II. Friedmann - Robertson - Walker models

2.1. Friedmann Equations: comoving coordinates and redshift

The Friedmann equation describes the background, not the formation of galaxies or other observable structures.

2.1.1. Robertson – Walker – Metric

Introduction

Friedmann was the first one to recast Einsteins equations into a form that allows a solution for a Universe that is either expanding or collapsing, but to do so he did not use the most general homogeneous and isotropic metric. This metric was derived ten years later by Robertson and Walker (1935).

Derivation

The starting point is the principle of relativity: the equations describing the laws of physics have the same form irrespective of the coordinate system. This is fulfilled by Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu},$$

which require a metric. This metric must be homogeneous and isotropic in space (cosmological principle) $\Longrightarrow$ Robertson – Walker metric.

Let us start with a general metric:

$$\boxed{ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu}, \quad \text{with} \begin{cases} x^\mu = (ct, x^i) \equiv \text{vector (coordinates)} \\ g_{\mu\nu} = \text{metric tensor, describes space} \end{cases}$$

↳ distance between two points

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ 0 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ 0 & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}$$

> **NOTE**
> $g_{it} = 0$
> If time depended on space, measurements of time could distinguish one place from another (not homogeneous or isotropic)

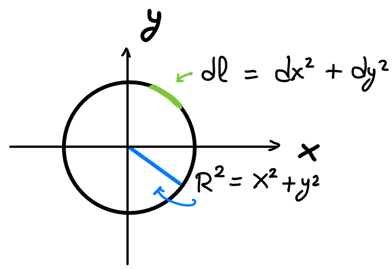
γ_{ij} should describe a homogeneous and isotropic 3D space, i.e. maximal symmetry for the 4D hypersurface described by x^i.

Metric of 1D curved space

1D space ≡ hypersurface in 2D (maximally symmetric in 2D)

↳ $dl^2 \Rightarrow$ 1D distance on 2D hypersurface

$R^2 \Rightarrow$ 2D hypersurface



We can get rid of the second dimension (e.g. y) as we live on 1D space (e.g. only x).

Taking the derivative of the second equation:

$$R^2 = x^2 + y^2 \longrightarrow 0 = x\,dx + y\,dy \longrightarrow dy = \frac{-x\,dx}{y}$$

└ re-arranging terms $\longrightarrow y^2 = R^2 - x^2$

$$\boxed{dl^2 = dx^2 + \frac{x^2 dx^2}{R^2 - x^2}}$$

Distance in curved 1D space

NOTE: We recover Euclidean distance when $R \to \infty$ $\quad (dl^2 = dx^2)$

Metric in 3D curved space (4D maximally symmetric hypersurface).

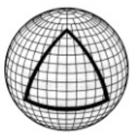
The calculation is similar to the 1D case.

We have three possible cases for homogeneous and isotropic metrics:

1. Spherical 3D space

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2$$

$$R^2 = x_1^2 + x_2^2 + x_3^2 + z^2$$

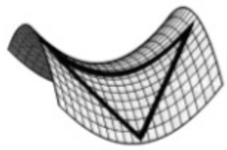


Positive Curvature

2. Hyperbolic 3D space:

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2$$

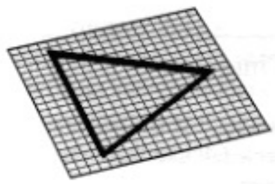
$$-R^2 = x_1^2 + x_2^2 + x_3^2 + z^2$$



Negative Curvature

3. Euclidean 3D space:

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2$$



Flat Curvature

3D space

Euclidean $\longrightarrow$ trivial

Spherical $\longrightarrow$ inherited from 4D symmetry

Hyperbolic $\longrightarrow$ Sec 13.2 in Weinberg's "Gravitation & Cosmology"

NOTE

Homogeneity: dl is invariant under translations

$$x_i \longrightarrow x_i + D$$

Isotropy: dl is invariant under rotations

$$x_i \longrightarrow R^k{}_i\, x_k$$

Differenciating in the expression for R^2: $0 = x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + z\,dz$ and using it to obtain an expression for z, one gets to the line element:

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - (x_1^2 + x_2^2 + x_3^2)} \quad \text{(Spherical 3D case)}$$

And changing to spherical coordinates:

