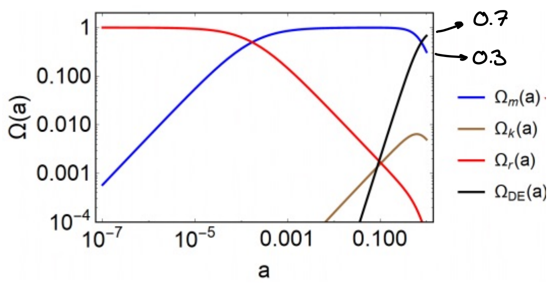
V. Inflation

5.1. The hot Big Bang and why we need Inflation.

Big Bang theory predictions

Initial singularity

We start studying the evolution of density parameters : $\Omega_i(a) = \dfrac{\Omega_{i,0}\, a^{-3(1+\omega)}}{H(a)^2}$



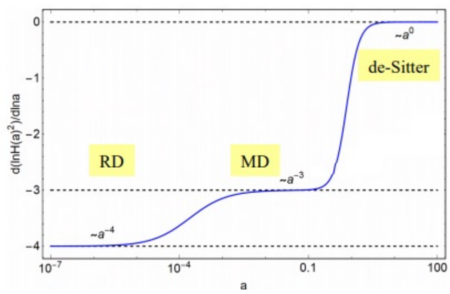
Starting with the values of the parameters measured today, we can use the Friedmann equation to study its evolution (propagating backwards in time). We observe that radiation dominated in the the Early Universe.

Since we know its density scales as $\rho_r(a) \propto a^{-4}$, Then when $a \to 0$ we have $\rho_r \to \infty$: singularity. And for the temperature: $\rho_r \sim T^4 \Rightarrow T(a \to 0) \to \infty$

This is also related to the emission of the CMB: it was formed when the density of the Universe was low enough for photons to travel freely.

Structure formation

Structure formation happens during the matter dominated era (as was predicted by the Big Bang Theory). However, this theory does not justify the seeds that created the large scale structure (we know when it is formed, but not how).



Problems in the Big Bang theory (and solutions)

Horizon problem

An expanding universe has particle horizons, that is, spatial regions beyond which causal communication cannot occur. The horizon distance can be defined as the maximum distance that light could have travelled since the origin of the Universe:

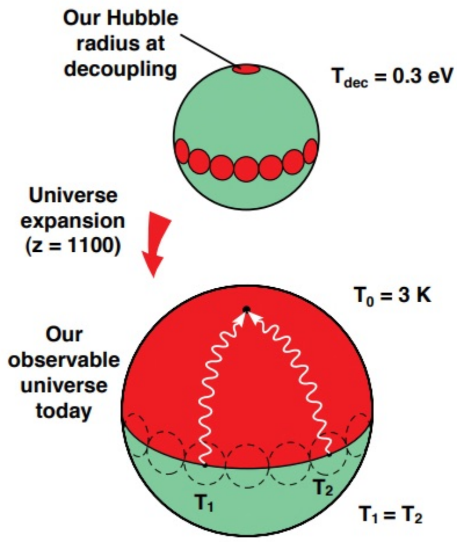
$$r_H = \int_0^t \frac{c\,dt}{R(t)} = a(t)\int_0^t \frac{dt'}{a(t')} \sim H^{-1}(t) \qquad \text{(comoving coordinates)}$$

In radiation domination, $a(t) \propto t^{1/2}$, so at late times:

$$D_H = R_0\, r_H \simeq \frac{6000}{\sqrt{\Omega_r}}\, h^{-1}\ \text{Mpc}$$

For instance, at the beginning of nucleosynthesis, the horizon distance is a few light-seconds, but grows linearly with time, and by the end of nucleosynthesis it is a few light-minutes, i.e. a factor 100 larger, while the scale factor has only increased a factor of 10 ($d_H \propto t$, $a \propto t^{1/2}$). The fact that the causal horizon increases faster than the scale factor implies that at any given time the Universe contains regions within itself that, according to the Big Bang theory, were never in causal contact before.

This is particularly sharp in the case of the observed cosmic microwave background (CMB). Information cannot travel faster than the speed of light, so the casual region at the time of photon decoupling could not be larger than $d_H(t_{dec}) \sim 3 \times 10^5$ light years across, or about 1°. However, we measure that regions that are separated by more than 1° degree have the same temperature up to $\sim 10^{-5}$ K variations, even if they are not supposed to be in causal contact when they were emitted.



This constitutes the so-called horizon problem.

This problem can be solved assuming a phase of accelerated expansion ($\omega \to -1$), where the scale factor grows exponentially.

$$\boxed{a(t) \simeq e^{H_I (t - t_{end})}}$$

and the horizon is: $D_H = e^N$, where N is defined as the number of e-folds:

$$\boxed{N = -\int d\ln a = \int H\, dt}$$

The horizon problem can be solved for $N \sim 60$.

Flatness problem

If we consider the sum of the densities today (without considering curvature), we get:

$$1 - \Omega_0 = -\frac{K^2 c^2}{R_0^2 H_0^2} < 0.01 \longrightarrow \text{Curvature} < 1\%$$

$\hookrightarrow$ observations

Considering its evolution with the history of the Universe:

$$1 - \Omega = -\frac{Kc^2}{R_0^2 a(t)^2 H(t)^2} \longrightarrow \boxed{1 - \Omega(t) = \frac{H_0^2 (1 - \Omega_0)}{H^2(t)\, a^2(t)}}$$

Assuming matter domination: $a \sim t^{2/3}$

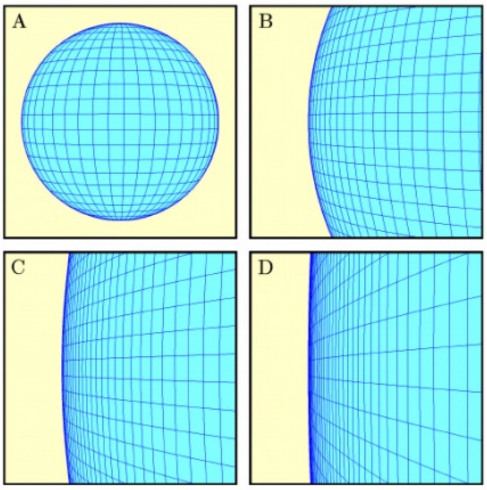
$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} \longrightarrow 1 - \Omega(t) = \frac{(1 - \Omega_0)\, a^2}{\Omega_{r,0} + \Omega_{m,0}\, a} = \frac{1 - \Omega_0}{\Omega_{m,0}} \left(\frac{t}{t_0} \right)^{2/3}$$

Which is an unstable solution: it is an increasing function of time, so it requires very fine-tuned initial conditions to obtain the values that we observe today.

