{a}}{a}\vec{u} = -\frac{1}{a}\nabla_x\Phi - \underbrace{\frac{1}{a}\frac{\nabla_x p}{\rho}}_{A}$$

A

$$\frac{\nabla p}{\rho} = \frac{c_s^2 \nabla \rho}{\rho} = \frac{c_s^2 \nabla(\bar{\rho}(1+\delta))}{\rho} = \frac{c_s^2[(1+\delta)\nabla\bar{\rho} + \bar{\rho}\nabla(1+\delta)]}{\rho} = \frac{c_s^2 \bar{\rho}\nabla(1+\delta)}{\rho} = \frac{c_s^2\bar{\rho}\nabla\delta}{\rho} = \frac{c_s^2\nabla\delta}{1+\delta}$$

adiabatic perturbations

$\nabla p = c_s^2 \nabla \rho$

Then:

- Poisson's equation: $\Delta_x \Phi = 4\pi G a^2 \bar{\rho}\delta$
- Continuity equation: $\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot [(1+\delta)\vec{u}] = 0$
- Conservation of momentum: $\frac{\partial \vec{u}}{\partial t} + \frac{1}{a}(\vec{u}\cdot\nabla)\vec{u} + \frac{\dot{a}}{a}u = -\frac{1}{a}\nabla\Phi - \frac{c_s^2}{a}\frac{\nabla\delta}{1+\delta}$
- Adiabatic perturbations $\nabla p = c_s^2 \nabla \rho$

Linear perturbation theory

Since we are considering small perturbations, we can take: $\delta << 1$. We can also consider $\frac{(\vec{u}\cdot\nabla)}{a} << H = \frac{\dot{a}}{a}$: the velocity of the perturbations is much smaller than the expansion of the Universe. We can discard some of the terms of the previous equations:

- Poisson's equatio : $\Delta_x \Phi = 4\pi G a^2 \bar{\rho}\delta$
- Continuity equation: $\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla\cdot[(1+\delta)\vec{u}] = 0$
- Conservation of momentum : $\frac{\partial\vec{u}}{\partial t} + \frac{1}{a}(\vec{u}\cdot\nabla)\vec{u} + \frac{\dot{a}}{a}u = -\frac{1}{a}\nabla\Phi - \frac{c_s^2}{a}\frac{\nabla\delta}{1+\delta}$
- Adiabatic perturbations $\nabla p = c_s^2 \nabla \rho$

We are left with:

- Poisson's equation: $\Delta_x \Phi = 4\pi G a^2 \bar{\rho}\delta$
- Continuity equation: $\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla\cdot\vec{u} = 0$
- Conservation of momentum : $\frac{\partial\vec{u}}{\partial t} + \frac{\dot{a}}{a}u = -\frac{1}{a}\nabla\Phi - \frac{c_s^2}{a}\nabla\delta$
- Adiabatic perturbations $\nabla p = c_s^2 \nabla \rho$

NOTE

$\bar{\rho}$ is the background density, and so depends on all the components of the Universe:

$$\bar{\rho}\delta = \bar{\rho}_{tot}\left(\frac{\bar{\rho}}{\bar{\rho}_{tot}}\delta + \frac{\bar{\rho}_X}{\bar{\rho}_{tot}}\delta_X + \frac{\bar{\rho}_Y}{\bar{\rho}_{tot}}\delta_Y + \dots\right) \quad \text{Source term}$$

The rest of the equations remain for the decoupled component of interest, the only quantity common to all possible components is the gravitational potencial.

We can combine some of the previous equations to eliminate u, ∇p and Φ. Taking the continuity equation and the momentum equation:

$$\left.\begin{aligned} A &= \frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla\cdot\vec{u} = 0 \\ B &= \frac{\partial\vec{u}}{\partial t} + \frac{\dot{a}}{a}u = -\frac{1}{a}\nabla\Phi - \frac{c_s^2}{a}\nabla\delta \end{aligned}\right\} \quad 0 = \frac{\partial A}{\partial t} - \frac{1}{a}\nabla\cdot B$$

Proof:

$$\begin{aligned}\frac{\partial A}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla\cdot\vec{u}\right) \\ &= \frac{\partial^2\delta}{\partial t^2} - \frac{\dot{a}}{a^2}\nabla\cdot\vec{u} + \frac{1}{a}\frac{\partial}{\partial t}\nabla\cdot\vec{u} \\ &= \frac{\partial^2\delta}{\partial t^2} + \frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} + \frac{1}{a}\frac{\partial}{\partial t}\nabla\cdot\vec{u}\end{aligned}$$

continuity equation

$$\begin{aligned}\frac{1}{a}\nabla\cdot B &= \frac{1}{a}\nabla\cdot\left(\frac{\partial\vec{u}}{\partial t} + \frac{\dot{a}}{a}\vec{u} + \frac{1}{a}\nabla\Phi + \frac{c_s^2}{a}\nabla\delta\right) \\ &= \frac{1}{a}\left(\frac{\partial}{\partial t}\nabla\cdot\vec{u} + \frac{\dot{a}}{a}\nabla\cdot\vec{u} + \frac{1}{a}\Delta\Phi + \frac{c_s^2}{a}\Delta\delta\right) \\ &= \frac{1}{a}\left(\frac{\partial}{\partial t}\nabla\cdot\vec{u} - \dot{a}\frac{\partial\delta}{\partial t} + \frac{1}{a}\Delta\Phi + \frac{c_s^2}{a}\Delta\delta\right)\end{aligned}$$

We obtain the equation that governs the evolution of the density contrast $\delta(x,t)$

$$\boxed{\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - 4\pi G\bar{\rho}\delta - \frac{c_s^2}{a^2}\Delta\delta = 0}$$

This equation is linear, and can be decomposed in different sine functions.

It is valid for arbitrary cosmologies, as well as for collisionless ($c_s = 0$) and collisional matter (coupled to itself, $c_s \neq 0$).

Cosmological expansion acts as a damping term.

Remember that we are describing matter perturbations well inside the Hubble radius.

If there are additional (grav. interacting) components, they only enter into the $4\pi G$- term (for example, radiation coupled gravitationally).

9.2. Growth of matter perturbations

Jeans limit

To study the evolution of the density contrast we need to solve:

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - 4\pi G\bar{\rho}\delta - \frac{c_s^2}{a^2}\Delta\delta = 0$$

We take a decomposition in plane waves as an ansatz for the solution:

$$\boxed{\delta(\vec{x},t) = \sum_k \delta_k(t)\, e^{i\vec{k}\cdot\vec{x}}} \leftarrow \text{Fourier space}$$

Doing this, we are separating space and time. Each wave is characterized by its wave number k.

Taking $\Delta\delta = -k^2\delta$, we can rewrite the evolution equation as:

$$\frac{\partial^2\delta_k}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta_k}{\partial t} + \left(\frac{c_s^2}{a^2}k^2 - 4\pi G\bar{\rho}\right)\delta_k = 0$$

where δ_k is the amplitude in k-space. This is the equation of a damped harmonic oscillator:

$$m\frac{d^2x(t)}{dt^2} + c\frac{dx(t)}{dt} + kx(t) = 0$$

It is necessary to solve the evolution equation for every wave (as characterized by its k) individually.

The term $\left(\frac{c_s^2}{a^2}k^2 - 4\pi G\bar{\rho}\right)$ reflects the balance between pressure support and gravity (note that we have allowed for baryonic/pressure gradients). We can cast this into a condition for the wavenumber k in terms of the background density.

- $\frac{K^2}{a^2} < \frac{4\pi G\bar{\rho}}{c_s^2} \Rightarrow (\ \ ) < 0 \longrightarrow$ gravitational collapse
- $\frac{K^2}{a^2} > \frac{4\pi G\bar{\rho}}{c_s^2} \Rightarrow (\ \ ) > 0 \longrightarrow$ oscillations (with decreasing amplitude due to damping term)

We define the Jeans limit as:

$$\boxed{\lambda_J = c_s\sqrt{\frac{\pi}{G\bar{\rho}}} \qquad M_{J,w} = \frac{4\pi}{3}\left(\frac{\lambda_J}{2}\right)^3 \bar{\rho}_w}$$

REMINDER

$K = \frac{2\pi a}{\lambda}$

where λ_J is the Jeans length and M_J is the Jeans mass.