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix} \longrightarrow dx_1^2 + dx_2^2 + dx_3^2 = g_r\,dr^2 + g_\theta\,d\theta^2 + g_\varphi\,d\varphi^2$$

$$\begin{cases} g_r = \left|\frac{\partial \vec{r}}{\partial r}\right| = 1 \\ g_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = r \\ g_\varphi = \left|\frac{\partial \vec{r}}{\partial \varphi}\right| = r\sin\theta \end{cases}$$

$$\Rightarrow \quad dl^2 = dr^2 \left[\frac{R^2}{R^2 - r^2}\right] + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \quad \text{(spherical 3D case)}$$

To take R as a prefactor, we introduce new rescaled (comoving) coordinates

$\boxed{r = Rx}$ $\quad$ $\boxed{dr = R\,dx}$ $\quad$ $r \equiv$ spherical coordinate

$$dl^2 = R^2\left(d\chi^2\left[\frac{1}{1-\chi^2}\right] + \chi^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right)$$ spherical 3D case

For the hyperbolic and Euclidean cases we just take into account the change of sign in (or disappearance of) R. This allows us to rewrite the metric as:

$$dl^2 = R^2\left(\frac{dx^2}{1-kx^2} + x^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right) \quad \text{with} \begin{cases} k=+1 & \text{spherical} \\ k=0 & \text{Euclidean} \\ k=-1 & \text{Hyperbolic} \end{cases}$$

$$\gamma_{ij} = \begin{pmatrix} \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & R^2x^2 & 0 \\ 0 & 0 & R^2x^2\sin^2\theta \end{pmatrix} \xrightarrow[\text{time components}]{\text{taking into account the}} g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2x^2 & 0 \\ 0 & 0 & 0 & R^2x^2\sin^2\theta \end{pmatrix}$$

Which gives us the Robertson-Walker metric:

$$\left(ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu\right)$$

$$\boxed{ds^2 = -(c\,dt)^2 + R^2\left[\frac{dx^2}{1-kx^2} + x^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]}$$

Robertson-Walker metric

NOTES

- Curvature of 3D space: $\mathcal{R} = 6\frac{k}{R^2}$

  Curvature of space-time: $\mathcal{R} = 6\frac{\ddot{R}}{R} + 6\left(\frac{\dot{R}}{R}\right)^2 + 6\left(\frac{k}{R^2}\right)$
- Any non-trivial g_{00} component can be absorbed by re-defining time: $dt' = \sqrt{g_{00}}\, dt$
- The inclusion of g_{0i} or g_{i0} components would break isotropy

2.1.2. Geodesics

General case.

We can built the geodesic equation from the position 4-vector and the metric of space-time:

$x^\mu = (ct, x^i)$

$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$ with $g_{\mu\nu} \equiv$ Robertson Walker metric

In absence of external force, the geodesic equation is:

$$\frac{d^2x^\mu}{ds^2} = -\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds}$$

$$\text{with } \Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu k}\left[\frac{\partial g_{k\nu}}{\partial x^\lambda} + \frac{\partial g_{k\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^k}\right]$$

And in terms of the proper velocity:

$$\boxed{0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda} \quad \text{with } u = \frac{dx^\mu}{ds}$$

Particularization to Robertson-Walker metric.

First, we calculate the Christoffel symbols:

$$\Gamma^0_{ij} = \frac{\dot{R}}{R}\gamma_{ij} \qquad \Gamma^i_{0j} = \dot{R}^2 R^2 \delta_{ij} \qquad \Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left(\frac{\partial \gamma_{kl}}{\partial x^j} + \frac{\partial \gamma_{jl}}{\partial x^k} - \frac{\partial \gamma_{jk}}{\partial x^l}\right)$$

All other components are zero

In FRW metric: $\mu = 0$, $\Gamma^0_{ij} = \frac{\dot{R}}{R}\gamma_{ij}$

Thus, we can write the geodesic equation as:

Physical 3D velocity $|u|^2 = \gamma_{ij} u^i u^j$

$$0 = \frac{du^0}{ds} + \Gamma^0_{\nu\lambda} u^\nu u^\lambda = \frac{du^0}{ds} + \Gamma^0_{ij} u^i u^j = \frac{du^0}{ds} + \frac{\dot{R}}{R}\gamma_{ij} u^i u^j = \frac{du^0}{ds} + \frac{\dot{R}}{R}|u|^2$$

$$\Rightarrow 0 = \frac{du^0}{ds} + \frac{\dot{R}}{R}|u|^2$$

We can do an additional change of variables to write $\frac{du^0}{ds}$ in terms of t:

$$u^0 = -c\frac{dt}{ds}, \qquad u^i = \frac{dx^i}{ds}$$

$$\frac{du^0}{ds} = \frac{du^0}{dt}\frac{dt}{ds} = -\frac{du^0}{dt}\frac{u^0}{c} \quad (1)$$