This can be solved again with inflation:

$$H = \text{const} \longrightarrow a \sim \exp(Ht)$$

$$\frac{a(t_f)}{a(t_i)} = e^N \longrightarrow N = H_i(t_f - t_i)$$



Then:

$$1 - \Omega(t) = \frac{c^2}{R_0^2 a^2 H^2} \longrightarrow |1 - \Omega(t_f)| = \exp(-2N)|1 - \Omega(t_i)|$$

which asymptotically goes to zero when N grows. Even starting with a very curved universe (A), it can become as flat as we want after a certain number of e-folds (D).

Monopole problem

Topological defects (like monopoles, cosmic strings, domain walls, ...) coming from GUTs ($\sim 10^{16}$ GeV) are created before inflation, finding at least one per horizon (distance).

Until know, we have not found anything like this (neither gravitational waves, nor traces in the CMB).

If there were any monopoles, according to GUT theories its contribution should be dominant:

$$\Omega_{mon}^{(0)} = \frac{M}{3H_0^2 M_{Pl}^2 [D_{mon}^{(0)}]^3} \simeq 10^{15} \longrightarrow \text{the Universe would collapse on itself}$$

Assuming that there was 1 monopole/horizon (before inflation), there would be diluted since the horizon expanded like $e^{60} \sim 10^{27}$

Origin of large scale structure

This will be discussed in further lectures, but inflation explains where did the seeds of large scale structure came from.

5.2. Scalar field models (and other curiosities).

Basics of Inflation model-building

First, we consider a phase of exponential expansion (de-Sitter). To find which componen of the Universe could produce that, we use the 2nd Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho(\alpha) + 3P(\alpha)) \Longrightarrow P < -\frac{\rho}{3} \quad \text{(condition for accelerated expansion)}$$

$$P = \omega\rho \begin{cases} \omega = 0 & \text{Non relativistic matter} \quad P \ll \rho \\ \omega = \frac{1}{3} & \text{Relativistic matter} \quad P = \frac{1}{3}\rho \\ \omega < -\frac{1}{3} & \longrightarrow \quad P < -\frac{\rho}{3} \; \text{(Expansion)} \longrightarrow \text{Cosmological constant } (\omega = -1) \end{cases}$$

The Cosmological constant can produce the expansion we desire, but there is no way to stop that expansion: the de-Sitter phase never ends, and dilutes everything. This results in an empty Universe, which would need a reheating process (since after inflation we are at radiation domination era.

Model conditions

Possible inflationary models must:

1. Solve "classical" problems: (horizon, flatness, monopole, etc).
2. End before radiation epoch, and be followed by reheating to create particles.
3. Set the initial conditions for large scale structure.
4. Make unique and testable predictions
5. Be motivated from high-energy physics (standard model or quantum gravity)

Scalar field inflation

One of the first approaches is adding a scalar field to the lagrangian. Scalar fields (bosons with spin 0) have already been observed, like the Higgs boson. They are already used in Dark Energy (which is also an accelerated phase) and their dynamics are very well understood.

Starting with the lagrangian of general relativity:

$$S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, R + S_m$$

we add the lagrangian of the scalar field:

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2\kappa^2} R + \mathcal{L}_\phi\right] + \underbrace{S_M}_{\text{standard model}}$$

$$\mathcal{L}_\phi = \underbrace{-\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi}_{\text{Kinetic term}} - \underbrace{V(\phi)}_{\text{Potential}}$$

From this lagrangian one can calculate the energy-momentum tensor (varying $\mathcal{L}$ with respect to $g^{\mu\nu}$):

$$T_{\mu\nu}^{(\phi)} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\,\mathcal{L}_\phi)}{\delta g^{\mu\nu}} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2} g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + V(\phi)\right]$$

Equation of state

From the energy-momentum tensor of the Scalar fields, one can write on its density and pressure:

$$P_\phi = \frac{1}{3} T_i^{i(\phi)} = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$$\rho_\phi = -T_0^{0(\phi)} = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$\xrightarrow[\text{state}]{\text{Equation of}} \boxed{w_\phi \equiv \frac{P}{\rho} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}}$$

In quintessence (the model that we are using for $\mathcal{L}_\phi$), $w(z)$ cannot cross -1. Using the continuity equation:

$$\dot{\rho}_\phi + 3H(\rho_\phi + P) = 0 \qquad \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P(\rho))$$

When $w \to -1$, $P(\rho) \to -\rho$: $\dot{\rho} \to 0$

Thus, $\lim_{w\to -1} \frac{d^n \rho(t)}{dt} = 0$ (higher order derivatives). $\Longrightarrow$ $w(z)$ goes asymptotically to -1

This can be used to discriminate simple scalar fields from more complicated modes (modifications of gravity, etc.).

Equations of motion

We can also analyse the equations of motion. Using the Friedmann equations:

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_M\right] \qquad \dot{H}^2 = -\frac{\kappa^2}{2}(\dot{\phi}^2 + \rho_M + P_M) \qquad \kappa^2 = 8\pi G$$

As well as the Klein-Gordon equation in an expanding Univere, which describes the dynamics of the field, and the continuity equation.

