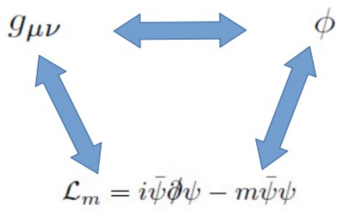
VI. Gauge Invariant Perturbations

6.1. Theory of Gauge Invariant perturbations

"Inflation practically is quantum mechanics on curved space time". Inflaton perturbations affect metric, which is coupled to matter. Thus, if the quantum fluctuations (ϕ) had an unique signature, this could be detected in $g_{\mu\nu}$ and $\mathcal{L}$ (it would be imprinted on them, e.g. on primordial gravitational waves).



Reparametrization invariant perturbation theory

Background

We can write the Robertson-Walker metric as:

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \gamma_{ij}\, dx^i dx^j\right]$$

$$\eta = \int dt/a(t) \quad \text{conformal time}$$

And using the scalar field $\phi = \phi(\eta)$ (which cannot depend on space because of homogeneity and isotropy) we can derive the Friedmann and Klein-Gordon equations

$$\mathcal{H}^2 = \frac{\kappa^2}{2}\left(\frac{1}{2}\phi'^2 + a^2 V(\phi)\right) \qquad \phi'' + 2\mathcal{H}\phi' + a^2 V'(\phi) = 0.$$

$$\mathcal{H}' - \mathcal{H}^2 = -\frac{\kappa^2}{2}\phi'^2 \qquad \mathcal{H} = aH \qquad \phi' = a\dot{\phi}$$

NOTATION

$\mathcal{H}$: conformal Hubble parameter

Perturbations and gauge potentials.

If the metric has any perturbations, it is important to be careful with the decomposition (since it is a tensor). Perturbations can be "broken" into Scalar - Vector - Tensor (SVT), and use this decomposition.

For a vector, we can decompose it as:

$$\omega_i = \omega_i^{\parallel} + \omega_i^{\perp}$$

$\omega_i^{\perp}$ ↳ transverse

$\omega_i^{\parallel}$ ↳ longitudinal

This components satisfy:
$$\begin{cases} \vec{\nabla} \times \vec{\omega}^{\parallel} = 0 \Rightarrow \omega_i^{\parallel} = \nabla_i A & A \equiv \text{potential} \\ \vec{\nabla} \cdot \vec{\omega}^{\perp} = 0 \end{cases}$$

Extrapolating to a tensor S_{ij} : Orthogonal, non interacting

$$S_{ij} = S_{ij}^{\parallel} + S_{ij}^{\perp} + \underbrace{S_{ij}^{T}}_{\text{(traceless)}} \longrightarrow \begin{cases} S_{ij}^{\perp} = \nabla_i S_j^{\perp} + \nabla_j S_i^{\perp} \\ \partial^i S_{ij}^{T} = 0 \\ S_{ij}^{\parallel} = \left(\nabla_i \nabla_j - \frac{1}{3} g_{ij} \nabla^2\right) B \end{cases}$$

During inflation, the quantum fluctuations of the scalar field will include metric perturbations which will backreact on the scalar field. Let us consider, in linear perturbation theory, the most general line element with both scalar and tensor metric perturbations, together with the scalar field perturbations (from FWR metric):

(A, B, $\mathcal{R}$, E gauge dependent functions)

$$ds^2 = a^2(\eta)\left[-(1+2A)\,d\eta^2 + 2B_{|i}\,dx^i d\eta + \{(1+2\mathcal{R})\gamma_{ij} + 2E_{|ij} + 2h_{ij}\}\,dx^i dx^j\right]$$

$$\phi = \phi(\eta) + \delta\phi(\eta, x^i)$$

$$h_{ij} = \frac{h}{3}\delta_{ij} + h_{ij}^{\parallel} + h_{ij}^{\perp} + h_{ij}^{T}$$

The indices $\{i, j\}$ label the three-dimensional spatial coordinates with metric γ_{ij}, and the $|i$ denotes covariant derivative with respect to that metric. The gauge invariant tensor perturbation h_{ij} corresponds to a transverse traceless grav. wave, $\nabla^i h_{ij} = h^i_i = 0$. The four scalar perturbations $(A, B, \mathcal{R}, E)$ are gauge dependent functions of (η, x^i).

The number of degrees of freedom of $g_{\mu\nu}$ are:

$$4\times 4 = 16 \xrightarrow{\text{symmetry}} 10 \longrightarrow 2\,(\text{propagating d.o.f.}) + 4\left(\begin{matrix}\text{gauge}\\ \text{freedom}\end{matrix}\right) + 4\left(\begin{matrix}\text{coordinate}\\ \text{freedom}\end{matrix}\right)$$

Under a general (gauge) transformation:

$$\tilde{\eta} = \eta + \xi^0(\eta, x^i)$$

$$\tilde{x} = x^i + \gamma^{ij}\xi_{|j}(\eta, x^i)$$

with arbitrary functions (ξ^0, ξ) the scalar and tensor perturbations transform, in linear order, as:

$$\tilde{A} = A - \xi^{0\prime} - \mathcal{H}\xi^0 \qquad \tilde{B} = B + \xi^0 - \xi'$$

$$\tilde{\mathcal{R}} = \mathcal{R} - \mathcal{H}\xi^0 \qquad \tilde{E} = E - \xi$$

$$\tilde{h}_{ij} = h_{ij} \quad (\text{invariant!})$$

where a prime denotes derivative with respect to conformal time.

