III. Thermal history of the Universe

3.1. The hot Big Bang Model

We have already seen that the Universe is expanding following the Friedmann equation, but we have not said anything about the temperature. The dominant component of the Universe were photons and so, when we refer to the temperature of the Universe we will be talking about the temperature of those photons.

We also know that entropy is being conserved ($TdS = dU + pdV = 0$), since there is not any heat flow. Having a heat flow would mean that energy is going from one place to another, which would create a prefered direction (and does not comply with the cosmological principle). The universe expands adiabatically, i.e. like a fluid in thermal equilibrium.

For barotropic fluids ($p = w\rho c^2$), this implies that: $\rho = R^{3(1+w)} + \text{const}$, thus:

$$\left.\begin{array}{llll} \text{Radiation:} & w = 1/3 & \rho \propto R^{-4} \longrightarrow & T \propto R^{-1} \\ \text{Matter:} & w = 0 & \rho \propto R^{-3} \longrightarrow & T \propto R^{-2} \end{array}\right\} \text{when decoupled}$$

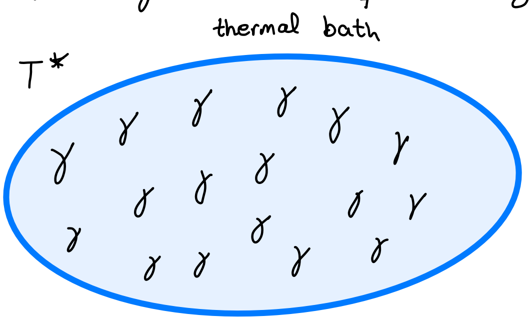
The Universe is cooling down when expanding.

The fact that we are working with barotropic fluids expanding adiabatically makes that the closer we get to the Big Bang (singularity), the hotter the Universe will be.

3.2. Thermal equilibrium.

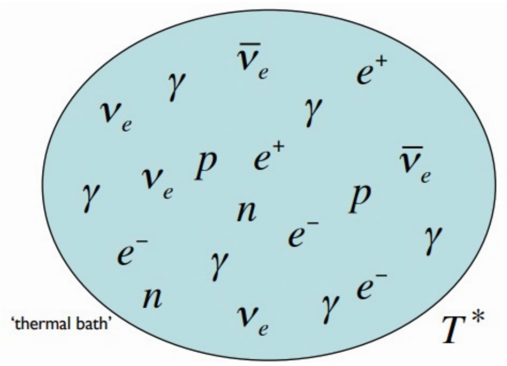
Thermal bath in equilibrium.

When photons and matter were coupled, they evolved in the same way, which was dictated by the dominant component (photons). In order to do calculations, we need a mathematical description of thermal equilibrium for radiation and matter.



Let us start with a thermal bad, assuming that there are only photons at a certain temperature T (they all do not need to have exactly the same energy, in fact their energies follow a Planck distribution).

Let us now add more "ingredients" to that thermal bath: neutrinos, electrons, positrons, protons...



Neutrinos, protons, electrons, neutrons and positrons are kept in equilibrium by the weak interaction and Thomson scattering.

- Weak interaction:

$$\nu_e + n \Leftrightarrow p + e^-$$
$$e^+ + n \Leftrightarrow p + \bar{\nu}_e$$
$$n \Leftrightarrow p + e^- + \bar{\nu}_e$$

- Thomson scattering

$$e^- + \gamma \longleftrightarrow e^- + \gamma$$

Since the photons outnumber everything, they dictate how the temperature of the thermal bath evolves.

Interaction rate: decoupling from the equilibrium

Particles can mantain that equilibrium as long as their own interaction rate is larger that the cosmic expansion rate. If the cosmic expansion rate is larger, then there is not enough time for the interaction to take place, thus, they decouple from the thermal bad. The interaction rate is defined as:

$$\boxed{\Gamma_c \propto n \sigma v}$$

$n \equiv$ number density $\quad \sigma \equiv$ interaction cross-section $\quad v \equiv$ relative velocity

Once a particle species is decoupled, it evolves indepently. Neutrinos were the first to decouple when $T_\nu(t_{dec}) = T(t_{dec})$ when the interaction rate of the weak interaction, Γ_W became $\Gamma_W < \Gamma_C$. This leaves us with a "relativistic neutrino background" in the Universe. Unless disturbed, the uncopled particles remain in their own equilibrium.

Characterization of particles in equilibrium: n, ρ, P.