Jeans length depends on all gravitating components, while Jeans mass is defined for a certain component w. If the mass of the perturbation is larger than M_J, it collapses.

Evolution of the density contrast $\delta(t)$ for dark matter

The dominant non-relativistic component is dark matter (with $c_s = 0$). Its evolution eqn. is given by:

no oscillations

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - 4\pi G\bar{\rho}\delta = 0$$

NOTE

We are working with δ_K

Matter-dominated Universe

We need to calculate how does the perturbation grows (in an expanding Universe) as a function of time. The most interesting period is matter domination, since baryonic matter decoupled from the CMB in this epoch. During matter domination:

- $\dot{a}/a = 2/3t$
- $4\pi G\bar{\rho} = 2/3t^2$

$\Omega_m = 1$ solution for $a(t)$

$$\rightarrow \frac{\partial^2\delta}{\partial t^2} + \frac{4}{3t}\frac{\partial\delta}{\partial t} - \frac{2}{3t^2}\delta = 0$$

Ansatz: the solution is a power law:

$$\delta = Ct^n$$

$$\dot{\delta} = nCt^{n-1}$$

$$\ddot{\delta} = n(n-1)Ct^{n-2}$$

Solving the differential equation:

$$\boxed{\delta = C_1 t^{2/3} + C_2 t^{-1}} \quad \text{(growing mode + decaying mode)}$$

We only consider the growing mode, since it is the one that produces structures.

For a $\Omega_m = 1$ universe we had $a \sim t^{2/3}$, thus: $\boxed{\delta \propto a}$

In the (early) era of matter domination, dark matter perturbations grow proportional to the scale factor.

Generalization for arbitrary cosmologies

We have to integrate the evolution equation, but considering:

$$4\pi G \bar{\rho} = 4\pi G \Omega_m \rho_{crit} = 4\pi G \,\Omega_m \frac{3H^2}{8\pi G} = \frac{3}{2}\Omega_m H^2$$

$$\rightarrow \frac{\partial \delta}{\partial t^2} + 2H\frac{\partial \delta}{\partial t} - \frac{3}{2}\Omega_m H^2 \delta = 0$$

Changing variables $t = t(a)$:

$$a^2 \frac{\partial^2 \delta}{\partial a^2} + (2-q)\frac{\partial \delta}{\partial a} - \frac{3}{2}\Omega_m \delta = 0 \qquad q = -\frac{\ddot{a}a}{\dot{a}^2}$$

It is possible to look for a exact solution solving:

$$\boxed{\delta(a) = \frac{5}{2}\Omega_{m,0}\frac{H}{H_0}\int_0^a \frac{1}{(aH)^3}\,da}$$

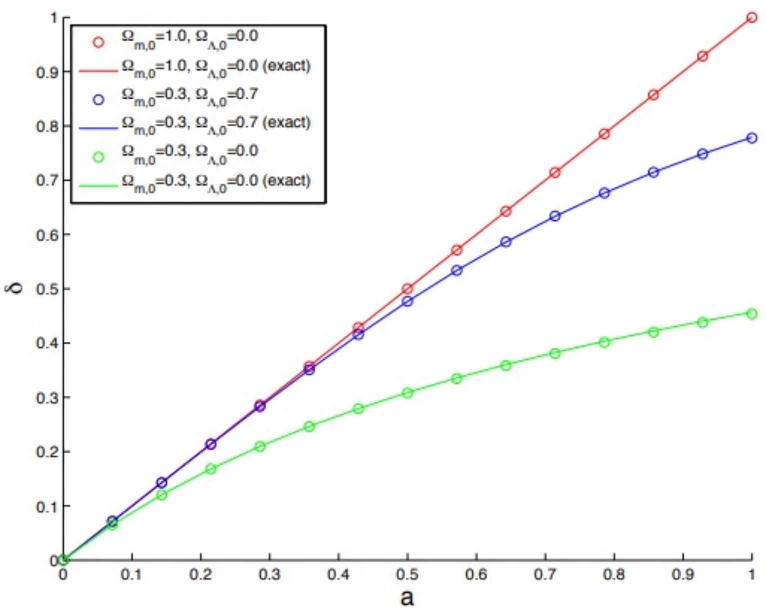
Or an approximate solution:

$$\boxed{\delta(a) \approx \frac{5a}{2}\Omega_m(a)\left[\Omega_m^{4/7}(a) - \Omega_\Lambda(a) + \left(1 + \frac{\Omega_m(a)}{2}\right)\left(1 + \frac{\Omega_\Lambda(a)}{70}\right)\right]^{-1}}$$

For a flat Universe consisting of a 2-component fluid (dark matter + something with $w = $ const, $w = -1 \rightarrow (1-\Omega_{m,0})\,a^0 \equiv \Omega_{m,0}$ or $w = -1/3 \Rightarrow (1-\Omega_{m,0})a^{-2} \equiv \Omega_{K,0}\,a^{-2}$):

$$H^2(a) = H_0^2\left[\Omega_{m,0}\,a^{-3} + (1-\Omega_{m,0})\,a^{-3(1+w)}\right] \qquad p = w\rho c^2, \quad w = \text{const}$$

$$\delta(a) = a\; {}_2F_1\left[\frac{w-1}{2w}, \frac{-1}{3w}, 1-\frac{5}{6w}, 1-\Omega_m^{-1}(a)\right] \qquad \text{(Bueno Belloso et al. 2011)}$$



$\rightarrow \Omega_m = 1$, $\delta \propto a$

All solutions converge to this at early times.

$\rightarrow$ There are various ways to quantify the growth:

- Growth factor: $g = \delta / a$ (compare to linear growth)
- Logarithmic growth rate: $f = d\ln\delta / d\ln a$

Perturbation evolution through time

We are interested on studying the evolution of matter perturbations during the following epochs:

Radiation domination

$$1+z_{eq} = \frac{\rho_{m,0}}{\rho_{r,0}} = 24000\,\Omega_{m,0}\,h^2$$

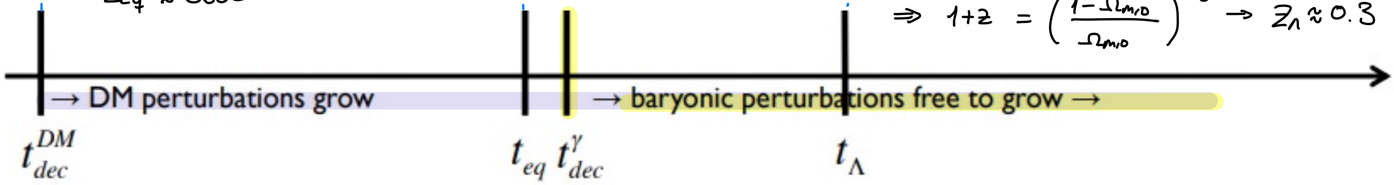
$z_{eq} \approx 3500$

Matter domination

Λ domination

$$1 = \frac{\Omega_m}{\Omega_\Lambda} = \frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}(1+z_\Lambda)^3$$

$$\Rightarrow 1+z = \left(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}\right)^{1/3} \rightarrow z_\Lambda \approx 0.3$$



DM perturbations already grew during radiation domination because they were not coupled to the thermal bath. Baryonic perturbations could only grow when they decoupled from the photons. Until then, they were oscillating without collapsing (as it was discussed before.