Since we look for measurable quantities, we need to construct objects that do not depend on the gauge. It is possible to construct two gauge-invariant gravitational potentials (Barden potentials).

$$\boxed{\begin{aligned}\Phi &= A + (B - E')' + \mathcal{H}(B - E')\\ \Psi &= \mathcal{R} + \mathcal{H}(B - E')\end{aligned}}$$

Two point correlation functions and perturbations.

The end goal of perturbation analysis is to calculate the two point function for scalar potentials $\Phi \sim \mathcal{R}_K$, where $\mathcal{R}_K$ is the curvature perturbation (which will be discused later).

$$\langle 0 | \mathcal{R}^*_K \mathcal{R}_{K'} | 0 \rangle = \frac{|u_K|^2}{z^2} \delta^3(\vec{K}-\vec{K}') \equiv \frac{\mathcal{P}_\mathcal{R}(K)}{4\pi K^3} (2\pi)^3 \delta^3(\vec{K}-\vec{K}')$$

$$\mathcal{P}_\mathcal{R}(K) = \frac{K^3}{2\pi^2} \frac{|u_K|^2}{z^2} = \frac{\kappa^2}{2\varepsilon} \left(\frac{H}{2\pi}\right)^2 \left(\frac{K}{aH}\right)^{3-2\nu} \equiv A_S^2 \left(\frac{K}{aH}\right)^{n_S - 1} \longrightarrow \text{Primordial power spectrum}$$

↳ inflaton perturbation, we need a solution
$$\begin{cases} u \equiv a\,\delta\phi + z\Phi \\ z = a\dfrac{\dot\phi}{\mathcal{H}} \end{cases}$$

And similarly for tensor perturbations:

$$\sum_\lambda \langle 0 | h^*_{K\lambda} h_{K'\lambda} | 0 \rangle = \frac{8\kappa^2}{a^2} |V_K|^2 \delta^3(\vec{K}-\vec{K}') \equiv \frac{\mathcal{P}_g(K)}{4\pi K^3} (2\pi)^3 \delta^3(\vec{K}-\vec{K}')$$

$$\mathcal{P}_g(K) = 8\kappa^2 \left(\frac{H}{2\pi}\right)^2 \left(\frac{K}{aH}\right)^{3-2\mu} \equiv A_T^2 \left(\frac{K}{aH}\right)^{n_T}$$

Barden potentials and Einstein equations.

Barden potentials are related through the perturbed Einstein equations:

$$\Phi = \Psi \leftarrow \text{No isotropic stress}$$

$$\frac{K^2 - 3K}{a^2} \Psi = \frac{\kappa^2}{2} \delta\rho \qquad K \equiv \text{Curvature of the Universe}$$

where $\delta\rho$ is the gauge-invariant density perturbation. The latter equation is the expression of the Poisson equation for the gravitational potential, written in relativistic form.

During inflation, the energy density is given in terms of a scalar field, and thus the gauge-invariant equations for the perturbations on comoving hypersurfaces (constant energy density hypersurfaces) are (rest of Einstein + Klein Gordon):

$$\Phi'' + 3\mathcal{H}\Phi' + (\mathcal{H}' + 2\mathcal{H}^2)\Phi = \frac{\kappa^2}{2}[\phi'\delta\phi' - a^2 V'(\phi)\delta\phi]$$

$$-\nabla^2\Phi + 3\mathcal{H}\Phi' + (\mathcal{H}' + 2\mathcal{H}^2)\Phi = -\frac{\kappa^2}{2}[\phi'\delta\phi' + a^2 V'(\phi)\delta\phi]$$

$$\Phi' + \mathcal{H}\Phi = \frac{\kappa^2}{2}\phi'\delta\phi$$

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi = 4\phi'\Phi' - 2a^2V'(\phi)\Phi - a^2V''(\phi)\delta\phi \swarrow \text{Scalar field pert. equation}$$

This system of equations seem difficult to solve at first sight. However, there is a gauge invariant combination of variables that allows one to find exact solutions.