We can characterise ensembles of particles through:

- number density: $n = \dfrac{g}{(2\pi\hbar)^3} \displaystyle\int f(p)\, 4\pi p^2\, dp$

- energy density: $\rho c^2 = \dfrac{g}{(2\pi\hbar)^3} \displaystyle\int E(p) f(p)\, 4\pi p^2\, dp$

- Pressure $\quad P = \dfrac{g}{(2\pi\hbar)^3} \displaystyle\int \frac{p^2 c^2}{3E} f(p)\, 4\pi p^2\, dp \qquad E^2 = |\vec{p}c|^2 + m^2c^4$

which can be determined if the phase space distribution function ($f(p)$) and the statistical weight are known. For the phase space distribution, one needs to solve the integro-differential equation (Boltzmann equation): $\dfrac{dn}{dt} + 3Hn = \displaystyle\int C[f(\vec{e})]\, d^3p$

If it is in kinetic equilibrium, this equation is easy to solve, and we obtain:

Relativistic:

$$f(p) = \frac{1}{e^{(E-\mu)/k_B T} \pm 1}$$

$+$: Fermi-Dirac distribution (Fermions)

$-$: Bose-Einstein distribution (Bosons)

Non-relativistic $(T < E - \mu)$

$$f(p) \approx e^{-(mc^2 + p^2/2mc^2 - \mu)/k_B T}$$

$$E = \sqrt{|\vec{p}c|^2 + m^2c^4} = mc^2\sqrt{p^2/2mc^2 + 1} \approx mc^2 + p^2/2mc^2$$

Relativistic particles in kinetic equilibrium $(m \ll T,\ \mu = 0)$

Introducing the expression for the phase space in the equations we get:

- number density: $$n = \frac{g}{(2\pi\hbar)^3}\int \frac{p^2}{e^{c\sqrt{p^2+m^2c^2}/k_B T} \pm 1}\,dp$$
- energy density: $$\rho c^2 = \frac{g}{(2\pi\hbar)^3}\int c\sqrt{p^2+m^2c^2}\,\frac{p^2}{e^{c\sqrt{p^2+m^2c^2}/k_B T} \pm 1}\,dp$$
- Pressure $$P = \frac{g}{(2\pi\hbar)^3}\int \frac{p^2c^2}{c\sqrt{p^2+m^2c^2}}\,\frac{p^2}{e^{c\sqrt{p^2+m^2c^2}/k_B T} \pm 1}\,dp$$

> **NOTE**
> In the Early Universe, $\mu \ll T$, $(\mu_\gamma = 0$ anyways$)$. Further, for relativistic particles which are continuously created and annihilated there is no net change in particle number, and hence their chemical potential can be neglected in general.

Taking $m \ll T$, this expressions reduce to: $(p^2 \gg m^2c^2)$

- number density: $$n = \frac{g}{(2\pi\hbar)^3}\int \frac{p^2}{e^{c\sqrt{p^2}/k_B T} \pm 1}\,dp$$
- energy density: $$\rho c^2 = \frac{g}{(2\pi\hbar)^3}\int c\sqrt{p^2}\,\frac{p^2}{e^{c\sqrt{p^2}/k_B T} \pm 1}\,dp$$
- Pressure $$P = \frac{g}{(2\pi\hbar)^3}\int \frac{p^2c^2}{c\sqrt{p^2}}\,\frac{p^2}{e^{c\sqrt{p^2}/k_B T} \pm 1}\,dp$$

And simplifying terms:

- number density: $$n = \frac{g}{(2\pi\hbar)^3}\int \frac{p^2}{e^{cp/k_B T} \pm 1}\,dp$$
- energy density: $$\rho c^2 = \frac{g}{(2\pi\hbar)^3}\int cp\,\frac{p^2}{e^{cp/k_B T} \pm 1}\,dp$$
- Pressure $$P = \frac{g}{(2\pi\hbar)^3}\int cp\,\frac{p^2}{e^{cp/k_B T} \pm 1}\,dp$$

Same integral. The equations can be combined to get $P = \ldots$

$$\Rightarrow \boxed{P = \frac{1}{3}\rho c^2}$$

Doing an "smart" coordinate transformation we can write the integrals as:

$$\xi = \frac{cp}{k_B T} \Rightarrow p = k_B T \xi / c \qquad dp = k_B T\, d\xi / c$$

- number density: $$n = \frac{g}{2\pi^2\hbar^3}\int \left(\frac{k_B T}{c}\right)^2 \frac{\xi^2}{e^\xi \pm 1}\,\frac{k_B T}{c}\,d\xi = \frac{g}{2\pi^2}\left(\frac{k_B}{\hbar c}\right)^3 T^3 \int \frac{\xi^2}{e^\xi \pm 1}\,d\xi$$
- Energy density: $$\rho c^2 = \frac{gc}{2\pi^2\hbar^3}\int \left(\frac{k_B T}{c}\right)^3 \frac{\xi^3}{e^\xi \pm 1}\,\frac{k_B T}{c}\,d\xi = \frac{g}{2\pi^2}\,\frac{k_B^4}{\hbar^3 c^3}\,T^4 \int \frac{\xi^3}{e^\xi \pm 1}\,d\xi$$

The solution to this integral are Γ-functions

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$$\int \frac{\xi^n}{e^\xi - 1} d\xi = \Gamma(n+1)\zeta(n+1) \quad (1)$$