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \quad , \quad V_{,\phi} \equiv dV/d\phi$$

$$\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0$$

Some example models that will be discussed later are:

Freezing models:

- $V(\phi) = M^{4+n}\phi^{-n} \qquad (n>0)$
- $V(\phi) = M^{4+n}\phi^{-n}\exp(\alpha\phi^2/m_{pl}^2)$

Thawing models:

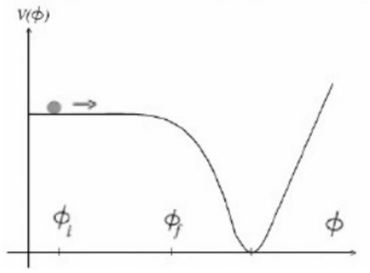
- $V(\phi) = V_0 + M^{4-n}\phi^n \qquad (n>0)$
- $V(\phi) = M^4\cos^2(\phi/f)$

One can reconstruct the potential from the Friedmann equation if $E(z) = H(z)/H_0$ is known:

$$\left.\begin{aligned} H^2 &= \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_M\right] \\ \dot{H} &= -\frac{\kappa^2}{2}(\dot{\phi}^2 + \rho_M + P_M) \end{aligned}\right\} \rightarrow \left\{\begin{aligned} \frac{\kappa^2}{2}\left(\frac{d\phi}{dz}\right)^2 &= \frac{1}{1+z}\frac{d\ln E(z)}{dz} - \frac{3\Omega_m^{(0)}}{2}\frac{1+z}{E^2(z)} \geq 0 \\ \frac{\kappa^2 V}{3H_0^2} &= E(z) - \frac{1+z}{6}\frac{dE^2(z)}{dz} - \frac{1}{2}\Omega_m^{(0)}(1+z)^3 \end{aligned}\right.$$

The condition for reconstruction is: $\frac{dH^2}{dz} \geq 3\Omega_m^{(0)} H_0^2 (1+z)^2 \longrightarrow \rho_\phi + P_\phi \geq 0$ (weak energy condition)

Slow roll inflation



Let us study how a generic potential satisfies all the previous constrains. This is related to the "slow roll" idea introduced by Lind et al. If the potential is flat, the model behaves like a cosmological constant (the Kinetic term will be ~ 0).

Now we can work with the equations of the model. Matter has not been created yet, so we can cross all the terms associated to it (/). Since it is slowly rolling, we can ignore the derivatives of ϕ:

$$H^2 = \frac{\kappa^2}{3}\left[\cancel{\frac{1}{2}\dot{\phi}^2} + V(\phi) + \cancel{\rho_M}\right]$$

$$\dot{H}^2 = -\frac{\kappa^2}{2}\left(\dot{\phi}^2 + \cancel{\rho_M} + \cancel{P_M}\right) \quad \Rightarrow$$

$$\cancel{\ddot{\phi}} + 3H\dot{\phi} + V_{,\phi} = 0$$

$$\boxed{\begin{aligned} 3H^2 &= \rho \simeq V \\ -2\dot{H} &= (P+\rho) = \dot{\phi}^2 \\ 3H\dot{\phi} &\simeq -\frac{\partial V}{\partial \phi} \end{aligned}}$$

Slow roll parameters and inflation predictions.

We can describe an inflation model in terms of its slow roll parameters. There are various definitions depending on the literature.

> **NOTE**
> log derivatives quantify the "slope" of functions

$$\varepsilon_1 \equiv -\frac{\dot{H}}{H^2} \simeq \frac{1}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \equiv \varepsilon_V$$

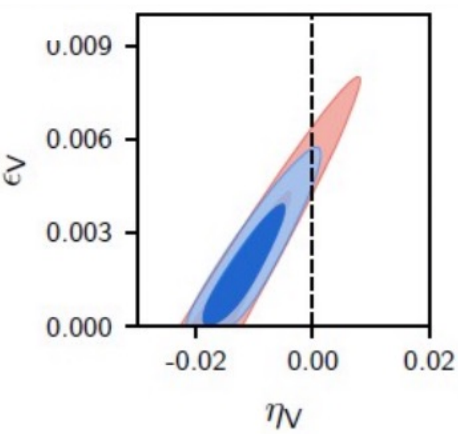
$$\varepsilon_2 \equiv \frac{d\ln(\varepsilon_1)}{dN} \simeq -2\frac{V_{,\phi\phi}}{V} + 2\left(\frac{V_{,\phi}}{V}\right)^2 \equiv -2\eta_V + 4\varepsilon_V$$

> **NOTATION**
> $V_{,\phi} = \partial V/\partial\phi$

or

$$\varepsilon = \frac{2}{\kappa^2}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 \simeq \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \equiv \varepsilon_V \ll 1$$

$$\delta = \frac{2}{\kappa^2}\frac{H''(\phi)}{H(\phi)} \simeq \frac{1}{\kappa^2}\frac{V''(\phi)}{V(\phi)} - \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \equiv \eta_V - \varepsilon_V \ll 1$$



Having a set of parameters, one can compare them to observations to get constrains on their values. For example, η_V and ε_V were measured using the anisotropies on the CMB. This is possible because the scalar field seeds scalar metric perturbations:

$$\langle 0|\mathcal{R}^*_k \mathcal{R}_{k'}|0\rangle = \frac{\mathcal{P}_\mathcal{R}(k)}{4\pi k^3}(2\pi)^3\,\delta^3(\vec{k}-\vec{k}') \quad \leftarrow \text{Spectrum of perturbations}$$

$$\mathcal{P}_\mathcal{R}(k) = \frac{\kappa^2}{2\varepsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \equiv A_s^2\left(\frac{k}{aH}\right)^{n_s-1}$$ where A_s is the amplitude of the perturbations.