Let us define:

$$\boxed{\begin{aligned} u &\equiv a\delta\phi + z\Phi \\ z &\equiv a\frac{\phi'}{\mathcal{H}} \end{aligned}}$$

WARNING

$z \neq$ redshift

Under this redefinition, the above equations simplify enormously to just three independent equations:

$$u'' - \nabla^2 u - \frac{z''}{z} u = 0$$

$$\nabla^2 \Phi = \frac{\kappa^2}{2} \frac{\mathcal{H}}{a^2} (z u' - z' u)$$

$$\left(\frac{a^2 \Phi}{\mathcal{H}} \right)' = \frac{\kappa^2}{2} z u$$

This equations can be solved analytically, e.g. in Matter domination, or numerically in general for the classical system. From the first equation we can find a solution $u(z)$, which substituted into the last one can be integrated to give $\Phi(z)$, and together with $u(z)$ allow us to obtain $\delta\phi(z)$

$$u'' - \nabla^2 u - \frac{z''}{z} u = 0 \longrightarrow u(z) \longrightarrow \left(\frac{a^2 \Phi}{\mathcal{H}} \right)' = \frac{\kappa^2}{2} z u \longrightarrow \Phi(z) \longrightarrow \delta\phi(z)$$

Quantum mechanics in curved space-time

Until now, we have treated the perturbations as classical, but we should in fact consider the perturbations Φ and $\delta\phi$ as quantum fields. Note that the perturbed action for the scalar mode u can be written as:

$$\delta S = \frac{1}{2} \int d^3x \, d\eta \left[(u')^2 - (\nabla u)^2 + \frac{z''}{z} u^2 \right]$$

Kinetic term ↗ ↖ potential with time dependent mass term

In order to quantize the field u in the curved background defined by the metric we wrote before, we can use the operator:

$$\hat{u}(\eta, x) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \left[u_k(\eta)\, \hat{a}_k\, e^{i\vec{k}\cdot\vec{x}} + u_k^*(\eta)\, \hat{a}_k^{+}\, e^{-i\vec{k}\cdot\vec{x}} \right]$$

annihilation operator ↑ ↖ creation operator

where the creation and annihilation operators satisfy the commutation relation of bosonic fields, and the scalar field's Fock space is defined through the vacuum condition:

$$\left[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}^{+} \right] = \delta^3(\vec{k} - \vec{k}') \qquad \hat{a}_k |0\rangle = 0$$

The equations of motion for each mode $u_k(\eta)$ are decoupled in linear perturbation theory:

$$\boxed{u_k'' + \left(k^2 - \frac{z''}{z}\right) u_k = 0}$$

Mode equation

The ratio z''/z acts like a time-dependent potential for this Schrödinger-like equation.

Introduction of the slow roll parameters

In order to find exact solutions to the mode equation, we will use the slow-roll parameters:

$$\varepsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = \frac{k^2}{2}\frac{z^2}{a^2}$$

$$\delta = 1 - \frac{\phi''}{\mathcal{H}\phi'} = 1 + \varepsilon - \frac{z'}{\mathcal{H}z}$$

$$\xi = -\left(2 - \varepsilon - 2\delta + \delta^2 - \frac{\phi'''}{\mathcal{H}^2\phi'}\right)$$

In terms of these parameters, the conformal time and the effective potential for the u_k mode can be writen as:

$$\eta = \frac{-1}{\mathcal{H}} + \int \frac{\varepsilon\, da}{a\mathcal{H}}$$

$$\frac{z''}{z} = \mathcal{H}^2\left[(1+\varepsilon-\delta)(2-\delta) + \mathcal{H}^{-1}(\varepsilon' - \delta')\right]$$

Note that the slow-roll parameters evolve slowly. ε and δ can be taken as constants to order ε^2.

$$\varepsilon' = 2\mathcal{H}(\varepsilon^2 - \varepsilon\delta) = \mathcal{O}(\varepsilon^2)$$

$$\delta' = \mathcal{H}(\varepsilon\delta - \xi) = \mathcal{O}(\varepsilon^2)$$

In that case, for constant slow-roll parameters, we can write:

$$\eta = \frac{-1}{\mathcal{H}}\,\frac{1}{1-\varepsilon}$$

$$\frac{z''}{z} = \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right), \qquad \text{where} \quad \nu = \frac{1+\varepsilon-\delta}{1-\varepsilon} + \frac{1}{2}$$

Mode equation solutions (scalar perturbations)

Now we are going to search for approximate solutions of the mode equation, where the effective potential is of order $z''/z \simeq 2\mathcal{H}^2$ in the slow approximation. In quasi-de Sitter, there is a charasteristic scale given by the (event) horizon size or Hubble scale during inflation H^{-1}.

- There will be modes u_k with physical wavelengths much smaller than this scale, $k/a \gg H$, that are well within the de Sitter horizon and therefore do not feel the curvature of space-time $\quad \lambda \ll 1/H \longrightarrow k \gg aH$

- There will be modes with physical wavelengths much greater than the Hubble Scale, $k/a \ll H$. $\quad \lambda \gg 1/H \longrightarrow k \ll aH$

In these two asymptotic regimes, the solutions can be written as:

$$\boxed{\begin{cases} u_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} & k \gg aH \\ u_k = C_1 z & k \ll aH \end{cases}}$$

Proof for $k \gg aH$:

$$u''_k + \left(k^2 - \frac{z''}{z}\right) u_k = 0 \xrightarrow{k \gg aH} u''_k + k^2 u_k = 0 \longrightarrow u_k = \frac{1}{\sqrt{2k}} e^{-ik\eta}$$