Evolution of dark matter perturbations

We have already discussed that, during matter domination: $\boxed{\delta_m(a) \propto a}$

To study the growth of δ during radiation domination we consider again the eqn. of structure formation:

$$\frac{\partial^2\delta_m}{\partial t^2} + 2\frac{\dot a}{a}\frac{\partial\delta_m}{\partial t} - 4\pi G\bar\rho\,\overbrace{\left(\frac{\bar\rho_m}{\bar\rho}\delta_m + \frac{\bar\rho_r}{\bar\rho}\delta_r\right)}^{\delta\ \rightarrow\ \text{Potential perturbations are sourced by all components}} = 0$$

It is necessary to consider ρ_r since it is the dominant term. In the equation, it appears with the radiation perturbation δ_r. But we have that $\delta_r \approx 0$ (they oscillate).

This can be proved using GR.

> NOTE: We don't have $\delta_r = \frac{4}{3}\delta_m$ (as for baryons) because they do not interact via Thomson scattering.

We can write the previous equation as:

$$\frac{\partial^2\delta_m}{\partial t^2} + 2\frac{\dot a}{a}\frac{\partial\delta_m}{\partial t} - 4\pi G\bar\rho_m\delta_m = 0$$

Since $4\pi G\bar\rho_m << 4\pi G\rho_r = \frac{3}{2}\left(\frac{\dot a}{a}\right)^2$, the potential term can be ignored. The eqn. becomes:

$$\frac{\partial^2\delta_m}{\partial t^2} + \frac{1}{t}\frac{\partial\delta_m}{\partial t} \approx 0$$

Dem: Radiation-dominated Universe: $a(t) \propto t^{2/3}$, $\dot a \propto \frac{2}{3}t^{-1/3}$, $\frac{\dot a}{a} = \frac{2}{3}\frac{t^{-1/3}}{t^{2/3}} = \frac{2}{3}\frac{1}{t}$

Solution using: $\frac{\partial y}{\partial t} + \frac{1}{t}y = 0 \qquad y = \frac{\partial\delta_m}{\partial t}$

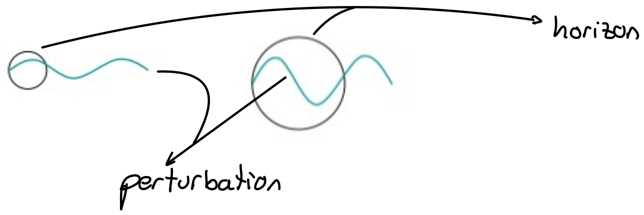
$\boxed{\delta_m(a) \propto \ln(t)}$ — radiation domination

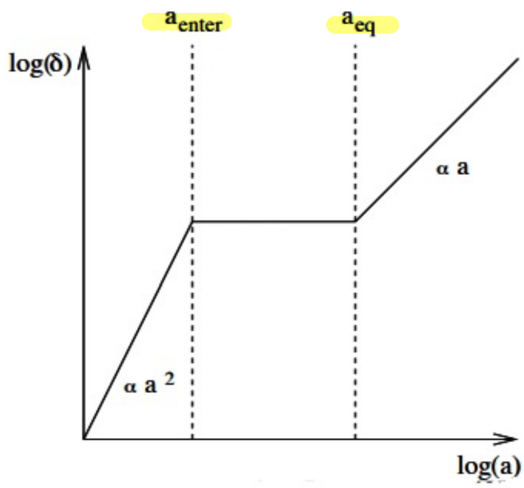
Dark matter perturbations are "not really growing" during radiation domination. This is true for perturbations inside the horizon. Perturbations outside the horizon are growing like $\delta(a) \propto a^2$ (relativistic treatment).

This is called the Meszaros effect:

- δ_m outside the horizon grows like a^2
- δ_m inside the horizon grows like $\ln(a)$

When a perturbation enters the horizon (because it has grown faster) it "freezes in" up to the point when matter starts to dominate.





This effect allows to predict the shape of the Power spectrum of primordial matter perturbations.

During Λ domination:

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2\frac{\dot a}{a}\frac{\partial \delta_m}{\partial t} - 4\pi G \bar\rho_m \delta_m = 0$$

As before, $\delta_\Lambda \approx 0$, $4\pi G \bar\rho_m \ll 4\pi G \bar\rho_\Lambda = \frac{3}{2}\left(\frac{\dot a}{a}\right)^2$

$\hookrightarrow$ as far as we know

Taking into account that $\frac{\partial^2 \delta_m}{\partial t^2} + 2H_0 \frac{\partial \delta_m}{\partial t} \approx 0$

Sol: $\frac{\partial y}{\partial t} + 2H_0 y = 0 \qquad y = \frac{\partial \delta_m}{\partial t}$

$$\boxed{\delta_m(a) \propto a^{-2}}$$

In Λ domination the perturbations decay away. There will not be structure formation from perturbations entering the horizon.

Evolution of baryonic perturbations

Baryons are not able to grow their perturbations until decoupling (before that, we have the baryonic acoustic oscillations. After decoupling, they will catch up with dark matter perturbations, and grow as $\delta_b(a) \propto a$:

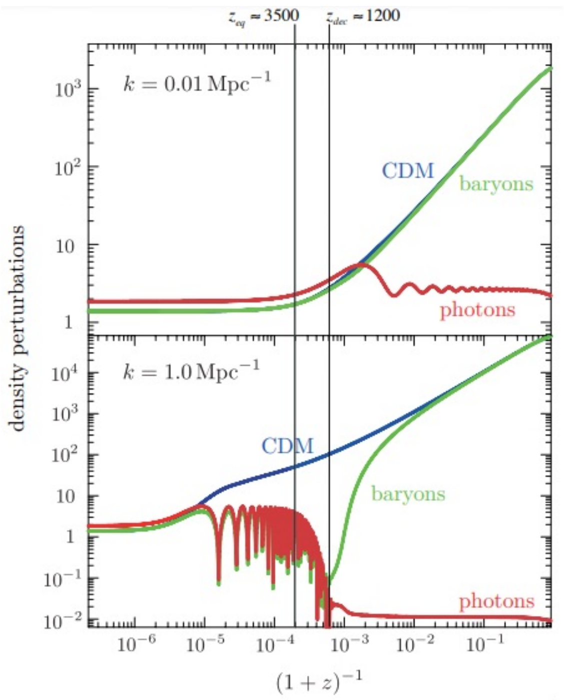
- Before decoupling

$$\boxed{\delta_b \approx \frac{3}{4}\delta_r}$$

- After decoupling

$$\boxed{\delta_b(a) \approx \delta_m\left(1 - \frac{a_{dec}}{a}\right)}$$

$\hookrightarrow$ re-scaling



These are detailed numerical integrations of:

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\frac{\dot a}{a}\frac{\partial \delta_k}{\partial t} + \left(\frac{c_s^2}{a^2}k^2 - 4\pi G\bar\rho\right)\delta_k = 0$$

for cold dark matter, photons and baryons and two wave numbers ($k = 0.01\,\mathrm{Mpc}^{-1}$ and $k = 1.0\,\mathrm{Mpc}^{-1}$). The upper panel shows a large scale perturbations. Baryons are barely coupled to photons, compared to the lower panel (small scale perturbation), where they are strongly coupled. After decoupling, baryons follow DM, forming the same structures in the end.