ζ: Riemann zeta function

$$\int \frac{\xi^2}{e^\xi + 1} d\xi = \frac{3}{4}\int \frac{\xi^n}{e^\xi - 1} d\xi \quad (2)$$

$n = 2$, $\Gamma(n) = (n-1)!$ $\Rightarrow$ $(1) = 2\zeta(3)$, $(2) = \frac{3}{4} 2\zeta(3)$

$$\int \frac{\xi^3}{e^\xi + 1} d\xi = \frac{7}{8}\int \frac{\xi^n}{e^\xi - 1} d\xi \quad (3)$$

$n = 3$, $\Gamma(n) = (n-1)!$ $\Rightarrow$ $(1) = 6\zeta(4) = 6\frac{\pi^4}{90}$, $(3) = \frac{7}{8} 6\zeta(4) = \frac{7}{8} 6 \frac{\pi^4}{90}$

We obtain:

Number density: $n = \left[\frac{3}{4}\right] \frac{\zeta(3)}{\pi^2} \left(\frac{K_B}{\hbar c}\right)^3 g T^3$

Energy density: $\rho c^2 = \left[\frac{7}{8}\right] \frac{\pi^2}{30} \frac{K_B^4}{\hbar^3 c^3} g T^4$ $\quad$ [ ] → only for fermions

Pressure: $P = \frac{1}{3}\rho c^2$

Non relativistic particles in kinetic equilibrium ($m \gg T$)

Now $f(p) \approx e^{-(mc^2 + p^2/2mc^2 - \mu)/K_B T}$

$$E = \sqrt{|\vec{p}c|^2 + m^2c^4} = mc^2\sqrt{p^2/2mc^2 + 1} \approx mc^2 + p^2/2mc^2$$

Solving the integrals one gets:

- Number density: $n = g\left(\frac{m k_B}{2\pi\hbar^2}\right)^{3/2} T^{3/2} e^{-(mc^2-\mu)/k_B T}$
- Energy density: $\rho c^2 = nmc^2 + \frac{3}{2} n k_B T$
- Pressure: $P = n k_B T$

Recap:	Non-degenerated relativistic gas ($k_B T \gg mc^2$, $\mu = 0$) BOSONS	FERMIONS	Non relativistic gas ($k_B T \ll mc^2$)
NUMBER DENSITY	$n = \frac{\zeta(3)}{\pi^2}\left(\frac{K_B}{\hbar c}\right)^3 g T^3$	$n = \frac{3}{4}\frac{\zeta(3)}{\pi^2}\left(\frac{K_B}{\hbar c}\right)^3 g T^3$	$n = g\left(\frac{m k_B}{2\pi\hbar^2}\right)^{3/2} T^{3/2} e^{-(mc^2-\mu)/k_B T}$
ENERGY DENSITY	$\rho c^2 = \frac{\pi^2}{30}\frac{K_B^4}{\hbar^3 c^3} g T^4$	$\rho c^2 = \frac{7}{8}\frac{\pi^2}{30}\frac{K_B^4}{\hbar^3 c^3} g T^4$	$\rho c^2 = nmc^2 + \frac{3}{2} n k_B T$
PRESSURE	$P = \frac{1}{3}\rho c^2$	$P = \frac{1}{3}\rho c^2$	$P = n k_B T$

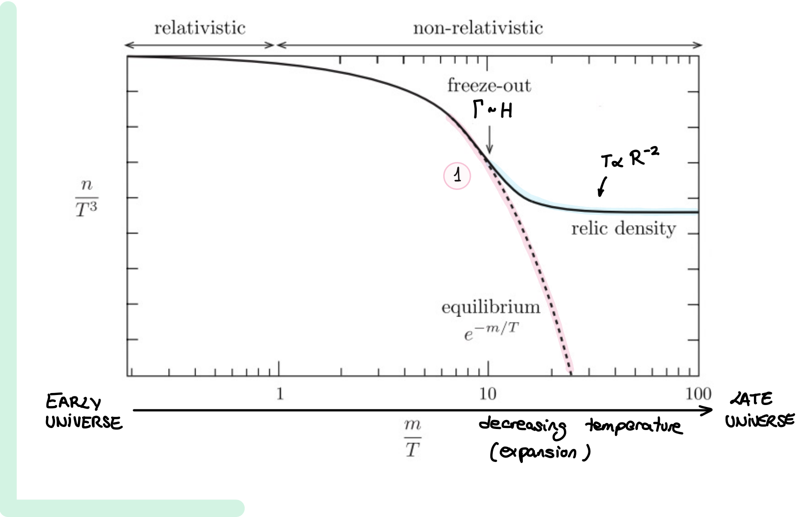
NOTE:

- The energy density of relativistic particles scales with temperature as T^4:

$$\boxed{\rho c^2 \propto T^4}$$

Stefan - Boltzmann law

1 The number density of the non-relativistic component is exponentially supressed. If it would have stayed in equilibrium with the temperature of the photon bath, it would be diluted.