Proof for $k \ll aH$:

$$u''_k + \left(k^2 - \frac{z''}{z}\right) u_k = 0 \xrightarrow{k \ll aH} u''_k - \frac{z''}{z} u_k = 0$$

$$\frac{z''}{z} = \frac{1}{\eta^2}\left(\nu - \frac{1}{4}\right) \longrightarrow z = C_1 \eta^{\frac{1}{2}-\nu} + C_2 \eta^{\frac{1}{2}+\nu} \qquad \text{(ODE for mass term)}$$

$$u_k = \tilde{C}_1 \eta^{\frac{1}{2}-\nu} + \tilde{C}_2 \eta^{\frac{1}{2}+\nu} \longrightarrow u_k = \alpha_1 z$$

In the limit $k \gg aH$ the modes behave like ordinary quantum modes in Minkowski space-time, approximately normalized, while the opposite limit, u/z becomes constant on superhorizon scales. For approximately constant slow-roll parameters one can find solutions to the mode equation that interpolate between the two asymptotic solutions:

$$\boxed{u_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\nu + 1/2)\pi/2} (-\eta)^{1/2} H_\nu^{(1)}(-k\eta)}$$

General solution

where $H_\nu^{(1)}(z)$ is the Hankel function of the first kind, and ν is given by $\nu = \frac{1+\varepsilon-\delta}{1-\varepsilon} + \frac{1}{2}$ (in terms of the slow-roll parameters). In the limit $k\eta \to 0$, the solution becomes:

$$|u_k| = \frac{2^{\nu - 3/2}}{\sqrt{2k}} \frac{\Gamma(\nu)}{\Gamma(3/2)} (-k\eta)^{\frac{1}{2}-\nu} = \frac{C(\nu)}{\sqrt{2k}} \left(\frac{k}{aH}\right)^{\frac{1}{2}-\nu}$$

$$C(\nu) = 2^{\nu - \frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} (1-\varepsilon)^{\nu - 1/2} \simeq 1 \quad \text{for } \varepsilon, \delta \ll 1$$

We can now compute Φ and $\delta\phi$ from the super-Hubble-scale mode solution ($k \ll aH$). Substituting into $\left(\frac{a^2\Phi}{\mathcal{H}}\right)' = \frac{k^2}{2} z u$, we find:

$$\boxed{\begin{aligned} \Phi &= C_1\left(1-\frac{\mathcal{H}}{a^2}\int a^2 d\eta\right) + C_2\frac{\mathcal{H}}{a^2} \\ \delta\phi &= \frac{C_1}{a^2}\int a^2 d\eta - \frac{C_2}{a^2} \end{aligned}}$$

The term proportional to C_1 corresponds to the growing solution, while that proportional to C_2 correspond to the decaying solution, which can soon be ignored. These quantities are gauge invariant but evolve with time outside the horizon, during inflation, and before entering again the horizon during the radiation or matter eras. We would like to write an expression for a gauge invariant quantity that is also constant for super horizon modes. Fortunatelly, in the case of adiabatic perturbations, there is such a quantity:

$$\left.\begin{aligned} \left(\frac{a^2\Phi}{\mathcal{H}}\right)' &= \frac{k^2}{2} z u \\ \varepsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} &= \frac{k^2}{2}\frac{z^2}{a^2} \end{aligned}\right\} \quad \zeta \equiv \Phi + \frac{1}{\varepsilon\mathcal{H}}\left(\Phi' + \mathcal{H}\Phi\right) = \frac{u}{z}$$

We can find a solution for Φ in radiation / matter domination:

$$\Phi = C_1\left(1-\frac{\mathcal{H}}{a^2}\int a^2 d\eta\right) \longrightarrow \Phi_k = \left(1-\frac{\mathcal{H}}{a^2}\int a^2 d\eta\right)\mathcal{R}_k$$

$$\begin{aligned} H^2 &= H_0^2\, a^{-3(1+\omega)} \\ \mathcal{H}^2/a^2 &= H_0^2\, a^{-3(1+\omega)} \\ a &\sim \eta^{2/(1+3\omega)} \end{aligned} \longrightarrow \boxed{\Phi_k = \frac{3+3\omega}{5+3\omega}\mathcal{R}_k = \begin{cases} \frac{2}{3}\mathcal{R}_k & \text{radiation era} \\ \frac{3}{5}\mathcal{R}_k & \text{matter era} \end{cases}}$$

One can calculate the initial curvature perturbation ζ (which is more or less constant in super horizon modes) and relate it to $\mathcal{R}_k$. We want to calculate the two-point function for ζ, which is related to de two-point function of Φ, and so to the two-point function of $\mathcal{R}_k$. This will be discussed in the Primordial Power spectrum section.