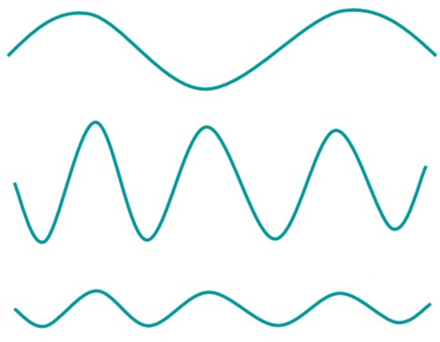
9.3. Statistics of perturbations

Perturbations in k-space

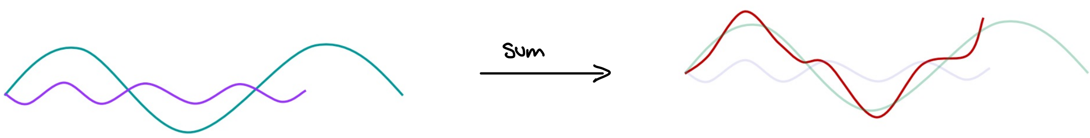
We can decompose perturbations into waves:

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot a}{a}\frac{\partial \delta}{\partial t} - 4\pi G\bar\rho\,\delta = 0$$

$$\longrightarrow \quad \delta(\vec x, t) = \delta_k(t)\, e^{i\vec k\cdot\vec x}$$



Since the evolution equation is linear, each $\delta_k(t)$ grow independently



We can take "moments" of the density contrast:

- First moment:

  $\langle \delta(\vec x, t)\rangle = 0$

  Because the average of $\langle \frac{\rho - \bar\rho}{\bar\rho}\rangle$ is automatically 0. This means that, in Fourier space: $\langle \delta_k(t)\rangle = 0$, $\delta_{\vec k}(t) = \delta_{-\vec k}(t)$

- We can take higher order moments:

  $\xi_2 = \langle \delta(\vec x_1, t)\,\delta(\vec x_2, t)\rangle \longrightarrow$ two-point correlation function

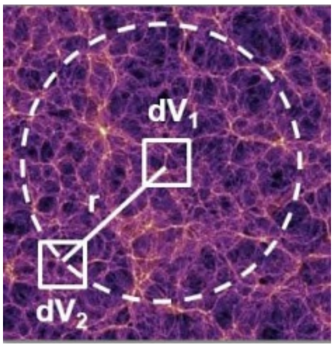
  $\xi_3 = \langle \delta(\vec x_1, t)\,\delta(\vec x_2, t)\,\delta(\vec x_3, t)\rangle$

  $\xi_4 = \ldots$

And, since we are working with an homogeneous and isotropic Universe: $\xi_2 = \xi_2(|\vec{x}_1 - \vec{x}_2|)$ (we can only work with differences, not positions).

Two-point correlation function and Power spectrum

The two-point correlation function is commonly used to quantify the strength of perturbations.



This second moment quantifies the likelihood of finding another point at a certain distance to a given point ($\equiv$ same density). This is compared to a random mass distribution.

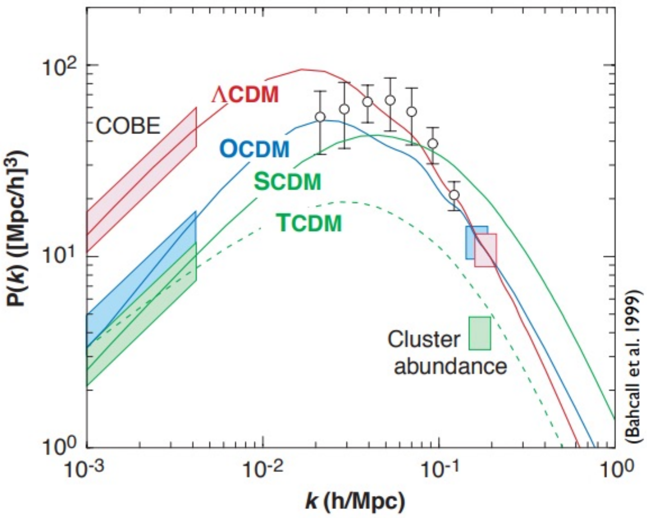
$$\xi_2(\vec{x}) = \frac{n_{pair}(\vec{x}+d\vec{x})}{n_{random}(\vec{x}+d\vec{x})} - 1 \qquad \text{(difficult to calculate)}$$

This correlation function has an analogous in Fourier space called the Power Spectrum.

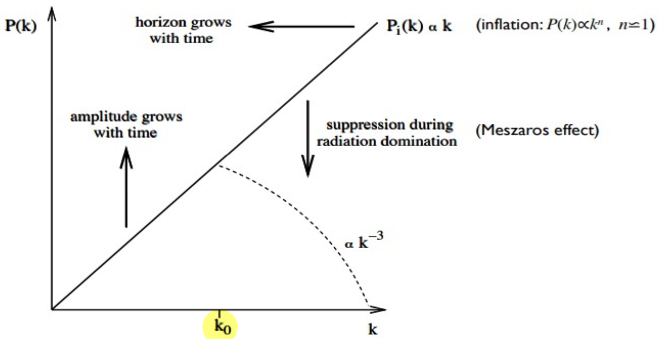
$$\xi_2(\vec{x}) = \frac{1}{(2\pi)^3}\int P(k)\, e^{-i\vec{k}\cdot\vec{x}}\, d^3k = \frac{1}{2\pi^3}\int P(k)\,\frac{\sin(\vec{k}\cdot\vec{x})}{\vec{k}\cdot\vec{x}}\, k^2\, dk$$

$P(k) = \langle|\delta_{\vec{k}}|^2\rangle_{|\vec{k}|=k} \leftarrow$ power spectrum

Knowing the Fourier amplitudes, it is easy to obtain the power Spectrum, and so ξ_2.

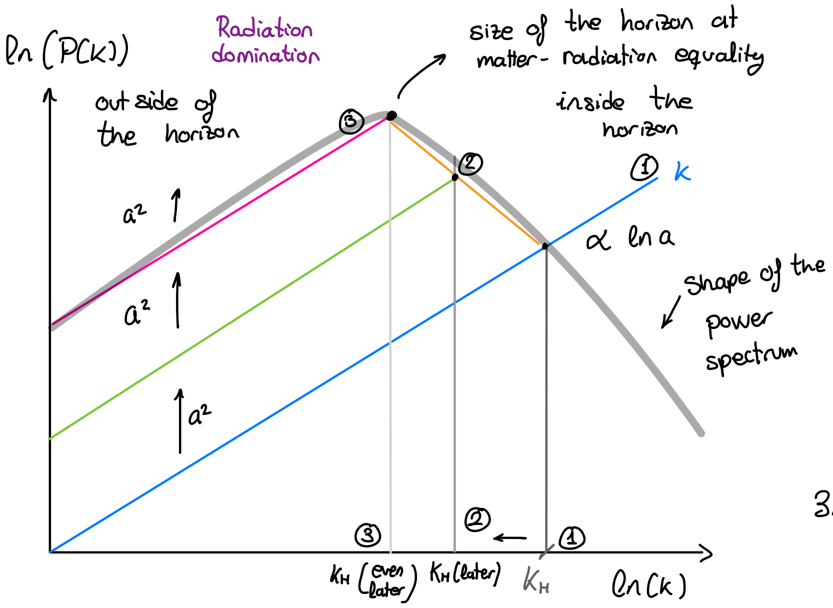


Taking models from different cosmologies, they all have the same shape. This is related to the Meszaros effect. (+ inflation)



From Inflation, we know that the initial spectrum of matter perturbations is $P_i(k) \propto k$.
Then, $P(k)$ evolves as a function of time going through radiation domination and entering matter domination. Since $P(k) = \langle\delta_k^2\rangle$, and we know how δ_k evolves, then:

1. We have k coming out of inflation. We also need to add the size of the horizon $k_H = \frac{2\pi}{r_H}$



2. We know that the horizon is growing, which means that k_H is moving to smaller k. If a perturbation is outside of the horizon, it is growing like a^2. If it is inside, it will not grow ($\propto \ln a$ in rad. dom).

3. The same will happen for sequent time steps.

This will happen until matter domination. As soon as we enter matter domination, modes inside the horizon ( ) will start to grow again as $\delta \propto a$. The peak in the power spectrum is the size of the horizon right at matter-radiation equallity.