2 If it is able to freeze-out, the species decouples from the photons and it remains at is own equilibrium:

$T \propto R^{-2}$

$n_{nr} \propto R^{-3} \Rightarrow n_{nr} \propto T^{3/2}$

(Without the exponential factor because it is not coupled to the photons anymore, i.e. not in equilibrium with them).

All the particles in the thermal bath share the same temperature, but they all have different distribution functions. Species are characterised by their mass, statistical weight g and maybe integration factors (bosons vs. fermions).
However, relativistic species can be combined via an effective g_*.

Energy densities and effective statistical weight

Let us focus first on energy density, considering all the possible contributions: coupled [not] relativistic and decoupled [not] relativistic particles.

$$\rho = \rho^{th}_{rel} + \rho^{th}_{nr} + \rho^{dec}_{rel} + \rho^{dec}_{nr}$$

Relativistic species in thermal equilibrium (ρ^{th}_{rel})

$\rightarrow$ effective statistical weight g_*

$$\rho_{rel}\, c^2 = \sum_i \rho_{rel,i}\, c^2 = \frac{\pi^2}{30} \frac{k_B{}^4}{\hbar^3 c^3}\, g_*(T)\, T^4 \qquad \boxed{g_*^{th}(T) = \sum_B g_i{}^B + \frac{7}{8}\sum_F g_i{}^F}$$

At some point in the Universe everything was relativistic and in thermal equilibrium. This was at temperatures above $k_B T >> 175$ GeV, thus, all the particles in the standard model of particle physics were in thermal equilibrium with each other.

We can calculate the effective statistical weight:

$$g_B = \text{gluons} + \text{photons} + W^{\pm} + Z_0 + \text{Higgs} = 8\times 2 + 2 + 3\times 3 + 1 = 28$$

$$g_F = \text{quarks} + \text{leptons} + \text{neutrinos} = 12\times 6 + 6\times 2 + 3\times 2 = 90$$

$$\boxed{g_* = 28 + \frac{7}{8}\, 90 = 106.75}$$

As T drops, various of those relativistic species become non-relativistic (and annihilate), and so they are removed from g_*.

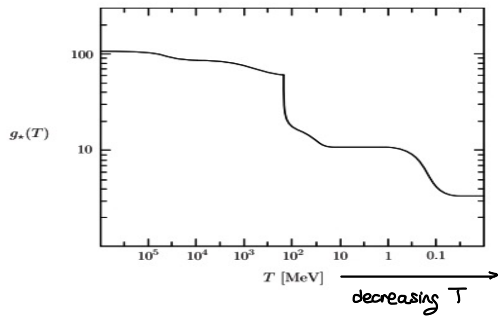
BUT neutrinos, for instance, continue to exist and remain relativistic after decoupling.

Relativistic species (coupled and decoupled, $\rho_{rel}^{th} + \rho_{rel}^{dec}$).

We can add the contribution of decoupled relativistic species to the total energy density to the effective statistical weight as:

$$\boxed{g_*^{dec}(T) = \sum_B g_i^B \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_F g_i^F \left(\frac{T_i}{T}\right)^4}$$

where T_i is the temperature of the decoupled species, which can be different to the temperature of the Universe (thermal/photon bath).



The graph shows how the effective weight of all the components in the thermal bath evolve as a function of the temperature of the Universe.
It is usually aproximed as a step function.

T decreasing

temperature	T	particles	g_*	$4g_*$
$T<T_{dec}$		γ's + 3 ν's	3.36	$13.45=4*((2+(7/8)*2*3*(4/11)^{(4/3)}))$
$T_{dec}<T<m_e$	0.5 MeV	γ's + 3 ν's	7.25	29=4*(2+(7/8)*2 * 3)
$m_e<T<m_\mu$	95 MeV	+ e^-, e^+	10.75	43=29 + 4*((7/8)*2 * 2)
$m_\mu<T<m_\pi$	139 MeV	+ μ^-, μ^+	44.25	57=43 + 4*((7/8)*2 * 2)
$m_\pi<T<T_{QCD}$	150 MeV	+ π^+, π^-, π^0	17.25	69=57 + 4*(3) **remark**: now the 3 pions annihilate again...
$T_{QCD}<T<m_c$	1.3 GeV	+ u,u, d,d, + g's − π^+, π^-, π^0	61.75	205= 69 + 4*(8*2 + (7/8)*(2*3*2*2) – 3*1) **remark**: the 3 pions (w/ g*=1) are formed!
$m_c<T<m_s$	see below*	s, s		247=205 + 4*((7/8)*1*3*2*2)
$m_s<T<m_\tau$	1.8 GeV	+ c, c	72.25	280=247 + 4*((7/8)*2*3 * 2)
$m_\tau<T<m_b$	4.2 GeV	+ τ^-, τ^+	75.75	303=289 + 4*((7/8)*2 * 2)
$m_b<T<m_{W,Z}$	85 GeV	+ b, b	86.25	345=303 + 4*((7/8)*2*3 * 2)
$m_{W,Z}<T<m_H$	125 GeV	+ $W^\pm$, Z^0	95.25	381=345 + 4*(3*3)
$m_H<T<m_t$	173 GeV	+ H	96.25	385=345 + 4*(1)
$m_t<T$		+ t, t	106.75	427=385 + 4*((7/8)*2*3 * 2)