Tensor perturbations

Let us now compute the tensor or gravitational wave metric perturbations generated during inflation. The perturbed action for the tensor mode can be written as:

kinetic term

$$\delta S = \frac{1}{2}\int d^3x\, d\eta\, \frac{a^2}{2k^2}\left[(h'_{ij})^2 - (\nabla h_{ij})^2\right]$$

with the tensor field h_{ij} considered as a quantum field:

Fourier modes

$$\hat{h}_{ij}(\eta,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \sum_{\lambda=1,2} \left[h_k(\eta)\, e_{ij}(\vec{k},\lambda)\,\hat{a}_{\vec{k},\lambda}\, e^{i\vec{k}\cdot\vec{x}} + h.c.\right]$$

where $e_{ij}(\vec{k},\lambda)$ are two polarization tensors, satisfying symmetric, transverse and traceless conditions.

$$e_{ij} = e_{ji} \qquad k^i e_{ij} = 0 \qquad e_{ii} = 0$$

$$e_{ij}(-\vec{k},\lambda) = e^*_{ij}(\vec{k},\lambda) \qquad \sum_\lambda e^*_{ij}(\vec{k},\lambda)\, e^{ij}(\vec{k},\lambda) = 4$$

while the creation and annihilation operators satisfy the usual commutation relation of bosonic fields. We can now redefine our gauge invariant tensor amplitude as:

$$v_k(\eta) = \frac{a}{\sqrt{2}\,k}\, h_k(\eta)$$

which satisfies the following evolution equation, decoupled for each mode $v_k(\eta)$ in linear perturbation theory:

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0$$

The ratio a''/a acts like a time-dependent potential for this Schrödinger like equation analogous to the term z''/z for the scalar meter perturbation. For constant slow-roll parameters, the potential becomes:

$$\frac{a''}{a} = 2\mathcal{H}^2\left(1-\frac{\varepsilon^2}{2}\right) = \frac{1}{\eta^2}\left(\mu^2 - \frac{1}{4}\right)$$

$$\mu = \frac{1}{1-\varepsilon} + \frac{1}{2}$$

We can solve the equation for v_k in the two asymptotic regimes:

$$\boxed{\begin{array}{ll} v_k = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} & k \gg aH \\ v_k = C a & k \ll aH \end{array}}$$

In the limit $k \gg aH$ the modes behave like ordinary quantum modes in Minkowski spacetime approximately normalized, while in the opposite limit the metric perturbation h_k becomes constant on superhorizon scales.

Primordial power spectrum

Not only we do expect to measure the amplitude of the metric perturbations generated during inflation and responsible for the anisotropies in the CMB and density fluctuations in large scale structure, but we should also be able to measure its power spectrum, or two-point correlation function in Fourier space. Let us consider first the scalar metric perturbations $\mathcal{R}_k$, which enter the horizon at $a = k/H$. Its correlator is given by:

Different modes are decoupled

$$\langle 0 | \mathcal{R}^*_k \mathcal{R}_{k'} | 0 \rangle \equiv \frac{|u_k|^2}{z^2} \delta^3(\vec{k}-\vec{k}') \equiv \frac{\mathcal{P}_\mathcal{R}(k)}{4\pi k^3} (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$\boxed{\mathcal{P}_\mathcal{R}(k) = \frac{k^3}{2\pi^2} \frac{|u_k|^2}{z^2} = \frac{\kappa^2}{2\varepsilon} \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu} \equiv A_s^2 \left(\frac{k}{aH}\right)^{n_s-1}}$$

where $\mathcal{P}_\mathcal{R}$ is the Primordial power spectrum and we have used that $\mathcal{R}_k \approx \zeta_k = \frac{u_k}{z}$

NOTE

1. If $n_s = 1$, we have equal power on all scales (flat spectrum)
2. n_s is determined from the inflationary model (since it is related to the slow roll parameters).

$$n_s - 1 \equiv \frac{d \ln \mathcal{P}_\mathcal{R}(k)}{d \ln k} = 3 - 2\nu = 2\left(\frac{\delta - 2\varepsilon}{1-\varepsilon}\right) \simeq 2\eta_V - 6\varepsilon_V \qquad \text{(small)}$$

3. A_s is the amplitude of inflation perturbations

The primordial power spectrum might have a "running" term [evolution of the slow roll parameters):

$$\frac{dn_s}{d\ln k} = -\frac{dn_s}{d\ln \eta} = -\eta \mathcal{H} \left(2\xi + 8\varepsilon^2 - 10\varepsilon\delta\right) \simeq 2\xi_V + 24\varepsilon_V^2 - 16\eta_V \varepsilon_V$$

Similarly, for tensor perturbations, we can calculate the two-point correlation function and find the power spectrum.

$$\sum_\lambda \langle 0 | h^*_{k\lambda} h_{k'\lambda} | 0 \rangle = \frac{8\kappa^2}{a^2} |v_k|^2 \delta^3(\vec{k}-\vec{k}') \equiv \frac{\mathcal{P}_g(k)}{4\pi k^3} (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

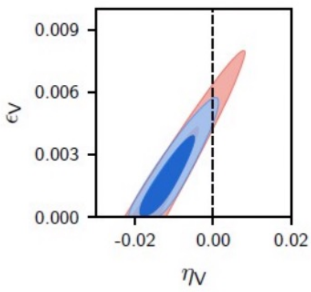
$$\mathcal{P}_g(k) = 8\kappa^2 \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\mu} \equiv A_T^2 \left(\frac{k}{aH}\right)^{n_T}$$

$$n_T \equiv \frac{d\ln \mathcal{P}_g(k)}{d\ln k} = 3 - 2\mu = \frac{-2\varepsilon}{1-\varepsilon} \simeq -2\varepsilon_V < 0$$