$-c^2 = u_\nu u^\nu \Rightarrow u^0 du^0 = -|u|\,d|u|$

$$\frac{du^0}{ds} = \frac{du^0}{dt}\frac{dt}{ds} \overset{(1)}{=} -\frac{du^0}{dt}\frac{u^0}{c} = -\frac{u^0 du^0}{c\,dt} = \frac{|u|\,d|u|}{c\,dt} = \frac{|u|}{c}|\dot{u}|$$

We obtain the geodesic equation:

$$\boxed{0 = \frac{|u|}{c}|\dot{u}| + \frac{\dot{R}}{R}|u|^2}$$

From it, we can read that $\frac{|\dot{u}|}{|u|} \propto -\frac{\dot{R}}{R}$. With one more change of variables we find:

$$\frac{|\dot{u}|}{u} \propto -\frac{\dot{R}}{R} \xrightarrow{\frac{d|u|}{dt} = \frac{d|u|}{dR}\dot{R}} \frac{1}{|u|}\frac{d|u|}{dR}\dot{R} \propto -\frac{\dot{R}}{R} \longrightarrow \frac{d|u|}{|u|} \propto -\frac{dR}{R} \longrightarrow \boxed{|u| \propto \frac{1}{R}}$$

magnitude of physical 3D velocity

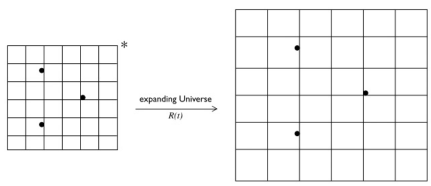
Thus, the momentum of particles scales like $1/R$

$$\boxed{|p| \propto \frac{1}{R}}$$

2.1.3. Comoving coordinates and redshift. Expanding Universe.

Expanding Universe

Cosmological expansion and redshift.



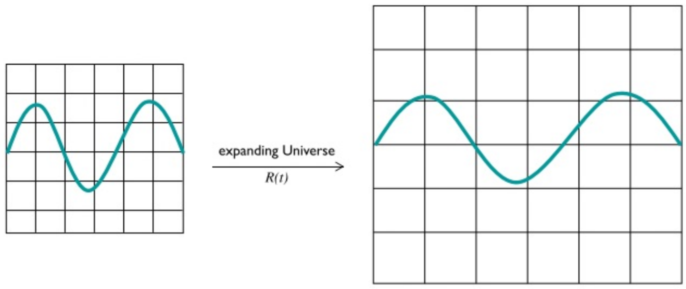
Let us look again at the Robertson-Walker metric.

$$ds^2 = -(c\,dt)^2 + R^2(t)\left[\frac{dx^2}{1-kx^2} + x^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]$$

Note that $R(t)$ can be an arbitrary function of time. Using $R(t)$, we had defined the comoving coordinates as:

$$\vec{r}(t) = R(t)\,\vec{x}$$

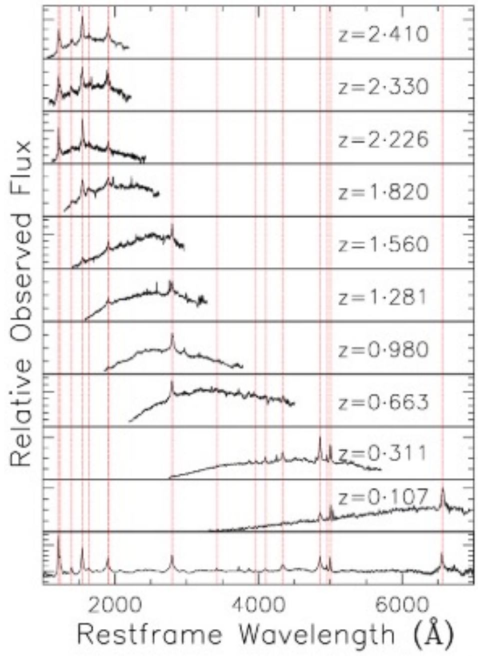
$\hookrightarrow$ Comoving coordinate



We want to find if there is any connection between $R(t)$ and observables.