$\nu = \dfrac{1+\varepsilon-\delta}{1-\varepsilon} + \dfrac{1}{2}$, ν wave number

The spectral index n_s is a prediction of inflation.

$$n_s - 1 \equiv \frac{d \ln \mathcal{P}_{\mathcal{R}}(k)}{d \ln k} = 3 - 2\nu = 2\left(\frac{\delta - 2\varepsilon}{1-\varepsilon}\right) \simeq 2\eta_V - 6\varepsilon_V , \qquad n_s - 1 \sim 0$$

Since $V(\phi)$ is not flat, the primordial power spectum might have a "running", i.e. higher order, prediction:

$$\frac{dn_s}{d\ln k} = -\frac{dn_s}{d\ln \eta} = -\eta \mathcal{H}\left(2\xi + 8\varepsilon^2 - 10\varepsilon\delta\right) \simeq 2\xi_V + 24\varepsilon_V^2 - 16\eta_V \varepsilon_V$$

The salar field also seeds tensor metric perturbations (which will be discussed in further lectures).

$$\sum_k \langle 0 | h_k^* \, h_{k'} | 0 \rangle = \frac{8\kappa^2}{a^2} |v_k|^2 \delta^3(\vec{k} - \vec{k}') \equiv \frac{\mathcal{P}_g(k)}{4\pi k^3} (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$\mathcal{P}_g(k) = 8\kappa^2 \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\mu} \equiv A_T \left(\frac{k}{aH}\right)^{n_T}$$

$$\longrightarrow n_T = \frac{d\ln \mathcal{P}_g(k)}{d\ln k} = 3 - 2\mu = \frac{-2\varepsilon}{1-\varepsilon} \simeq -2\varepsilon_V < 0$$

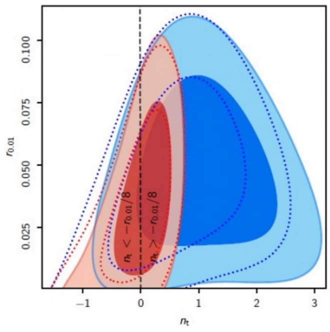
Primordial power spectrum for tensors might also have a "running":

$$\frac{dn_T}{d\ln k} = -\frac{dn_T}{d\ln \eta} = -\eta \mathcal{H}\left(4\varepsilon^2 - 4\varepsilon\delta\right) \simeq 8\varepsilon_V^2 - 4\eta \varepsilon_V$$

In single field slow-roll models, $n_t \sim -r/8$, where $r = \mathcal{P}_t / \mathcal{P}_s$

$$\ln \mathcal{P}_s(k) = \ln \mathcal{P}_0(k) + \frac{1}{2}\frac{d\ln n_s}{d\ln k} \ln(k/k_*)^2 + \frac{1}{6}\frac{d^2 \ln n_s}{d\ln k^2} \ln(k/k_*)^3 + \dots$$

$$\ln \mathcal{P}_t(k) = \ln(r A_s) + n_t \ln(k/k_*) + \dots$$



Constrains based on CMB measurements

An additional parameter is the number of e-folds, which tell us how many times does the Universe has grown exponentially (until the end of inflation, when ϕ reaches the end of the plateau: $\varepsilon = 1$).

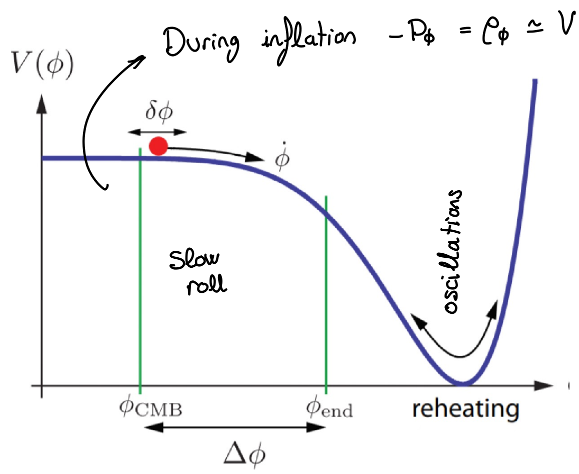
$$N(t) \equiv -\int_{a_f}^{a} d\ln \hat{a} = -\int_{t_f}^{t} H(\hat{t})\, d\hat{t} \simeq \int_{\phi_f}^{\phi} \frac{V(\hat{\phi})}{V_{,\phi}(\hat{\phi})} d\hat{\phi}$$

Reheating

After inflation, Universe is empty and cold, it is necessary to reheat it. This can be done by friction term in Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

During the oscillations, the scalar field releases energy in the Universe and converts energy from inflation to SM particles.



Example calculations of slow roll parameters

Let us consider a simple exponential toy model:

$$V(\phi) = V_0 e^{\lambda \kappa^2 \phi^2}$$

We can calculate the slow roll parameters as:

$$\varepsilon = \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 = 2\kappa^2\lambda^2\phi^2$$

$$\eta = \frac{1}{\kappa^2}\left(\frac{V''(\phi)}{V(\phi)}\right) = 2\lambda\left(2\kappa^2\lambda\phi^2 + 1\right)$$

The end of inflation happens when $\varepsilon = 1$, thus:

$$\phi_{end} = \frac{1}{\sqrt{2}\,\kappa\lambda} \quad \text{(Keeping the positive branch)}$$

Then, the number of e-folds is:

$$N_{inf} = \int_{\phi_{end}}^{\phi} \kappa \frac{1}{\sqrt{2\varepsilon(\phi)}}\, d\phi = \frac{\log\left(2(\kappa\phi\lambda)^2\right)}{4\lambda} \Rightarrow \phi(N) = \frac{e^{2\lambda N_{inf}}}{\sqrt{2}\,\kappa\lambda}$$

Now N will be our variable. We can write the slow roll parameters as a function of the number of e-folds:

$$\varepsilon = e^{4\lambda N_{inf}}$$

$$\eta = 2\left(\lambda + e^{4\lambda N_{inf}}\right)$$

From those parameters we can derive the inflation predictions:

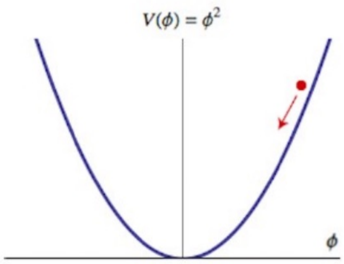
$$\left.\begin{aligned} n_s &= \eta - 6\varepsilon + 1 = 4\lambda - 2e^{4\lambda N_{inf}} + 1 \\ r &= 16\varepsilon = 16 e^{4\lambda N_{inf}} \end{aligned}\right\}$$

Cosmological model $\Lambda CDM+r$	Parameter	*Planck* TT,TE,EE +lowEB+lensing
	r	< 0.11
	$r_{0.002}$	< 0.10
	n_s	0.9659 ± 0.0041

From the values obtained from Planck observations, we know that $n_s \sim 0.96$. From here, we can get the number of e-folds as a function of λ. Additional constrains on the parameters can be obtained from the value of r. This helps us to accept or rule out models

Specific models

Chaotic inflation

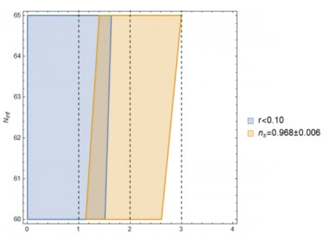


Proposed by Linden et al. in ~1980.