There will also be a running for the primordial power spectrum for tensors.

$$\frac{dn_T}{d\ln k} = -\frac{dn_T}{d\ln \eta} = -\eta \mathcal{H} \left(4\varepsilon^2 - 4\varepsilon\delta\right) \simeq 8\varepsilon_V^2 - 4\eta_V \varepsilon_V$$

Constrains on the slow roll parameters

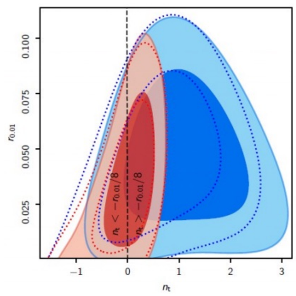


As it was discussed in the previous lecture, the CMB constrains the values of the slow roll parameters. We can define the amplitude of the scalar and tensor perturbations and the ratio between them (r).

In single field slow-roll models, $n_t \sim -r/8$, where $\boxed{r = \mathcal{P}_t / \mathcal{P}_s}$

$$\ln \mathcal{P}_s(k) = \ln \mathcal{P}_0(k) + \frac{1}{2}\frac{d\ln n_s}{d\ln k} \ln(k/k_*)^2 + \frac{1}{6}\frac{d^2 \ln n_s}{d\ln k^2} \ln(k/k_*)^3 + \dots$$

$$\ln \mathcal{P}_t(k) = \ln(r A_s) + n_t \ln(k/k_*) + \dots$$



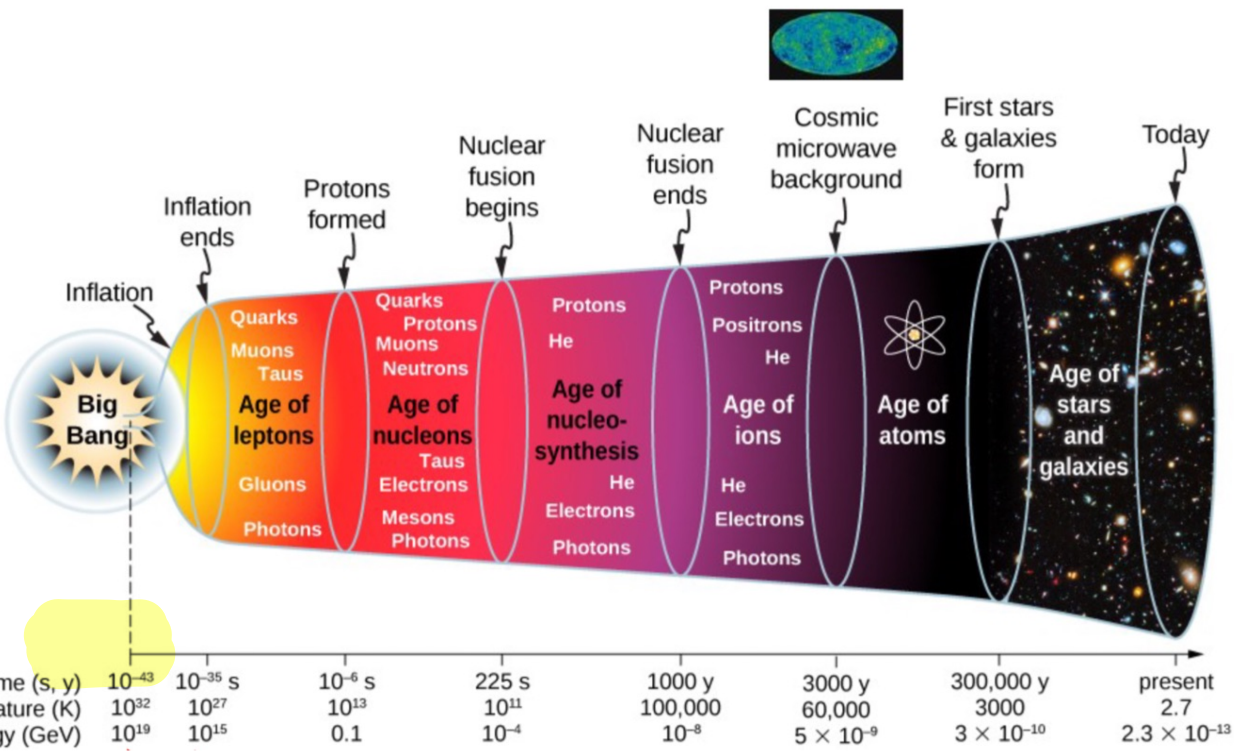
The Planck 2018 constrains on scalar parameters and the running are:

Parameter	TT,TE,EE+lowE+lensing
$\Omega_b h^2$	0.02237 ± 0.00015
$\Omega_c h^2$	0.1200 ± 0.0012
$100\theta_{MC}$	1.04092 ± 0.00031
τ	0.0544 ± 0.0073
$\ln(10^{10} A_s)$	3.044 ± 0.014
n_s	0.9649 ± 0.0042
H_0	67.36 ± 0.54
Ω_m	0.3153 ± 0.0073
σ_8	0.8111 ± 0.0060