Since all lengths scale like $R(t)$, this will also affect to waves. We can define redshift as:

$$\boxed{z = \frac{\lambda_0 - \lambda_E}{\lambda_E}} \qquad \begin{cases} \lambda_0 \equiv \text{observed wavelength} \\ \lambda_E \equiv \text{emitted wavelength} \end{cases}$$



If we are able to find a relation between $R(t)$ and z, we would have a connection between $R(t)$ and an observable.

This relation is given by:

- De Broglie relation: $p = h/\lambda$
- FRW geodesics: $p \propto \frac{1}{R}$

$$\boxed{\frac{\lambda_0}{R_0} = \frac{\lambda_E}{R_E}}$$

Expansion factor and redshift.

Working with our definition of redshift and the relation between $R(t)$ and the wavelength, we can write:

$$\Longrightarrow \frac{R_E}{R_0} = \frac{\lambda_E}{\lambda_0} = \frac{1}{\lambda_0/\lambda_E} = \frac{1}{\frac{\lambda_0}{\lambda_E} - 1 + 1} = \frac{1}{z+1} \qquad \boxed{\frac{R(t_E)}{R(t_0)} = \frac{1}{1+z}}$$

> NOTE
> This relation is very important because it does not depend on cosmology

And defining the scale factor $a(t)$ as the normalization of $R(t)$ by $R(t_0)$ $(a(t_0) = 1)$:

$$\boxed{\underbrace{a(t)}_{\text{calculable}} = \frac{R(t)}{R(t_0)} = \underbrace{\frac{1}{1+z}}_{\text{observable}}}, \qquad a(t) = \text{cosmic expansion factor}$$

z observable, $a(t)$ related to cosmology (need to learn how to calculate it)

2.2. Perfect fluids: equation of state and adiabatic expansion

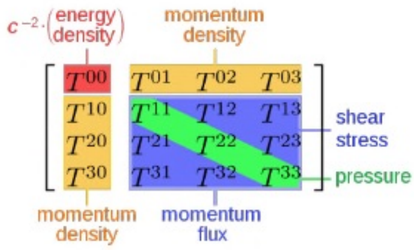
2.2.1. Derivation of Friedmann Equations

Friedmann equations and general relativity.

As we have already seen, Einstein's equations govern the behaviour of spacetime:

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu}}_{\text{geometry}} = \underbrace{8\pi G T_{\mu\nu}}_{\text{properties of matter}}$$

properties of matter,
↳ described by the energy - stress tensor $T_{\mu\nu}$.



geometry → described by the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R.

$$R_{\mu\nu} = f(g_{\mu\nu}) \qquad R = h(R_{\mu\nu})$$

Assuming isotropy / homogeneity for space (Robertson Walker metric) and that matter is a perfect fluid:

$$T_{\mu\nu} = -p g_{\mu\nu} + (\rho c^2 + p) u^\mu u^\nu,$$

one can obtain the Friedmann equations.

$$\boxed{\begin{aligned} \left(\frac{\dot{R}}{R}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{Kc^2}{R^2} + \frac{c^2\Lambda}{3} \\ \frac{\ddot{R}}{R} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3} \end{aligned}}$$

(Both equations are conected via Energy conservation)

Friedmann equations

From this equations we can obtain $R(t)$ (or $a(t)$) and study its dependences.

Critical density

Assuming no curvature and Λ, we can define the critical density as:

$$\boxed{\rho \mathrel{\hat{=}} \rho_{crit} = \frac{3H^2}{8\pi G}} \qquad H = \frac{\dot{R}}{R}$$

It will be used to normalise the Friedmann equation.

2.2.2. Equation of state

Equation of state of barotropic fluids

Solving the Friedmann Equations still requires an equation of state. We will restrict ourselves to barotropic fluids (i.e. linear relation between p and ρ):

$$\boxed{p = w(T)\rho c^2}$$

This expression is derived from the equation of state of an ideal gas:

$$p = nk_BT = \frac{k_BT}{m_pc^2}\rho_m c^2 = \frac{k_BT}{m_pc^2}\frac{\rho c^2}{\left(1 + \frac{k_BT}{(\gamma-1)m_pc^2}\right)} = w(T)\rho c^2$$

number density → mass density → energy density
↓ assume main contribution is coming from protons
↓ need to include the thermal energy.