Evolution of the Power Spectrum during matter domination

As we have mentioned before, the Power Spectrum is the square of the amplitudes of the Fourier modes δ_k, which were defined as:

$$\delta(\vec{x}, a) = \sum_{\vec{k}} \delta_{\vec{k}}(a) e^{i\vec{k}\cdot\vec{x}}$$

This modes evolve, with respect to the linear growth factor:

$$0 = \frac{\partial^2 D}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial D}{\partial t} - 4\pi G\bar{\rho} D \quad \leftarrow \text{matter domination}$$

Thus, recalling the definition of the power spectrum:

$$\boxed{P(k) = \langle |\delta_k|^2 \rangle_{|\vec{k}|=k}}$$

We can find its dependence with time as:

$$\boxed{P(k) = \left(\frac{D(a)}{D(a_0)}\right)^2 P_0(k)}$$

DEFINITION + REMINDER

$$\delta(\vec{x}, a) = \frac{D(a)}{D(a_0)} \delta_{\vec{k}}(a_0)$$

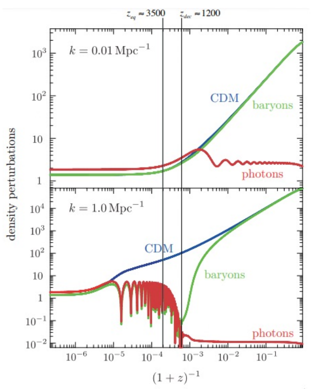
Taking $a_0 = 1 \rightarrow D(a=1) = 1$

- For $\Omega_m = 1$, $a \propto t^{2/3}$, must have $D(a) \propto a$, i.e. for small perturbations, overdensities grow as the scale factor everywhere.
- For $\Omega_m < 1$, $D(a) \propto a$ at early times, at late times (when curvature or Λ term dominates expansion rate) $D(a)$ increases more slowly → structure growth is slower.

See: Evolution of δ for arbitrary cosmologies (δ vs a)

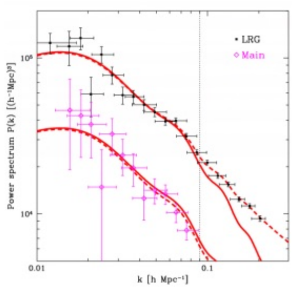
Baryonic oscillations in P(k) and $\xi(k)$

Oscillations leave distinct features in the power spectrum and the two-point correlation function:

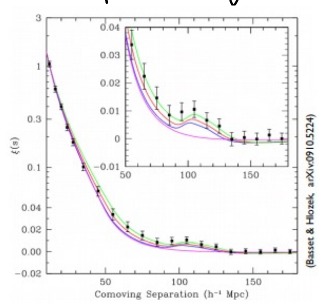


→

1. Power spectrum



2. Two-point c. f.



9.4. Non Linear structure formation

Analytical extrapolations of linear theory

Until now, we were considering perturbations with $\delta(\vec{x},t) < 1$, but the structures that we observe have (obviously) $\delta(\vec{x},t) \gg 1$. This is the territory of computational cosmology. However, extrapolating linear theory results when $\delta \gg 1$ provides an useful insight on structure growth. Some existent analytical (quasi-linear) approaches are the Zel'dovich approximation, the Spherical Top-Hat Collapse model and the Press-Schechter halo mass function.

Zel'dovich approximation

REMINDER

- Eulerian viewpoint:

(we measure the flow...)

- Lagrangian viewpoint:

(we follow mass elements...)

The Zel'dovich approximation is applying first order perturbation theory, but using a Lagrangian viewpoint (until now, we were taking an Eulerian viewpoint).

Let us consider an overdense region that is collapsing. The position changes as a (linear) function of time:

$$\vec{x}(t) = \vec{q} + D(t)\,\vec{S}(\vec{q})$$

- $D(t)\,\vec{S}(\vec{q})$ → temporal and spatial evolution
- $\vec{q}$ → initial (unperturbed) position, background
- $\vec{x}(t)$ → updated position

- Displacement field $\vec{S}(\vec{q})$

To obtain the displacement field, we take the first derivative of the Z.A:

$$\dot{\vec{x}} = \dot{D}\,\vec{S}(\vec{q})$$

- Temporal evolution:

$$\delta(\vec{x},t) = D(t)\,\delta(\vec{x}_0) \quad , \ \delta(\vec{x}_0) \text{ initial perturbation}$$

$$0 = \frac{\partial^2 D}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial D}{\partial t} - 4\pi G\bar{\rho} D$$

And using the definition of the peculiar velocity field:

$$\dot{\vec{x}} = \frac{1}{a}\vec{u}$$

we can combine them to obtain an equation for the evolution of $\vec{u}$:

$$\left.\begin{aligned} \vec{u} &= \dot{D} a\, \vec{S}(q) \\ \frac{\partial \vec{u}}{\partial t} &= \dot{a}\dot{D}\,\vec{S}(\vec{q}) + a\ddot{D}\,\vec{S}(\vec{q}) \end{aligned}\right\} \ (2a\dot{a}\dot{D} + a^2\ddot{D})\,\vec{S}(\vec{q}) = -\nabla\Phi$$

↑ conservation of momentum, Eulerian perturbation theory $\frac{\partial \vec{u}}{\partial t} + \frac{\dot{a}}{a}\vec{u} = -\frac{1}{a}\nabla\Phi$

We have the equations:

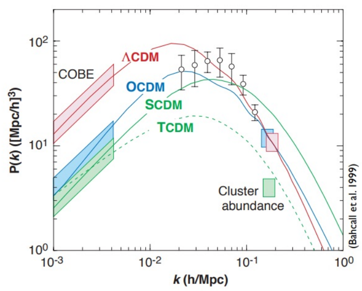
$$\begin{cases} 0 = \ddot{D} + 2\frac{\dot{a}}{a}\dot{D} - 4\pi G\bar{\rho} D \\ (2a\dot{a}\dot{D} + a^2\ddot{D})\,\vec{S}(\vec{q}) = -\nabla\Phi \end{cases} \quad + \quad \begin{aligned} &\Delta\Phi = 4\pi G a^2 \bar{\rho}\delta && \text{(Poisson equation)} \\ &\Phi = 4\pi G a^2 \bar{\rho}\Psi && \text{(Definition of peculiar potential)} \end{aligned}$$

And from them we can obtain:

$$\boxed{\begin{array}{c}\vec{S}(\vec{q}) = -\nabla\Psi \\ \Delta\Psi = \delta(\vec{x}_0)\end{array}}$$

"force" coming from the peculiar potential

Ψ sourced by initial perturbations



Since we know the solution for $D(a)$:

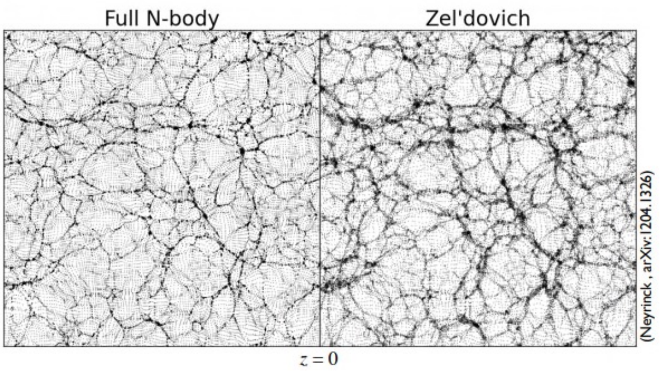
$$\boxed{D(a) = \frac{5}{2}\Omega_{m,0}\, H \int_0^a \frac{1}{(\Omega_{m,0}\, a^{-3} + (1-\Omega_{m,0}-\Omega_{\Lambda,0})a^{-2} + \Omega_{\Lambda,0}}\, da}$$

We can study the evolution if we know $\delta(\vec{x}_0)$. This can be obtained from the power spectrum, so we need some analytical calculations for $P(k)$ (e.g. from a Boltzmann solver).

$$\delta_0(k) = \sqrt{P_0(k)}\, R_{\vec{k}}\, e^{i\theta_{\vec{k}}}$$

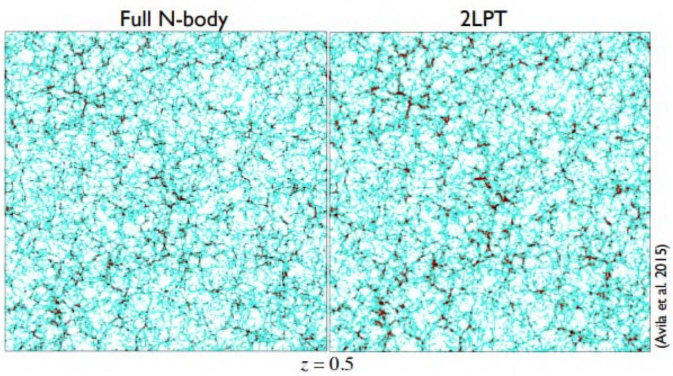
Once we have obtained δ_0, we can evolve it using the equations above.