Neutrino decoupling

Electron-positron annihilation

*The mass of the strange is 95Mev at the 1GeV scale and in general is of course running with energy. So, at the QCD transition scale ~175MeV it is quite higher, eg around 125MeV or so. The transition scale is a bit fuzzy, ie it's not a step function happening at one value only, so without a very difficult numerical simulation we cannot say exactly how/where it happens exactly. The system is strongly coupled, so counting degrees of freedom in the range of Tc to the bottom quark mass doesn't make much sense anyway (in his opinion). Also, any simulation (as mentioned in point 3) is very difficult to do to begin with.

Non relativistic species

We need add their contribution to the energy density of the Universe.

$$\rho_{nr} c^2 \propto \sum_i m_i c^2 n_i + \frac{3}{2} n_i k_B T$$

$$n_i^{th} = g_i \left(\frac{m_i k_B}{2\pi\hbar^2}\right)^{3/2} T^{3/2} e^{-(m_i c^2 - \mu)/k_B T}$$

$$n_i^{dec} \propto T_i^{3/2}$$

3.3. Entropy of the Universe

Total entropy

We have already seen that the entropy is a conserved quantity. It is expected that it is proportional to the scale factor and the temperature.

We start with the second law of thermodynamics: $TdS = dU + pdV = 0$

And writing dV and dU in terms of the scale factor:

$$\left.\begin{aligned} dV &= d(R^3) \\ dU &= d(V\rho c^2) = d(R^3 \rho c^2) \end{aligned}\right\} \quad dS = \frac{1}{T}[dU + pdV]$$

$$pd(R^3) = d(pR^3) - R^3 dp$$

$$dS = \frac{1}{T}[dU + pdV] = \frac{1}{T}\left[d(R^3\rho c^2) + pd(R^3)\right] =$$

$$= \frac{1}{T}\left[d(R^3(\rho c^2 + p)) - R^3 dp\right] =$$

Replace in favour of dT: Cauchy-Riemann integrability condition: $\frac{\partial^2 S}{\partial R^3 \partial T} = \frac{\partial^2 S}{\partial T \partial R^3} \Rightarrow dp = (\rho c^2 + p)\frac{dT}{T}$

$$= \frac{1}{T}\left[d(R^3(\rho c^2 + p)) - \frac{1}{T} R^3(\rho c^2 + p)\, dT\right] =$$

$$= \frac{1}{T} d\left(R^3(\rho c^2 + p)\right) - \frac{R^3}{T^2}(\rho c^2 + p)\, dT =$$

Integration by parts

$$= d\left[\frac{(\rho c^2 + p)R^3}{T} + \text{const}\right]$$

$$\Rightarrow \boxed{S(T) = R^3 \frac{(\rho c^2 + p)}{T} = \text{const}}$$

Relativistic species

$$p = \frac{1}{3}\rho_{rel} c^2 \Rightarrow$$

$$S(T) = \frac{R^3}{T}\left(1 + \frac{1}{3}\right)\rho_{rel} c^2 = \frac{4R^3}{3T} \frac{\pi^2}{30} \frac{k_B^4}{\hbar^3 c^3} g_{*s} T^4 = \frac{2\pi}{45} \frac{k_B^4}{\hbar^3 c^3} g_{*s} (RT)^3$$

$$\boxed{S(T) = \frac{2\pi}{45} \frac{k_B^4}{\hbar^3 c^3} g_{*s} (RT)^3} \xrightarrow{S(T) = \text{const}} \boxed{T \propto \underbrace{g_{*s}^{-1/3}}_{\text{changes in time !!}} R^{-1}}$$