$n \equiv$ number density
$\rho_m \equiv$ mass density
$\rho \equiv$ energy density

Dem: Mass density $\longrightarrow$ Energy density:

$$\rho c^2 = \rho_m c^2 + \mathcal{E} = \rho_m + \frac{P}{(\gamma-1)} = \rho_m c^2 + \frac{n k_B T}{(\gamma-1)} = \rho_m c^2 + \frac{\rho_m c^2 k_B T}{(\gamma-1) m_p c^2} = \rho_m c^2 \left(1 + \frac{k_B T}{(\gamma-1) m_p c^2}\right)$$

$p = (\gamma-1)\mathcal{E}$, γ = adiabatic coefficient , $\mathcal{E}$ thermal energy

Barotropic fluids in cosmology

All the terms in Friedmann equations can be interpreted as densities, and so have an associated equation of state.

- radiation: $\omega = 1/3$ (photons)
- collisionless matter: $\omega = 0$ $\rightarrow$ absence of collissions cannot build up pressure
- Vacuum energy: $\omega = -1$
- Curvature: $\omega = -1/3$

2.2.3. Adiabatic expansion

Energy conservation and Friedmann equations.

Almost all solutions predict an expansion or collapse of the Universe. This expansion is adiabatic (not isothermal or any other process).

In an adiabatic expansion there is a balance between the change in temperature (or energy) and pressure work. In adiabatic processes, entropy is conserved.

From the combination of 1st and 2nd Friedmann equations (also from $T^{\mu\nu}_{;\mu} = 0$) we can obtain energy conservation equation:

1. Friedmann equation

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - Kc^2 + \frac{c^2\Lambda}{3}R^2 \quad \frac{d}{dt}$$

$$2\dot{R}\ddot{R} = \frac{8\pi G}{3}(\dot{\rho}R^2 + 2\rho R\dot{R}) + 2R\dot{R}\frac{c^2\Lambda}{3}$$

$$\ddot{R} = \frac{4\pi G}{3}\left(\dot{\rho}\frac{R^2}{\dot{R}} + 2\rho R\right) + R\frac{c^2\Lambda}{3}$$

$$\frac{\ddot{R}}{R} = \frac{4\pi G}{3}\left(\dot{\rho}\frac{R}{\dot{R}} + 2\rho\right) + \frac{c^2\Lambda}{3}$$

2. Friedmann equation

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}$$

Equating both expressions:

$$\frac{4\pi G}{3}\left(\dot{\rho}\frac{R}{\dot{R}} + 2\rho\right) + \frac{c^2\Lambda}{3} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + c^2\frac{\Lambda}{3}$$

$$\left(\dot{\rho}\frac{R}{\dot{R}} + 2\rho\right) = -\left(\rho + \frac{3p}{c^2}\right)$$

$$\dot{\rho}\frac{R}{\dot{R}} + 2\rho = -\rho - \frac{3p}{c^2}$$

$$0 = \dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right)$$

$$\boxed{0 = \dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right)}$$

Energy conservation

Adiabatic expansion and energy conservation.

Now it is interesting to look at the second law of thermodynamics:

$TdS = dU + pdV$

The change in entropy is given by the change in energy and the pressure work. Let us take a look at the differentials:

- Volume element: $dV = d(R^3)$

  scales like R^3 since $r = Rx$ (comoving coordinates) $\longrightarrow V = R^3 V_c \longrightarrow dV = \overset{\text{constant}}{V_c}\, d(R^3)$

- Energy: need to multiply energy density by the volume, which scales like R^3

  $dU = d(V\rho c^2) = d(R^3 \rho c^2)$

Dividing by dt the expression of the 2nd law of thermodynamics we find:

$$\frac{dU}{dt} + p\frac{dV}{dt} = \frac{d(R^3\rho c^2)}{dt} + p\frac{d(R^3)}{dt} = 3R^2\dot{R}\rho c^2 + R^3\dot{\rho}c^2 + 3pR^2\dot{R} = 0$$

From energy conservation equation:

$$0 = R^3\dot{\rho}c^2 + 3\dot{R}R^2\rho c^2 + 3\dot{R}R^2 p$$

$$\longrightarrow T\frac{dS}{dt} = 0 \longrightarrow TdS = 0$$

During cosmic expansion entropy is conserved, and hence the expansion is adiabatic.