The potential goes as:

$$V(\phi) = \Lambda^{4-n}\phi^n$$

Allowed parameters are shown on the right.

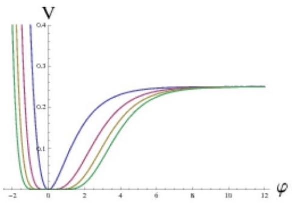


It is not exactly slow rolling: while going down the potential there is some friction, which will result in the loss of energy, and so a reheating.
The parameters of the model are given by:

$$\left.\begin{aligned} \varepsilon &= \frac{n^2}{2\kappa\phi^2} = \frac{n}{n+4N_{inf}} \\ \eta &= \frac{(n-1)n}{\kappa^2\phi^2} = \frac{2(n-1)}{n+4N_{inf}} \end{aligned}\right\} \quad \begin{aligned} n_s &= 1 - \frac{2(n+2)}{n+4N_{inf}} \\ r &= \frac{16n}{n+4N_{inf}} \end{aligned}$$

$$N_{inf} = \int_{\phi_{inf}}^{\phi} d\phi \; \kappa \frac{1}{\sqrt{2\varepsilon}} = \frac{\kappa^2\phi^2}{2n} - \frac{n}{4}$$

Plateau models



Proposed by Stewart et al in 1995
The potential goes as:

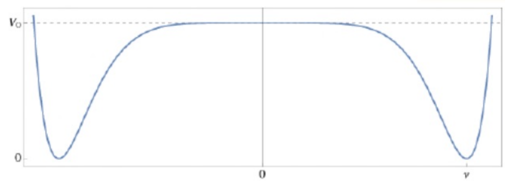
$$V(\phi) = \Lambda^4[1 - \exp(-\gamma\kappa\phi)]^2$$

(Flat and slowly decays to the vacuum).

The slow roll and observables give:

$$N \simeq \frac{\exp(\gamma\phi)}{2\gamma^2} \Longrightarrow n_s = 1 - \frac{2}{N} , \quad r = \frac{8}{\gamma^2 N^2}$$

Hilltop models



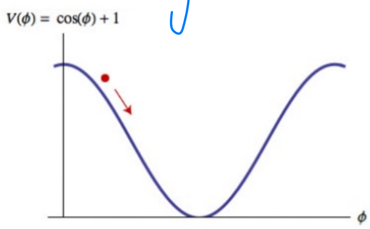
Proposed by Boubekeur et al (2005)
The potential goes as:

$$V(\phi) = \Lambda^4\left[1 - \left(\frac{\phi}{v}\right)^p\right]^2$$

The scalar field starts at the top of the hill, falls to one of the sides and evaporates there.
The relevant quantities are:

$$N \simeq \frac{\kappa^2 v^2}{2p(p-2)}\left(\frac{\phi}{v}\right)^{2-p} \Longrightarrow n_s \simeq 1 - \frac{2(p-1)}{(p-2)N} \qquad r \simeq \frac{32p^2}{\kappa^2 v^2}\left[\frac{2p(p-2)}{\kappa^2 v^2} N\right]^{\frac{2p-2}{2-p}}$$

Natural inflation



Proposed by Freese et al. (1993)
The potential goes as:

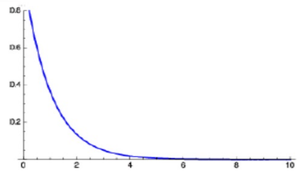
$$V(\phi) = \Lambda^4\left[1 + \cos\left(\frac{\phi}{v}\right)\right]$$

The scalar field falls and starts oscillating.

The observables are:

$$N \simeq -2\kappa^2 v^2 \ln\left[\sin\left(\frac{\phi}{2v}\right)\right] \longrightarrow n_s \simeq 1 - \frac{1}{\kappa^2 v^2}\frac{\exp\left(\frac{N}{\kappa^2 v^2}\right)+1}{\exp\left(\frac{N}{\kappa^2 v^2}\right)-1} , \quad r \simeq \frac{8}{\kappa^2 v^2}\exp\left[\left(\frac{N}{\kappa^2 v^2}\right)-1\right]^{-1}$$

Power law inflation



(Zucchin, 1985)

$V(\phi) = \Lambda^4 \exp(-\lambda \kappa \phi)$

Observables are independent from N:

$n_s = 1 - \lambda \qquad r = 8\lambda^2$

Since it will always stay in a flat plateau, it is important to make sure that they don't get to dominate the Universe. Parameters must be well tuned to make it either evaporate or decay into the standard model.

K-essence inflation

K-essence models can have scalar fields with generic kinetic terms $(X, X^2, \dots)$.

$$X \equiv -\frac{1}{2}(\partial\phi)^2 \longrightarrow S = \int d^4x \sqrt{-g}\, P(\phi, X) \longrightarrow S_E = \int d^4x \sqrt{-g}\left[\frac{1}{2}R + K(\phi)X + L(\phi)X^2 + \dots\right]$$

We can calculate $T_{\mu\nu}$ and the equation of state as before:

$$T_{\mu\nu}^{(\phi)} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}} = P_{,X}\,\partial_\mu\phi\,\partial_\nu\phi + g_{\mu\nu}P \qquad \begin{cases} P_\phi = P \\ \rho_\phi = 2XP_{,X} - P \end{cases}$$

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{P}{2XP_{,X} - P}$$ ← can cross $w = -1$: even more aggressive expansion (less N)

Modified gravity

Instead of explaining inflation with an scalar field, which is an extra particle that nobody knows where it came from, one can say that there is a modification in gravity, some more general theory that solves the problems with GR.