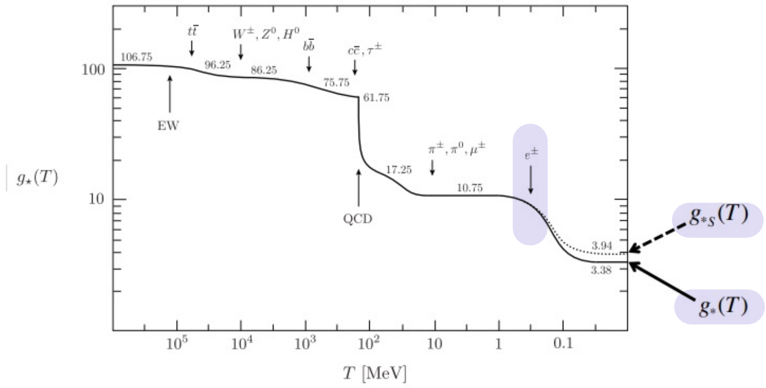
NOTE

$$g_{*s}^{th} = g_*^{th}(T)$$

$$g_{*s}^{dec} = \sum_B g_i^B \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_F g_i^F \left(\frac{T_i}{T}\right)^3$$

$$g_{*s}^{dec} \neq g_*^{dec}(T)$$

The expression for $S(T)$ that we obtained before from the second law of thermodynamics takes into account everything that is contained in the Universe, coupled or decoupled from the thermal bath.



After electron-positron annihilation something happens to the thermal bath. The two effective statistical weights start to differ. Whatever decouples at that point dissappears.

Temperature evolution

$T \propto g_{*S}^{-1/3} R^{-1}$ → when particles become non-relativistic and decouple, g_{*S} drops and its entropy is transfered to heat bath, which increases its temperature (Entropy of decoupled species is conserved separately).

Usually what decouples becomes non-relativistic or annihilated. The only remaining relativistic decoupled species were neutrinos.

- $\Omega_r = 1$:

$$\boxed{\frac{T}{1\,\mathrm{MeV}} \simeq 1.5\, g_{*S}^{-1/4} \left(\frac{1\,s}{t}\right)^{1/2}}$$

(time in seconds after de Big Bang)

Dem: From FRW lecture, $\Omega_r = 1$:

$R(t) \propto t^{1/2} \Rightarrow H \propto 1/t$

$$H = \sqrt{\frac{8\pi G}{3}\rho_r} = \sqrt{\frac{8\pi G}{3}\frac{\pi^2}{30}\frac{k_B^4}{\hbar^3 c^3} g_*(T)\, T^4} \propto \sqrt{g_*(T)}\, T^2 \propto \frac{1}{t}$$

Particle numbers

$$n_i = \left[\frac{3}{4}\right] \frac{\zeta(3)}{\pi^2}\left(\frac{k_B}{\hbar c}\right)^3 g_i T^3 \Rightarrow \frac{n_i}{S} = \left[\frac{3}{4}\right]\frac{45\zeta(3)}{2\pi^4}\frac{g_i}{g_{*S}}\frac{1}{R^3}$$

$\left[\frac{3}{4}\right]$ for Fermions

3.4. Decoupling

Generalities

As it was mentioned before, particles drop out of thermal equilibrium when their interaction rate is less than the expansion rate of the Universe. The interaction rate was defined as:

$$\boxed{\Gamma_c \propto n \sigma v}$$

$n \equiv$ number density $\sigma \equiv$ interaction cross-section $v \equiv$ relative velocity

The interaction rate depends on the carrier of the interaction:

- Interaction mediated by massless gauge bosons: $\Gamma_c \propto T$ (gluon, photon)
- Interaction mediated by massive gauge bosons $(T < M_X)$: $\Gamma_c \propto T^5$ (W, Z)

Universe expansion rate

We aim to compare the interaction rate to the expansion rate of the Universe, which is the Hubble parameter.

$$\boxed{\Gamma_e \propto H}$$

We would like to know how it scales with T.

1. Radiation domination: $T \propto R^{-1}$

   Friedmann equation:

   $$H^2 = H_0^2 \Omega_{r,0} (1+z)^4 \xrightarrow[R=(1+z)^{-1}]{\Omega_{r,0}=1} H \propto R^{-2}$$

   $$\Rightarrow H \propto T^2$$

2. Matter domination: $T \propto R^{-2}$

   Friedmann equation: $H \propto R^{-3/2}$

   $$\Rightarrow H \propto T^{4/3}$$

Condition for particles to remain in thermal equilibrium

- Interaction mediated by massless gauge bosons: $\frac{\Gamma_c}{H} \propto T^{-1} > 1$
- Interaction mediated by massive gauge bosons ($T < M_X$): $\frac{\Gamma_c}{H} \propto T^3 > 1$

The quantitative calculation requires actual $\Gamma_c = n\sigma v$ and $H =$ Friedmann equation, thus, we need to know the evolution of the number density using the Boltzmann equation:

$$\frac{dn}{dt} + 3Hn = -\langle\sigma v\rangle (n^2 - n_{eq}^2) \longrightarrow n(t) \text{ decoupled}$$

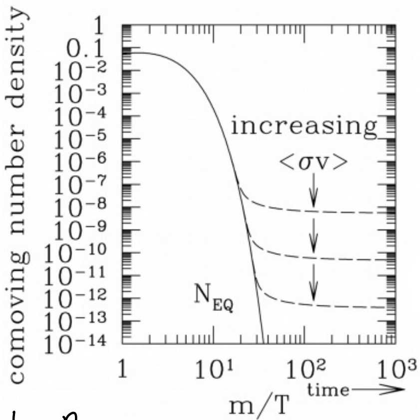
↳ still in equilibrium

cross section keeping species in eq.

$$\left.\begin{aligned} n < n_{eq} &\Rightarrow \text{r.h.s.} > 0 \\ n < n_{eq} &\Rightarrow \text{r.h.s.} < 0 \end{aligned}\right\} n \rightarrow n_{eq} \longrightarrow \text{Tries to be again in equilibrium}$$

The larger the cross section, the longer an species remains in equilibrium.