$dU + pdV = 0$

Any change in energy must be compensated by pressure work: The volume is changing, so energy for the expansion must come from somewhere. It comes from the change in energy of the contents of the Universe: the photons are redshifted.

Implication for barotropic fluids

Let us remember the equation of state for this fluids:

$p = w\rho c^2$

Using again dU and dV, but substituting p by the equation of state:

$$\left.\begin{aligned} dU &= d(R^3 \rho c^2) \\ dV &= d(R^3) \end{aligned}\right\} \quad d(R^3\rho c^2) = -p\, d(R^3) = -w\rho c^2 d(R^3)$$

$$0 = R^3 d\rho + \rho\, d(R^3) + w\rho\, d(R^3) = R^3 d\rho + (1+w)\rho\, d(R^3)$$

$$\frac{1}{\rho} d\rho = -(1+w)\frac{d(R^3)}{R^3}$$

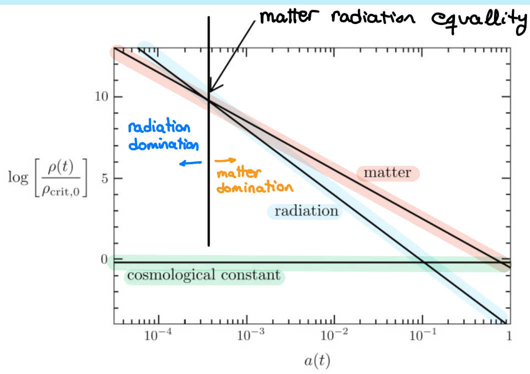
This differential equations gives the relation between ρ and the scale factor. If we solve it, we find:

$$\boxed{\rho R^{3(1+w)} = \text{const}} \qquad \rho R^\alpha = \rho_0 R_0^\alpha$$

Choose scaling $R_0^\alpha = 1$

- radiation: $w = 1/3 \longrightarrow \rho_r \propto R^{-4}$
- collisionless matter: $w = 0 \longrightarrow \rho_m \propto R^{-3}$
- Vacuum energy: $w = -1 \longrightarrow \rho_\Lambda = \text{const}$
- Curvature: $w = -1/3 \longrightarrow \rho_k \propto R^{-2}$

NOTE
- Matter density scales like $1/R^3$ because it is mass/volumen
- The additional $1/R$ factor for radiation is due to de energy change produced by z



2.2.4. Density parameters

Definition. Rewriting Friedmann equations

First Friedmann equation

The introduction of density parameters leads to the most memorable form for the Friedmann equations. We start with Friedmanns equation:

$$\left(\frac{\dot R}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{c^2\Lambda}{3}$$

where $\rho = \sum_w \rho_w(R(t))$ $\begin{cases} \text{Radiation: } w = 1/3 \longrightarrow \rho(R) \propto R^{-4} \\ \text{Collisionless matter: } w = 0 \longrightarrow \rho(R) \propto R^{-3} \end{cases}$

We could also include Λ and k in the density term, but we will focus first on radiation and dark matter.

Since we know how energy densities scale, we can rewrite the Friedmann equation in terms of the present density content and R_0.

$$\longrightarrow \left(\frac{\dot R}{R}\right)^2 = \frac{8\pi G}{3}\left[\rho_{m,0}\left(\frac{R}{R_0}\right)^{-3} + \rho_{r,0}\left(\frac{R}{R_0}\right)^{-4}\right] - \frac{kc^2}{R_0^2}\left(\frac{R}{R_0}\right)^{-2} + \frac{c^2\Lambda}{2}$$

We can use the critical density to normalise the densities and elliminate the prefactor. We can define:

> **REMINDER**
> Critical density:
> $$\rho_{crit,0} = \frac{3H_0^2}{8\pi G}$$

THE COSMOLOGICAL PARAMETERS

- Matter density: $\Omega_{m,0} = \frac{8\pi G}{3H_0^2}\rho_{m,0} = \frac{\rho_{m,0}}{\rho_{crit,0}}$
- Radiation density: $\Omega_{r,0} = \frac{8\pi G}{3H_0^2}\rho_{r,0} = \frac{\rho_{r,0}}{\rho_{crit,0}}$
- Vacuum energy: $\Omega_{\Lambda,0} = \frac{c^2}{3H_0^2}\Lambda$
- Curvature: $\Omega_{K,0} = -\frac{c^2}{H_0^2 R_0^2}k$
- Expansion rate: $H_0 = \left(\frac{\dot R}{R}\right)_0$