The simplest thing that can be added to GR (on LHS) is $R \to f(R)$. This is just a scalar degree of freedom that has be used in Dark Energy models. Dynamics are well understood, and it has been shown that it has a very reach phenomenology. This model is inspired on high energy physics.

$$S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, R + S_m \longrightarrow S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, f(R) + S_m$$

The simplest example of $f(R)$ is ΛCDM:

$$f(R) \simeq f(R_0) + f'(R_0)\, R + \dots$$

$$S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, f(R) + S_m \longrightarrow S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\,(R - 2\Lambda) + S_m$$

High energy physics motivate this theory because new terms appear when one tries to renormalize GR at one-loop order: [Birrel & Davis, 1986]

$$R \Rightarrow R + \alpha\left[\frac{1}{180} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{1}{180} R_{\mu\nu}R^{\mu\nu} - \frac{1}{6}\left(\frac{1}{5}-\xi\right)\Box R + \frac{1}{2}\left(\frac{1}{6}-\xi\right)^2 R^2 + \dots\right]$$

The most general (pure) modified gravity is of the form:

$$R \Rightarrow f(R, P, Q, \Box^n, G) \quad \begin{cases} R = g_{\mu\nu}R^{\mu\nu} \\ P = R_{\mu\nu}R^{\mu\nu} \\ Q = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \\ \Box = g^{\mu\nu}\nabla_\mu\nabla_\nu \leftarrow \text{D'Alembertian in curved space} \\ G = Q - 4P + R^2 \leftarrow \text{Gauss-Bonet term (topological invariant in 4D)} \end{cases}$$

$f(R)$ models

We can get the $f(R)$ equations of motion by varying the action with respect to the metric ($F = f'(R)$)

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\, f(R) + S_m$$

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu} \;\Leftrightarrow\; \tilde{g}^{\mu\nu} = g^{\mu\nu} - \delta g^{\mu\nu}$$

$$\begin{aligned}
\delta\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}g^{\lambda a}\left(\nabla_\mu\delta g_{a\nu} + \nabla_\nu\delta g_{a\mu} - \nabla_a\delta g_{\mu\nu}\right)\\
\delta R^\nu_{k\lambda a} &= \nabla_\lambda\delta\Gamma^\nu_{ka} - \nabla_a\delta\Gamma^\nu_{k\lambda}\\
\delta R_{\mu\nu} &= \frac{1}{2}\left(-\Box\delta g_{\mu\nu} + \nabla_a\nabla_\mu\delta g^a_\nu + \nabla_a\nabla_\nu\delta g^a_\mu - \nabla_\mu\nabla_\nu\delta g^a_a\right)\\
\delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}\,g_{ab}\delta g^{ab}\\
\delta R &= \delta(g^{\mu\nu}R_{\mu\nu}) = \delta g^{\mu\nu}R_{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}
\end{aligned}$$

$$F G_{\mu\nu} - \frac{1}{2}\left(f(R) - RF\right)g_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)F = \kappa T^{(m)}_{\mu\nu} \leftarrow \text{Equation of motion}$$

The conservation equation is given by:

$$S = \int d^4x\sqrt{-g}\,\mathcal{L}$$

$$\delta S = \int d^4x\sqrt{-g}\left[\frac{\sqrt{-g}\,\mathcal{L}}{\delta g^{\mu\nu}}\frac{1}{\sqrt{g}}\right]\delta g^{\mu\nu} = \int d^4x\sqrt{-g}\,S_{\mu\nu}\,\delta g^{\mu\nu} \Longrightarrow$$

$$\begin{aligned}
\delta g_{\mu\nu} &= \pounds_V g_{\mu\nu} = V^\sigma\nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda)g_{\lambda\nu} + (\nabla_\nu V^\lambda)g_{\mu\lambda}\\
&= \nabla_\mu V_\nu + \nabla_\nu V_\mu
\end{aligned}$$

$$\begin{aligned}
\delta S &= \int d^4x\sqrt{-g}\left[(V_\mu\nabla_\nu)S^{\mu\nu} + (V_\nu\nabla_\mu)S^{\mu\nu}\right]\\
&= -\int d^4x\sqrt{-g}V_\nu\left[\nabla_\mu S^{\mu\nu}\nabla_\mu S^{\mu\nu}\right] \Rightarrow
\end{aligned}$$

$$\boxed{\nabla_\mu S^{\mu\nu} = 0}$$

Assuming the Robertson-Walker metric, one can find the MoG version of the Friedmann equations:

$$3FH^2 = \rho_m + \rho_{rad} + \frac{1}{2}\left(FR - f\right) - 3H\dot{F}$$

$$-2F\dot{H} = \rho_m + \frac{4}{3}\rho_{rad} + \ddot{F} - H\dot{F}$$

Properly choosing F and $\dot{F}$ can give acceleration.

Since $f(R)$ modifies Newton's constant, we need to talk about G_{eff} (perturbations)

$$f(R) \simeq f(R_0) + f'(R_0)R + \dots$$

$$S = \frac{1}{8\pi G_N}\int d^4x\sqrt{-g}\,f(R) \simeq \frac{1}{8\pi G_N}\int d^4x\sqrt{-g}\left[f(R_0) + f'(R_0)R\right] \simeq \frac{1}{8\pi G_{eff}}\int d^4x\sqrt{-g}\left[R - 2\Lambda\right]$$

$\hookrightarrow G_{eff} \sim G_N / f(R_0)$

Doing a conformal transformation (Jordan $\to$ Einstein frame $(\sim)$), one gets that $f(R)$ is just a scalar field.