Cosmological model	Parameter	*Planck* TT,TE,EE +lowEB+lensing	*Planck* TT,TE,EE +lowE+lensing+BK14	*Planck* TT,TE,EE +lowE+lensing+BK14+BAO
ΛCDM+r	r	< 0.11	< 0.070	< 0.070
	$r_{0.002}$	< 0.10	< 0.064	< 0.065
	n_s	0.9659 ± 0.0041	0.9653 ± 0.0041	0.9670 ± 0.0037
$+dn_s/d\ln k$	r	< 0.16	< 0.079	< 0.076
	$r_{0.002}$	< 0.16	< 0.077	< 0.072
	n_s	0.9647 ± 0.0044	0.9640 ± 0.0043	0.9658 ± 0.0038
	$dn_s/d\ln k$	-0.0085 ± 0.0073	-0.0071 ± 0.0068	-0.0065 ± 0.0066

6.2. Transplanckian physics and the power spectrum

There is a very brief window for inflation to act (after the Planck epoch), so it is easy to get above Planck scales (10^{19} GeV). At these scales, quantum gravity becomes important (and we do not have that theory).



Since we do not have a theory for quantum gravity, we can consider the operators that are not part of the standard model (because they break Lorentz invariance,...) but could be important in high energy physics (Planck scales). We distinguish between relevant (dim < d, restore symmetry at high energies) and irrelevant operators (dim > d, do not restore symmetry). The procedure is to consider various operators (quad in φ) and add them to the inflaton Lagrangian.

$\mathcal{D} \equiv (h^{\mu\nu}\nabla_\mu\nabla_\nu - Kn^\mu\nabla_\mu)^{1/2}$ Momentum projection operator (Lorentz invariance violating)

$\frac{1}{3}K\varphi^2, \quad \varphi\mathcal{D}\varphi \leftarrow$ Dimension 3

$\frac{1}{9}K^2\varphi^2, \ \frac{1}{3}K\varphi\mathcal{D}\varphi, \ -h^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi \leftarrow$ Dimension 4

$K_{\mu\nu}\,dx^\mu dx^\nu = -a^2 H\, d\vec{x}\cdot d\vec{x}$

Once they are added to the Lagrangian, it is possible to find their effect on the Primordial power spectrum (and thus, finding predictions on inflation).

$$\mathcal{L}_{NR} = \frac{d_1}{M}H^3\varphi^2 + \frac{d_2}{aM}H^2\varphi(-\vec{\nabla}\cdot\vec{\nabla})^{1/2}\varphi + \frac{d_3}{a^2M}H\vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + \frac{d_4}{a^3M}\varphi(-\vec{\nabla}\cdot\vec{\nabla})^{3/2}\varphi$$

$\downarrow$ obtain power spectrum

$$\langle O(\eta)|\varphi(\eta,\vec{x})\varphi(\eta,\vec{y})|O(\eta)\rangle = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}(\vec{x}-\vec{y})}\left[\frac{2\pi^2}{k^3}P_k(\eta)\right]$$

$$|O(\eta)\rangle = Te^{-i\int_{\eta_0}^{\eta}d\eta' H_I(\eta')}|0\rangle$$

$$H_I(\eta) = -\int d^3\vec{x}\sqrt{-g}\,\mathcal{L}_{NR}$$

The corrections introduced in the power spectum are:

$$K^3\varphi^2 \longrightarrow H^3\varphi^2 \Rightarrow P_k(\eta) = \frac{H^2}{4\pi^2}\left[1+\frac{4}{3}d_1\frac{H}{M}[\ln|2k\eta| - 2+\gamma] + \dots\right] = \frac{H^2}{4\pi^2}\frac{2^\nu\Gamma(\nu)}{2\pi}|k\eta|^{3-2\nu} + \dots$$

$$K^2\varphi\mathcal{D}\varphi \longrightarrow H^2\varphi(-\vec{\nabla}\cdot\vec{\nabla})^{1/2}\varphi \Rightarrow P_k(\eta) = \frac{H^2}{4\pi^2}\left[1+d_2\frac{H}{M}\left[\pi+\frac{\cos(2k\eta_0)}{k\eta_0}\right]+\dots\right]$$

$$Kh^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi \longrightarrow H\vec{\nabla}\varphi\vec{\nabla}\varphi \Rightarrow P_k(\eta) = \frac{H^2}{4\pi^2}\left[1+d_3\frac{H}{M}\left[3+\cos\left(2\frac{k}{k_*}\frac{M}{H}\right)\right]+\dots\right]$$

$$\varphi\mathcal{D}^3\varphi \longrightarrow \varphi(-\vec{\nabla}\cdot\vec{\nabla})^{3/2}\varphi \Rightarrow P_k(\eta) = \frac{H^2}{4\pi^2}\left[1-d_4\frac{k}{k_*}\cos\left(2\frac{k}{k_*}\frac{M}{H}\right)+\dots\right]$$

If we get to measure any of this terms, it would mean that inflation starts a bit above the Planck scale.