Decouple: $\sigma = 0 \Rightarrow n \propto R^{-3}$ (freeze-out)



↳ n/R^3

Neutrino decoupling (during radiation domination).

Neutrinos were coupled to the thermal bath via the weak interaction:

$$n + \nu \longleftrightarrow p + e^- \qquad p + \bar{\nu} \longleftrightarrow n + e^+$$

Its interaction rate is given by:

$\Gamma_\nu = 3.66\ G_F^2 T^5$, where G_F is the Fermi constant

We want to figure out at what temperature did neutrinos decouple.

- Radiation domination: $H^2 = H_0^2 \Omega_{r,0} R^{-4} = H_0^2 \frac{\rho_{r,0}}{\rho_{crit,0}} R^{-4} = \frac{8\pi G}{3} \rho_{r,0} R^{-4} = \frac{8\pi G}{3} \rho_r = \frac{8\pi G}{3} \frac{\pi^2}{30} g_* T^4$

$$\boxed{\frac{\Gamma_\nu}{H} = \frac{3.6\ G_F^2 T^5}{\left(\frac{8\pi G}{3} \frac{\pi^2}{30} g_*\right)^{1/2} T^2}}$$

This can be written as: $\frac{\Gamma_\nu}{H} \approx \frac{2}{3} M_P G_F^2 T^3$, G_F: Fermi constant, M_P: Planck mass

And, to find the decoupling temperature:

$\frac{\Gamma_\nu}{H} = 1 \Rightarrow \boxed{T_\nu^{dec} \approx 0.8 \text{ MeV}}$

This is larger than the rest mass of the electrons (0.5 MeV), so electrons and positrons are still around being created in the thermal bath.

$T > 0.8 \text{MeV}$, $T_\nu^{dec} \propto R^{-1}$

The temperature of the decoupled (still relativistic) neutrinos drops like R^{-1}. After ν decoupling (range $T \in [0.8, 0.511]$ MeV) the thermal bath contained photons, electrons and positrons (all the rest has decoupled). We can calculate the effective statistical weights

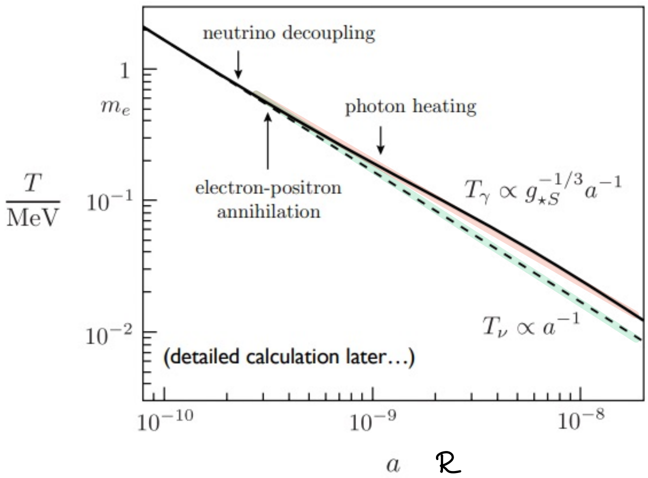
- $T \in [0.8, 0.511]$ MeV $\longrightarrow g_{*s} = \underbrace{2}_{\text{photons}} + \frac{7}{8}\underbrace{4}_{e^- \text{ and positrons}}$
- $T < 0.511$ MeV $\longrightarrow g_{*s} = 2$ (electron-positron annihilation)

Since $T \propto g_{*s}^{-1/3} R^{-1}$, the temperature of the thermal bath raised due to the change in g_{*s}. Through entropy conservation:

$$\frac{T_0}{T_\nu} = \left(\frac{11}{4}\right)^{1/3} \xrightarrow[T_0 = 2.725\text{K}]{} \boxed{T_\nu = 1.945 \text{ K}}$$

--- – Neutrinos: R^{-1}

—— – Thermal bath: reheated



NOTE

At some point neutrinos become non relativistic and start scaling like R^{-2}

Photon decoupling

Photons were coupled to the thermal bath via Thomson scattering with remaining electrons

$e^- + \gamma \longleftrightarrow e^- + \gamma$

Following the same steps as before:

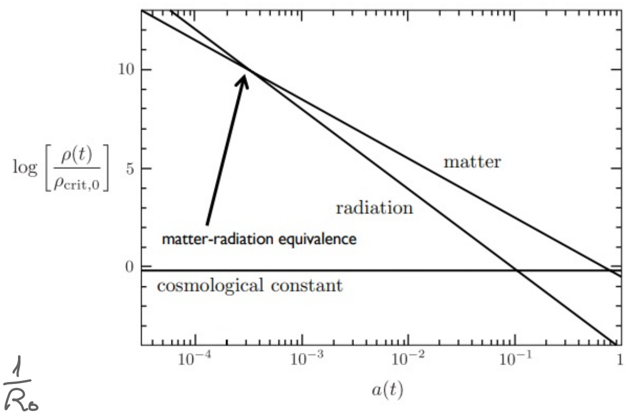
- Matter domination: $H^2 = H_0^2 \Omega_{m,0} \left(\frac{R}{R_0}\right)^{-3} \longrightarrow H = H_0 \Omega_{m,0}^{1/2} (R_0/R)^{3/2}$

$$\boxed{\frac{\Gamma_\gamma}{H} = \frac{n_e \sigma_T c}{H_0 \Omega_{m,0}^{1/2} (R_0/R)^{3/2}}}$$

Decoupling condition: $\frac{\Gamma_\gamma}{H} = 1$ $\xrightarrow[\text{Non-rel electrons}: \; n_e = g_e \left(\frac{m_e kT}{2\pi\hbar^2}\right) e^{-(m_e-\mu_e)c^2/kT}]{\text{Relativistic photons}: \; T \propto R^{-1}}$ $\boxed{T_\gamma^{dec} = 0.27 \text{ eV}}$

3.5. Matter-radiation equality

For this calculation we consider the scalings of matter and radiation (not the cosmological constant).



The equality time is defined as:

$$\rho_{rel}(R_{eq}) = \rho_{nr}(R_{eq}) \Rightarrow \boxed{1+z = \frac{\rho_{m,0}}{\rho_{r,0}}}$$

$$\left.\begin{aligned} \rho_{nr} R^3 &= \rho_{nr,eq} R_{eq}^3 = \rho_{nr,0} R_0^3 \\ \rho_{rel} R^4 &= \rho_{rel,eq} R_{eq}^4 = \rho_{rel,0} R_0^4 \end{aligned}\right\} \frac{1}{R_{eq}} = \frac{\rho_{nr,0}}{\rho_{rel,0}} \frac{1}{R_0}$$

NOTE

$\rho_{nr} \equiv \rho_m$

$\rho_{rel} \equiv \rho_r$

$H_0 = 100\, h\ \mathrm{Km/s/Mpc}$

We just need to calculate today's radiation and matter density:

- Matter density: $\rho_{m,0} = \Omega_{m,0} \dfrac{3H_0^2}{8\pi G} = 1.9 \times 10^{-29}\, \Omega_{m0} h^2\ \mathrm{g/cm^3}$
- Radiation density:

$$\rho_{r,0} = \rho_{CMB,0} + \rho_{\nu,0} \quad \left\{\begin{aligned} \rho_{CMB,0}\, c^2 &= \frac{\pi^2}{30} \frac{k_B^4}{\hbar^3 c^3} g_{CMB} T_{CMB}^4 \\ \rho_\nu c^2 &= \frac{7}{8} \frac{\pi^2}{30} \frac{k_B^4}{\hbar^3 c^3} g_\nu \left(\left(\frac{4}{11}\right)^{1/3} T_{CMB}\right)^4 \end{aligned}\right.$$

photons, neutrino species, $N_\nu = 3$

$$\Rightarrow \rho_{r,0} = \frac{1}{c^2} \frac{\pi^2}{30} \frac{k_B^4}{\hbar^3 c^3} T_{CMB}^4 \left(2 + \underbrace{\frac{7}{8} \times 2N_\nu \times \left(\frac{4}{11}\right)^{4/3}}_{\text{neutrinos}}\right) = 7.8 \times 10^{-34}\ \mathrm{g/cm^3}$$

$$1 + z_{eq} = \frac{\rho_{m,0}}{\rho_{r,0}} = 24000\, \Omega_{m,0} h^2 \longrightarrow \boxed{z_{eq} \simeq 3440} \xrightarrow[T_\gamma \propto (1+z)]{T_{\gamma,0} = 2.73\,K} \boxed{T_{\gamma,eq} \cong 0.8\,\mathrm{eV}}$$

(Planck cosmology)

Event	time t	redshift z	temperature T
Inflation	10^{-34} s (?)	–	–
Baryogenesis	?	?	?
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Dark matter freeze-out	?	?	?
Neutrino decoupling	1 s	6×10^9	1 MeV
Electron-positron annihilation	6 s	2×10^9	500 keV
Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV
Reionization	100–400 Myr	11–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	13.8 Gyr	0	0.24 meV

↑ Radiation domination

↓ Matter domination