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \longrightarrow R^2 = \Omega^2 (\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\partial_\mu \omega \partial_\nu \omega) \qquad \omega = \ln\Omega$$

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2\kappa^2} F\Omega^{-2}(\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega) - \Omega^{-4}U\right] + \int d^4x \mathcal{L}_M(\Omega^{-2}\tilde{g}_{\mu\nu}, \Psi_M) \qquad \Omega^2 = F \qquad U = \frac{FR - f}{2\kappa^2}$$

Redefining the field: $\kappa\phi \equiv \sqrt{3/2}\ \ln F$

> **NOTE**
> This is a mathematical trick, it does not have a real physical meaning because the Einstein frame is not a physical frame.

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2\kappa^2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + \int d^4x \mathcal{L}_M(F^{-1}(\phi)\tilde{g}_{\mu\nu}, \Psi_M)$$

Quintessence

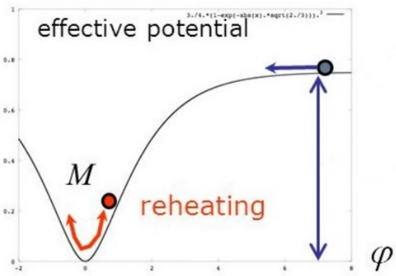
Potential
$$V(\phi) = \frac{U}{F^2} = \frac{FR - f}{2\kappa F^2}$$

Non-minimal coupling

Starobinsky inflation

Starobinsky proposed a simple $f(R)$ model. The R^2 term caused acceleration in the Universe, behaving like inflation.

$$f(R) = R + R^2/(6M^2) \longrightarrow V(\phi) = \Lambda^2\left(1 - e^{-\sqrt{2/3}\,\phi/M_{pl}}\right)^2$$



The slow-roll parameters of the model are:

$$n_s \simeq 1 - \frac{2}{N} \qquad r \simeq \frac{12}{N^2}$$

It behaves like the plateau models.

Modified gravity and ghosts.

After introducing the corrections trying to renormalize gravity, one finds lagrangians of the form:

$$S = \int d^4x \sqrt{-g}\, f(R, P, Q)$$

$$P \equiv R_{ab}R^{ab}$$

$$Q \equiv R_{abcd}R^{abcd}$$

$$F \equiv \frac{\partial f}{\partial R} \qquad f_P = \frac{\partial f}{\partial P} \qquad f_Q = \frac{\partial f}{\partial Q}$$

$$\begin{aligned} FG_{\mu\nu} = {} & \frac{1}{2}g_{\mu\nu}(f - R\,F) - (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)F \\ & -2\left(f_P R^a_\mu R_{a\nu} + f_Q\, R_{abc\mu}R^{abc}{}_\nu\right) \\ & -g_{\mu\nu}\nabla_a\nabla_b(f_P R^{ab}) - \Box(f_P R_{\mu\nu}) \\ & +2\nabla_a\nabla_b\left(f_P\, R^a{}_{(\mu}\delta^b{}_{\nu)} + 2f_Q\, R^a{}_{(\mu\nu)}{}^b\right) \end{aligned}$$

$\hookrightarrow$ Fourth order derivative $\to$ Ghost

These lagrangians result in equations of motion with 4th order derivatives, which might give ghosts. Linearizing and finding the propagator $G(k)$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\eta_{\mu\nu} + \eta_{\mu\nu}h_f$$

$$\longrightarrow \left(k^2 + \frac{k^4}{m^2_{spin2}}\right)\bar{h}_{\mu\nu} = 0 \longrightarrow G(k) \propto \underbrace{\frac{1}{k^2}}_{\text{massless}} - \underbrace{\frac{1}{k^2 + m^2_{spin}}}_{\text{ghost, makes vacuum decay}}$$

$$\Box h_f = m_s^2 h_f$$

$$m^2_{spin2} \equiv -\frac{F_0}{f_{P0} + 4f_{Q0}} \qquad m_s^2 \equiv \frac{1}{3}\frac{F_0}{F_{R0} + \frac{2}{3}(f_{P0} + f_{Q0})}$$

Other MoG models

There are also models with extra dimensions: the Kaluza-Klein models, Assuming an extra dimension, y, which is compactified with cylindrical boundary conditions, then the metric g_{MN} satisfies:

$$f(x,y) = f(x, y+2\pi) \longrightarrow \frac{\partial g_{MN}}{\partial y} = 0 \qquad \text{Similar to } U(1) \text{ symmetry}$$

Expanding the 5D metric in Fourier modes:

$$g_{MN}(x,y) = \sum_n g_{MN}^{(n)}(x)\, e^{iny/r} \longrightarrow g_{MN}^{(0)} = \phi^{-1/3} \begin{pmatrix} g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix}$$

Performing a dimensional reduction, one obtains:

$$S = \frac{1}{16\pi G_N^5} \int d^4x\, dy \sqrt{-g^{(5)}}\, R^{(5)} = \underbrace{\frac{1}{16\pi G_N^4}}_{G_N^{(4)} = G_N^{(5)}/2\pi r} \int d^4x \sqrt{-g}\, \overbrace{\left(R + \frac{1}{4}\phi F_{\mu\nu}F^{\mu\nu} + \frac{1}{6\phi^2}\partial^\mu\phi\,\partial_\mu\phi\right)}^{\text{4D GR + Maxwell + scalar field}}$$

Adding a scalar field:

$$S_\Phi = \int d^4x\, dy \sqrt{-g^{(5)}} \left(g_{MN}^{(0)}\, \partial_M \Phi\, \partial_N \Phi\right) =$$

$$= (2\pi r) \sum_n \int d^4x \sqrt{-g^{(4)}} \left[g^{\mu\nu} \left(\partial_\mu + \frac{in}{r} A_\mu\right) \Phi_n \left(\partial_\nu + \frac{in}{r} A_\nu\right) \Phi_n - \frac{n^2}{\phi r^2} \Phi_n^2 \right]$$

$$Q_n = \frac{8\pi G_N^{(4)} n}{r} \sqrt{\frac{2}{\phi}} \qquad M_n = |n|/r\sqrt{\phi}$$

$Q_n \sim M_n$ Problem

5.3. CMB constrains and inflation predictions

Constrains to inflation models.