This approach can be compared to the result of N-body simulations:



Reproduces structure, do not resolves properties of clusters (~10^7 hours)

(Need parallel computing)



2nd order Lagrangian perturbation theory

$$\vec{x}(a) = \vec{q} - D(a)\nabla\Psi + D^{(2)}\Delta\Psi^{(2)}$$

Resolves structure (~10^0 hours)

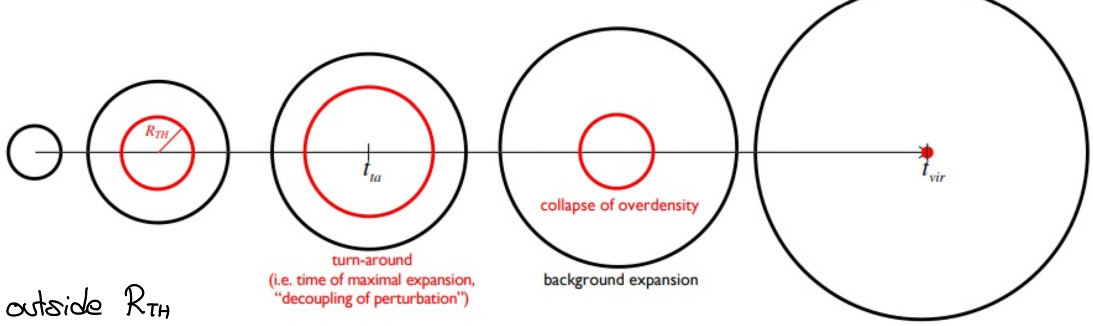
Spherical Top-Hat collapse

We want to predict the density contrast of a collapsing object. It is only possible to form structures if the overdensity decouples from the expanding background. The density contrast needs to be enough to expand with the Universe and then collapse due to its self-gravity, forming a virialised object.

Spherical top-hat overdensity:

$$\boxed{\begin{array}{l}R_{TH} = R_{TH}(t) \\ M_{TH} = \text{const}\end{array}}$$

$\to$ no spilling of mass outside R_{TH}



Let us treat the perturbation as a "closed Universe" with $K=1$ (curvature) i.e. a Universe that collapses. We can analyse its evolution using the Fredmann equations:

$$\dot{R}_{TH}^2 = \frac{8\pi G}{3} \underbrace{\rho_{TH}}_{R_{TH}\text{- dependent}} \underbrace{R_{TH}^2}_{\text{not constant}} - Kc^2 = \frac{8\pi G}{3} \frac{\overbrace{M_{TH}}^{\text{constant}}}{\frac{4}{3} R_{TH}^3} R_{TH}^2 - Kc^2 = \frac{2GM_{TH}}{R_{TH}} - Kc^2$$

$$\frac{1}{2}\dot{R}_{TH}^2 - \frac{GM_{TH}}{R_{TH}} = -|K|c^2 < 0 \qquad (K=1 \text{ for TH overdensity} \rightarrow \text{closed Universe})$$

The solution to this equation was given on the FRW lecture:

$$\boxed{\begin{cases} \dfrac{R_{TH}}{R_{ta}} = \dfrac{1}{2}(1-\cos\eta) \\ \dfrac{t}{t_{ta}} = \dfrac{1}{\pi}(\eta - \sin\eta) \end{cases}} \qquad \left.\begin{aligned} R_{ta} &= \frac{2GM_{TH}}{c^2} \\ t_{ta} &= \frac{\pi R_{ta}}{2c} \end{aligned}\right\} \text{All is characterized by } M_{TH}, \text{ the mass enclosed in } R_{TH}.$$

$\eta \in [0, 2\pi]$

We are not interested in the absolute value of the overdensity, we want to compare it to the background density. Thus, we calculate the density contrast:

$$\delta_{TH} = \frac{\rho_{TH} - \bar{\rho}}{\bar{\rho}} \qquad \rho_{TH} \rightarrow \text{Mass divided by volume} \qquad \bar{\rho} \rightarrow \text{flat model with } \Omega_m = 1$$

$$\left.\begin{aligned} \rho_{TH} &= \frac{3M_{TH}}{4\pi R_{TH}^3} = \dots = \frac{6M_{TH}}{\pi R_{ta}^2}(1-\cos\eta)^{-3} \\ \bar{\rho} &= \frac{1}{6\pi G t^2} = \dots = \frac{\pi^2}{6\pi G t_{ta}^2}(\eta - \sin\eta)^{-2} \end{aligned}\right\} \quad \boxed{1+\delta_{TH} = \frac{\rho_{TH}}{\bar{\rho}} \overset{*}{=} \dots = \frac{9}{2}\frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3}}$$

* Proof:

$$\frac{\rho_{TH}}{\bar{\rho}} = \frac{6M_{TH}}{\pi R_{ta}^3} \frac{6\pi G t_{ta}^2}{\pi^2} \frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3} = \frac{t_{ta}^2}{R_{ta}^3} \frac{36GM_{TH}}{\pi^2} \frac{(\eta-\sin\eta)^2}{(1-\sin\eta)^3} = \frac{\pi^2 R_{ta}^2}{4c^2 R_{ta}^3} \frac{36GM_{TH}}{\pi^2} \frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3} =$$

$$= \frac{1}{R_{ta}} \frac{36GM_{TH}}{4c^2} \frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3} = \frac{c^2}{2GM_{TH}} \frac{9GM_{TH}}{c^2} \frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3} = \frac{9}{2}\frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3}$$

We want to calculate the density contrast at the beginning (turnaround) and the end of collapse (formation of the final object). Thus, we need to determine η_{ta} and η_{vir}.

Plotting $(\eta, R_{TH}(\eta)/R_{ta})$ and $(\eta, t(\eta))$, it is possible to find $\eta_{ta} = \pi$ and $\eta_{vir} = 2\pi$. ($t_{vir} = 2t_{ta}$).

At turnaround:

$$1+\delta_{TH}(t_{ta}) = \frac{9\pi^2}{16} \approx 5.5$$

Once the collapse has finished:

$$1+\delta_{TH}(t_{vir}) = \infty \rightarrow \text{Singularity}$$

However, singular collapse is not a particularly useful model (it is unrealistic to assume purely radial motions). We can assume that the overdense region forms a virialized halo at collapse time.

The energy of the halo will be given by:

$$E_{ta} = U_{ta} \qquad U_{ta} = -\frac{3}{5}\frac{GM_{TH}^2}{R_{ta}}$$

$$E_{vir} = T_{vir} + U_{vir} \qquad U_{vir} = -\frac{3}{5}\frac{GM_{TH}^2}{R_{vir}}$$

+ virial theorem: $0 = 2T_{vir} + U_{vir}$

At turnaround, $T_{vir} \simeq 0$, so the total energy will be $\boxed{E_{vir} = U_{ta}}$

(or) $0 = 2T_{vir} + U_{vir} \Rightarrow 2E_{vir} = \underbrace{2T_{vir} + U_{vir}}_{=0 \text{ Th vir}} + U_{vir} = 2U_{ta}$

Then, $\boxed{R_{vir} = \frac{R_{ta}}{2}}$

Thus, we will have for virialized overdensity:

$$\rho_{TH}(t_{vir}) = \frac{3M_{TH}}{4\pi R_{vir}^3} = 8\rho_{TH}(t_{ta})$$

And for the background density:

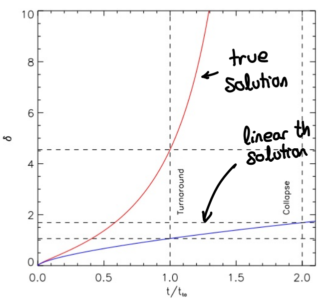
$$t_{vir} = 2t_{ta} \xrightarrow[\bar{\rho} = \frac{1}{6\pi G t^2}]{} \bar{\rho}(t_{vir}) = \frac{1}{2^2}\bar{\rho}(t_{ta})$$