6.3. Primordial black holes and inflation

Dark matter and primordial BH.

As it was discussed in the first lecture, roughly a 27% of the Universe is dark matter, which is collissionless and non relativistic. We only know that it interacts gravitationally. All the evidences for its existence are indirect (gravitational lensing, galactic rotation curves, ...)

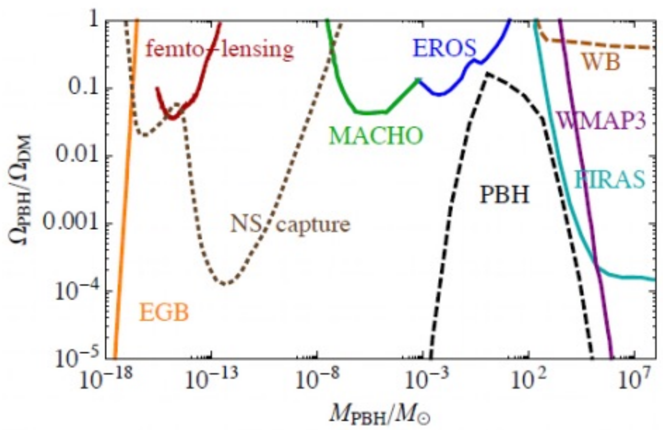
There are many "candidates" to be dark matter, like axions, WIMPS, neutralinos, etc, but nowadays primordial black holes are becoming suitable candidates. These are quantum fluctuations that are really strong, collapse and form a blackhole. They would behave like DM because they are collissionless, numerous and non relativistic.

Primordial BH

This is an old idea (García-Bellido, Linde and Wands 1996) that became hot recently after GWs discovery. PBH are formed after inflation when broad peaks in the primordial curvature power spectrum P(k) collapsed gravitationally during the radiation era and formed clusters of BH that merge and increase in mass after recombination until today. Masses range from 0.01 to 10^5 $M_\odot$ and could jump-start structure formation.

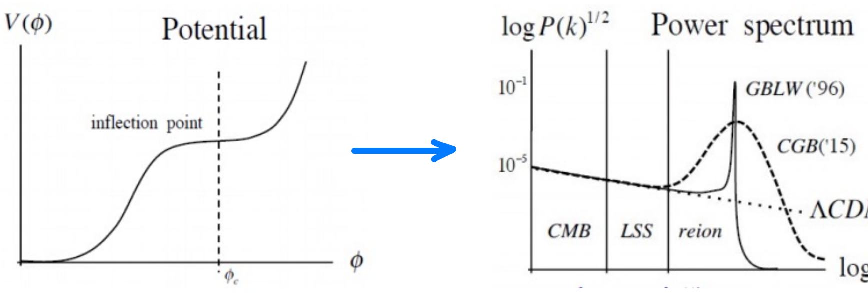
PBHs could make up almost all DM with a non-monochromatic distribution.

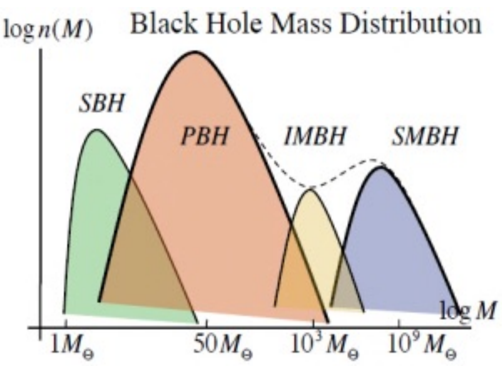
Based on experimental constrains:



Other constains come from extragalactic photon background (orange), femtolensing (red), microlensing by MACHO (green) and EROS (blue), from wide binaries (light brown), and CMB distortions by FIRAS (cian) and WMAP3 (purple).

Peaks in the spectrum can be formed by inflection points in the potential ($P \sim 1/\varepsilon$, $\varepsilon \to 0$ $\to P \gg 1$)

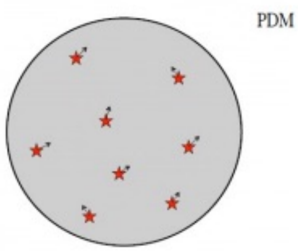


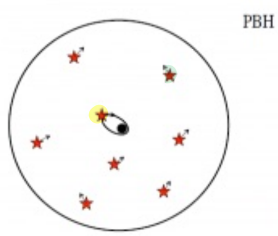


PBHs can have a range of masses (are not "mono-chromatic"). They would fill the gap between stellar BH and Super massive black holes in the centre of galaxies

They have many potential signatures and side effects.

- Since they were created by curvature perturbations, PBH have no spin.
- Emissions of GW in binaries and hyperbolic encounters
- Microlensing of type Ia Supernovae : Possible explanation for superluminal SnIa (or super-Chandrasekhar).
- Missing baryon problem : PBH might have eaten up the baryons
- Stochastic background of GW : uniform distribution of GW sources creates a background that could be visible by LISA
- Anomalous motion of stars : compare PDM vs. PBH-DM (could be seen by GAIA)





Orbiting around "nothing"

Not orbiting