On the cosmic microwave background we can see the inprint of inflaction, since inflation is the seed of the fluctuations. Thus, by measuring the CMB we can obtain constrains on inflation by analysing its power spectrum (this will be discussed in further lectures).

For the simplest model one can create needs the following 6 parameters. Two of them depend on inflation (n_s, $\ln 10^{10} A_s$), two of them depend on the kinematics of the universe and two of them are related to the matter content.

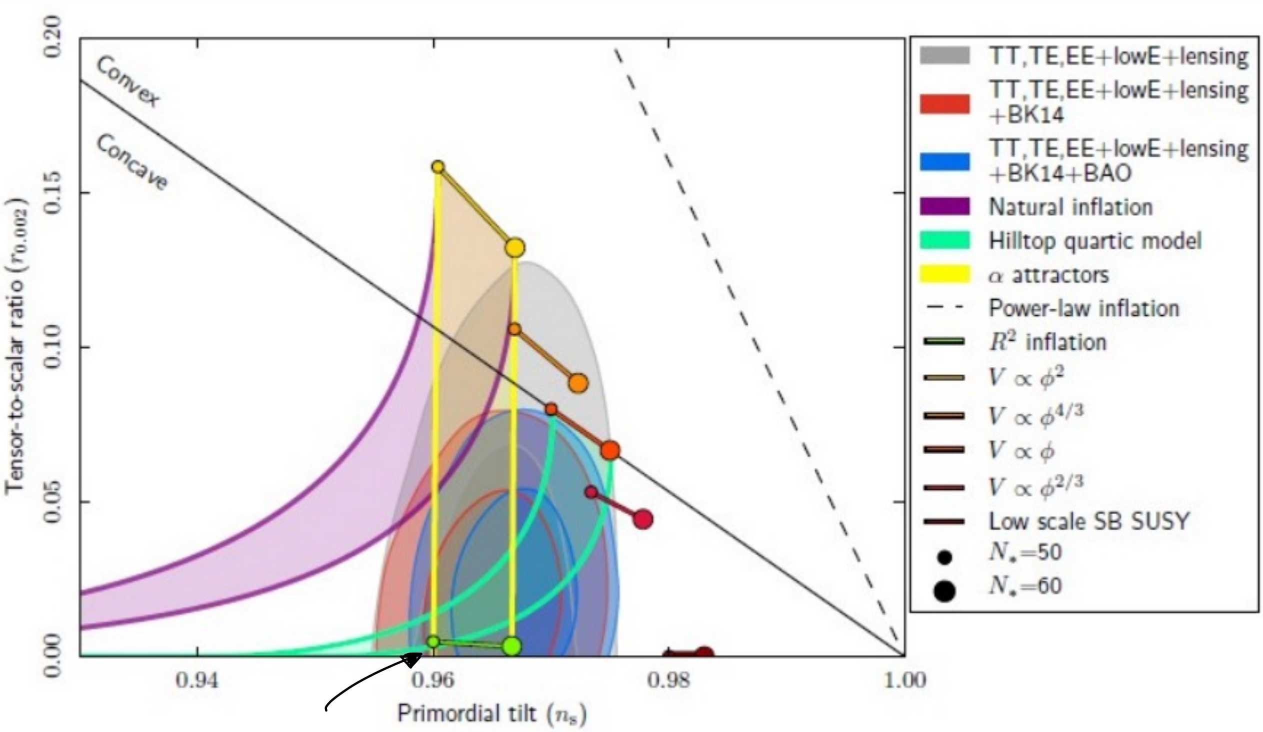
Parameter	TT+lowE 68% limits	TE+lowE 68% limits	EE+lowE 68% limits	TT,TE,EE+lowE 68% limits	TT,TE,EE+lowE+lensing 68% limits	TT,TE,EE+lowE+lensing+BAO 68% limits
$\Omega_b h^2$	0.02212 ± 0.00022	0.02249 ± 0.00025	0.0240 ± 0.0012	0.02236 ± 0.00015	0.02237 ± 0.00015	0.02242 ± 0.00014
$\Omega_c h^2$	0.1206 ± 0.0021	0.1177 ± 0.0020	0.1158 ± 0.0046	0.1202 ± 0.0014	0.1200 ± 0.0012	0.11933 ± 0.00091
$100\theta_{MC}$	1.04077 ± 0.00047	1.04139 ± 0.00049	1.03999 ± 0.00089	1.04090 ± 0.00031	1.04092 ± 0.00031	1.04101 ± 0.00029
τ	0.0522 ± 0.0080	0.0496 ± 0.0085	0.0527 ± 0.0090	$0.0544^{+0.0070}_{-0.0081}$	0.0544 ± 0.0073	0.0561 ± 0.0071
$\ln(10^{10}A_s)$	3.040 ± 0.016	$3.018^{+0.020}_{-0.018}$	3.052 ± 0.022	3.045 ± 0.016	3.044 ± 0.014	3.047 ± 0.014
n_s	0.9626 ± 0.0057	0.967 ± 0.011	0.980 ± 0.015	0.9649 ± 0.0044	0.9649 ± 0.0042	0.9665 ± 0.0038
H_0 [km s^{-1} Mpc^{-1}]	66.88 ± 0.92	68.44 ± 0.91	69.9 ± 2.7	67.27 ± 0.60	67.36 ± 0.54	67.66 ± 0.42
Ω_Λ	0.679 ± 0.013	0.699 ± 0.012	$0.711^{+0.033}_{-0.026}$	0.6834 ± 0.0084	0.6847 ± 0.0073	0.6889 ± 0.0056

Reminder

A_s - amplitud of the perturbations

n_s - spectral index

The plot shows comparations with the different models. It can be seen that some models are ruled out by observations.



(Starobinsky Inflation model fits the observations)

Inflation predictions (side-effects and prospects)

1. Production of gravitational waves (GWs)
2. Production of primordial black holes (PBHs)
3. Inflation probes high-energy physics (GUT+), not in reach of experiments
4. Inflation can be used to test for BSM physics
5. B-modes of CMB prove inflation.