And so, for the density contrast:

$$1+\delta_{TH}(t_{vir}) = \frac{8\rho(t_{ta})}{\bar{\rho}(t_{ta})/4} = 32\,(1+\delta(t_{ta})) = 32\,\frac{9\pi^2}{16} = 18\pi^2 \approx 178 \longrightarrow \boxed{1+\delta_{TH}(t_{vir}) = 18\pi^2 \approx 178}$$

Density contrast of a virialized object

Another possible approach is taking the linear theory solution for the density contrast.
Taylor-expanding $\cos\eta$ and $\sin\eta$ gives:



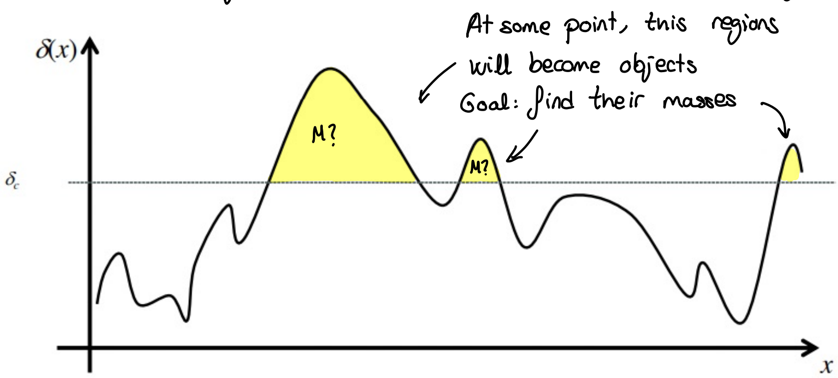
$$R_{TH}(t) = \frac{R_{ta}}{4}\left(\frac{6\pi t}{t_{ta}}\right)^{2/3}\left[1 - \frac{1}{20}\left(\frac{6\pi t}{t_{ta}}\right)^{2/3} + \dots\right] \Rightarrow \delta(t) = \frac{3}{20}\left(\frac{6\pi t}{t_{ta}}\right)^{2/3}$$

$$\delta(t_{vir} = 2t_{ta}) \approx \frac{3}{20}(12\pi)^{2/3} \approx 1.686 \Rightarrow \boxed{\delta_{lin}(t_{vir}) \approx 1.686 \equiv \delta_c}$$

Linearized density contrast of collapsed object

Press-Schechter halo mass function

Once we have estimations of the threshold for structure formation, we want to calculate the number of objects that will form for a given mass, $N(M)$.



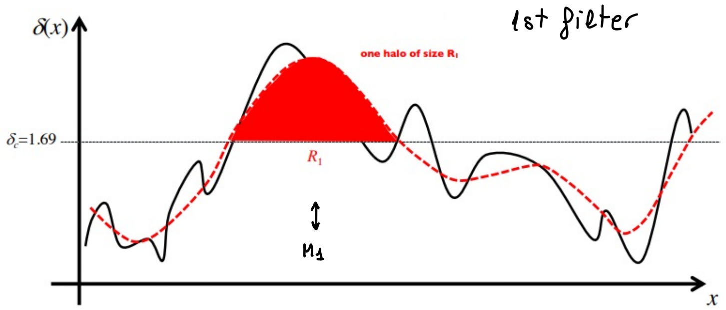
A halo has formed when its linear density contrast $\delta(x,a)$ has reached $\delta_c = 1.69$

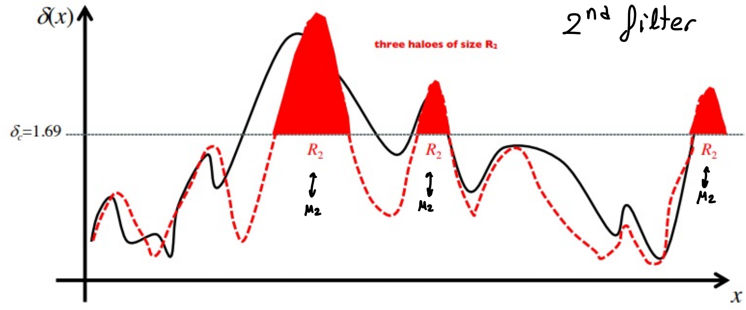
NOTE

Time dependence is given by $\delta(x,t) \propto D(a)$

To find the mass of each object (i.e. inside each peak), one can filter the density constrast. This is done convolving it with a filter that peaks out a particular mass. This is done with several filters, counting the peaks above the theshald for each filter.
We consider perturbations on a certain scale R^*, which can be related to halo mass via $M = \Omega_m \rho_{crit} \frac{4\pi}{3} R^3$:

$$\delta_R(\vec{x}, a) = \int \delta(\vec{x}', a)\, W_R(\vec{x} - \vec{x}')\, d^3x'$$





It is important to bear in mind that some of the peaks are being counted twice, so the will need to be substracted (whenever the filter mass is smaller that its mass).
The next assumption is that the density contrast $\delta_R(x)$ is a Gaussian field with variance σ_R :

$$p(\delta_R) = \frac{1}{\sqrt{2\pi\sigma_R^2}}\, e^{-\frac{1}{2}\left(\frac{\delta_R}{\sigma_R}\right)^2} \qquad \sigma_R^2 = \frac{1}{2\pi^2}\int_0^{+\infty} P(k)\, \hat{W}^2(kR)\, k^2\, dk$$

where $P(k)$ is the power spectrum of the density fluctuation (all waves inside R-window affect σ_R).

$$P(k) = \left(\frac{D(a)}{D(a_0)}\right)^2 P_0(k)$$

initial power spectrum of density perturbations at decoupling can be scaled using the growth factor

$p(\delta_R)$ is the probability of finding a peak with that particular value of δ_R. Thus, the probability of finding a peak above δ_R will be given by:

$$F_{>\delta_c}(R) = \int_{\delta_c}^{\infty} p(\delta_R)\, d\delta_R$$

An the number of peaks in rande $[R, R+dR]$ is given by:

$$dN \propto F_{>\delta_c}(R) - F_{>\delta_c}(R+dR)$$

This can be related to the mass M as:

$$M = \Omega_m \rho_{crit} \frac{4\pi}{3} R^3$$

Following this steps, one arrives to the Press-Schechter formula:

$$\boxed{\frac{dn}{dM}\, dM = \sqrt{\frac{2}{\pi}}\, \frac{\bar{\rho}}{M}\, \frac{\delta_c}{\sigma_M} \left|\frac{d\ln\sigma_M}{d\ln M}\right| \exp\left(\frac{-\delta_c^2}{2\sigma_M^2}\right) \frac{dM}{M}}$$

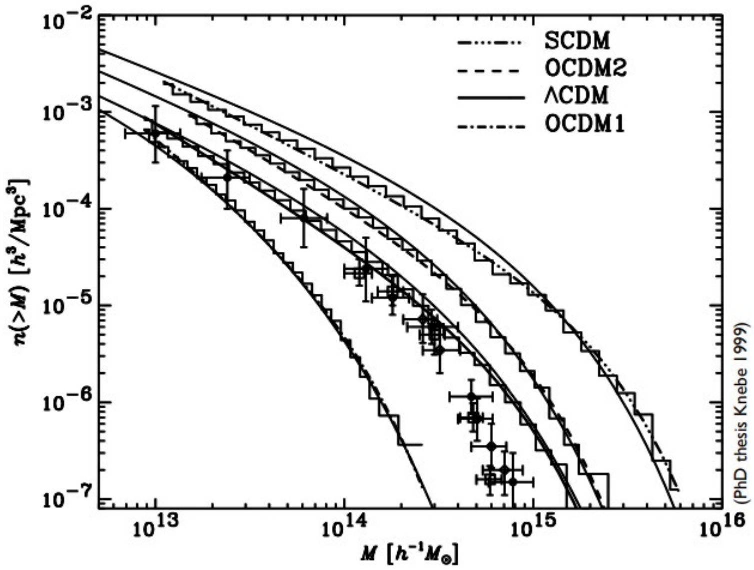
Press - Schechter function

where δ_c is the threshold density contrast of collapsed structures according to linear perturbation theory, $\bar{\rho}$ is the background (mean) density of the Universe and σ_M is the variance of the mass on scale corresponding to $M = (4/3)\,\Omega_m \rho_{crit} R^3$

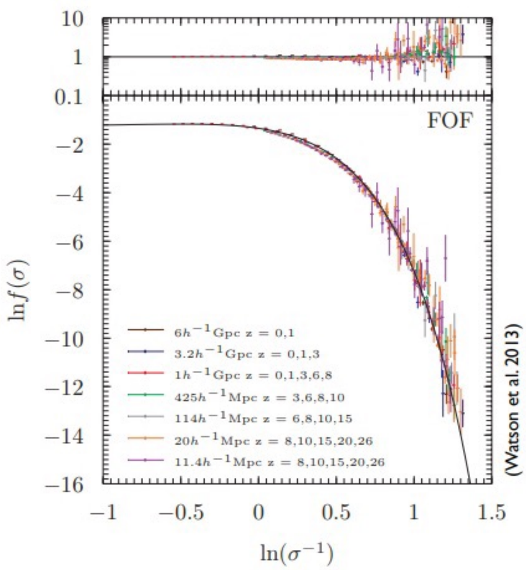
$$\sigma_R^2 = \frac{1}{2\pi^2}\int_0^{+\infty} P(k)\,\hat{W}^2(kR)\,k^2\,dk$$

It is common to use a spherical top hat filter ($\hat{W}^2(kR)$ is its Fourier transform):

$$\hat{W}(x) = \frac{3}{x^3}\left(\sin(x) - x\cos(x)\right)$$



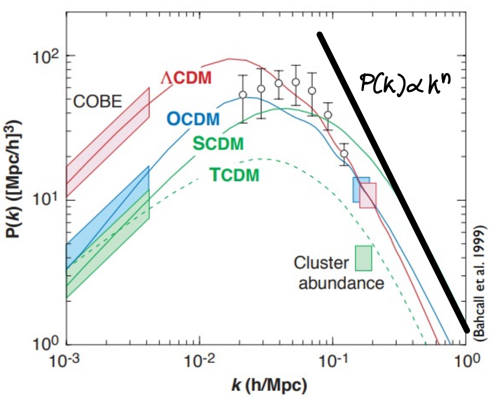
This can be taken to $z=0$ evolving $P(k)$ with $D(a)$. Solid lines represent the Press-Schechter function at $z=0$. Other lines are coming out of cosmological simulations starting at decoupling (Zel'dovich + hydrodinamical equations). ⌶ are observational data.



The agreement between the model and simulations was improvable, but new models and calculations had not introduced new physics, just introduced new parameters.

$$\frac{dn}{dM}\,dM = f(\sigma_M)\,\frac{\bar{\rho}}{M}\,\frac{d\ln\sigma_M^{-1}}{dM}\,dM$$

$$f(\sigma_M) = A\left[\left(\frac{\beta}{\sigma_M}\right)^{\alpha} + 1\right]\exp(-\gamma/\sigma_M^2)$$



The mass function for a scale-free power spectrum is given by:

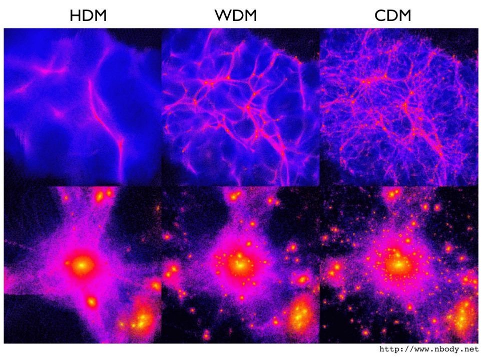
$$\sigma_M^2 = \sigma_0^2\left(\frac{4\pi\bar{\rho}}{3}\right)^{\frac{n+3}{3}} M^{-\frac{n+3}{3}} = \left(\frac{M}{M_*}\right)^{-\frac{n+3}{3}}$$

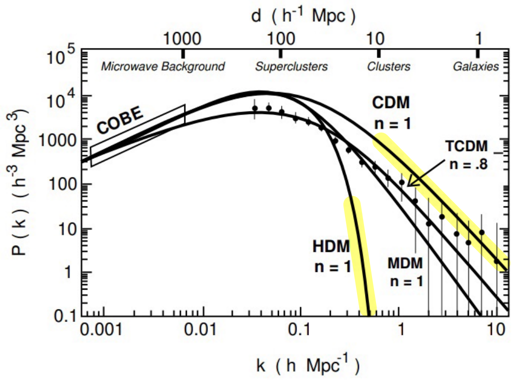
$$\frac{dn}{dM}\,dM = \frac{n+3}{\sqrt{2\pi}}\,\frac{\bar{\rho}}{M^2}\,\delta_c\left(\frac{M}{M_*}\right)^{\frac{n+3}{6}}\exp\left[-\frac{1}{2}\delta_c^2\left(\frac{M}{M_*}\right)^{\frac{n+3}{3}}\right]dM$$

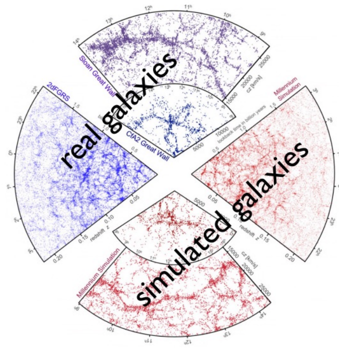
where M_* is the typical mass of an object at a given redshift.

Computational cosmology

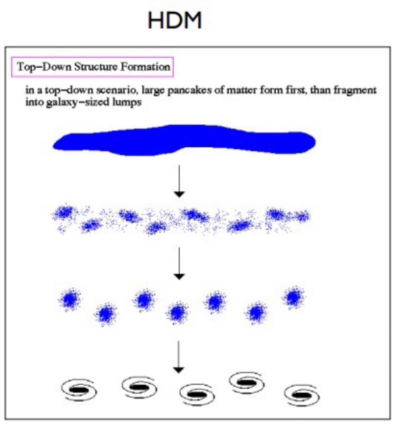
Numerical simulations are necessary to obtain information about the internal properties or the spatial distribution of objects. It is possible to try different models (for example, for dark matter). Comparing this models to observations, they are validated or rejected (structures formed, power spectrum, etc.)

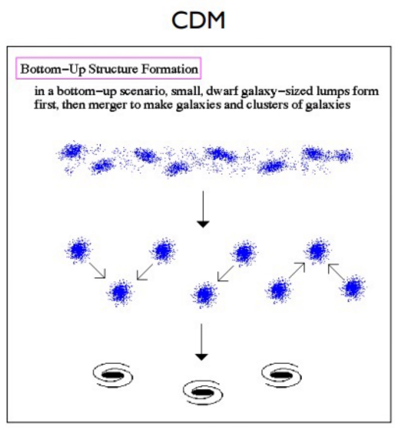






HDM power spectrum is suppressed for small wavelengths. Thus, tiny objects and structures are not formed from overdensities. This difference gives rise to two formation scenarios in terms of the "nature" of dark matter: bottom-up (cold dark matter) and top-down (hot dark matter) structure formation.





Simulations vs. observations

Galaxy redshift surveys cover Gpc^3 volumes. Simulations cannot follow full set of physics in such large volumes. However, there are multiple approaches:

(Biased) galaxy formation models require:

- SAM - Semi-analytical galaxy formation modeling
- HOD - Halo occupation distribution
- CLF: Conditional luminosity function
- (S)HAM: (Subhalo) Halo abundance Matching
- Biasing model: $P_{gal}(k) = b^2(k)\, P_{DM}(k)$

\+ other assumptions