Substituting this on the Friedmann equation and remembering that $\frac{R}{R_0} = \frac{1}{1+z}$:

$$\boxed{H^2 = H_0^2\left[\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{K,0}(1+z)^2 + \Omega_{\Lambda,0}\right]}$$ → Uses density parameters at present time

Note that these cosmological density parameters are the values measured todady. Calculating the Hubble parameter at today's time gives the known cosmic sum rule:

$1 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{K,0} + \Omega_{\Lambda,0}$

This holds at any time: $\boxed{1 = \Omega_r(z) + \Omega_m(z) + \Omega_K(z) + \Omega_\Lambda(z)}$ (for decoupled matter and radiation.

Second Friedmann equation

barotropic fluids

$$\frac{\ddot R}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3} = -\frac{4\pi G}{3}\left(\sum\rho_i + \frac{3\sum w_i\rho_i c^2}{c^2}\right) + \frac{c^2\Lambda}{3} = -\frac{4\pi G}{3}\left(\sum\rho_i + 3\sum w_i\rho_i\right) + \frac{c^2\Lambda}{3} =$$

only considering radiation and matter

$$= -\frac{4\pi G}{3}\sum(1+3w_i)\rho_i + \frac{c^2\Lambda}{3} = -\frac{H^2}{2}\left(\sum(1+3w_i)\Omega_i(z)\right) + H^2\Omega_\Lambda(z) = -\frac{H^2}{2}\left(\Omega_m(z) + 2\Omega_r(z)\right) + H^2\Omega_\Lambda(z) =$$

Multiplying by $-\frac{1}{H^2}$:

$$-\frac{\ddot R R}{\dot R^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

$$\boxed{q = -\frac{\ddot R R}{\dot R^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)}$$ → Uses density parameters at redshift z

Deceleration parameter

$\Omega(z)$: evolution of density parameters

To calculate the evolution of the density parameters we just need to look at their definition, adding how each term evolves with redshift:

$$\Omega(z) = \frac{8\pi G}{3H^2(z)}\,\rho(z) \overset{\text{barotropic matter}}{=} \frac{8\pi G}{3H^2(z)}\,\rho_0\,(1+z)^{3(1+\omega)}$$

$$\Omega(z) = \frac{8\pi G}{3H^2(z)}\,\rho(z) \overset{\text{1. Friedmann equation}}{=} \frac{8\pi G}{3H_0^2(z)}\,\frac{\rho_0\,(1+z)^{3(1+\omega)}}{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{K,0}(1+z)^2+\Omega_{\Lambda,0}}$$

REMINDER

Radiation: $\omega = 1/3$

Matter: $\omega = 0$

Vacuum energy: $\omega = -1$

$$\boxed{\Omega(z) = \frac{8\pi G}{3H^2(z)}\,\rho(z) \overset{\rho_0 \to \Omega_0}{=} \frac{\Omega_0(1+z)^{3(1+\omega)}}{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{K,0}(1+z)^2+\Omega_{\Lambda,0}}}$$

NOTE:

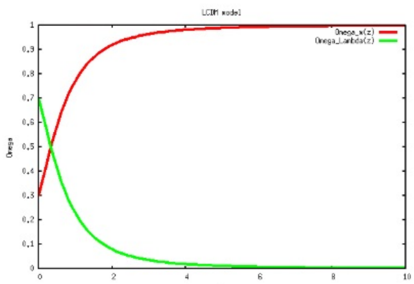
$$\Omega(z) = \frac{c^2\Lambda}{3H^2(z)} = \frac{\Omega_\Lambda}{\Omega_{r,0}(1+z)^4+\Omega_{m,0}(1+z)^3+\Omega_{K,0}(1+z)^2+\Omega_{\Lambda,0}}$$; same result as for $\omega = -1$

Ignoring radiation (since it is not important in that redshift interval):

ΛCDM model:

$\Omega_m(z) \longrightarrow 1$

$\Omega_\Lambda(z) \longrightarrow 0$



NOTE FOR LATER

$$\frac{\Omega_i}{\Omega_j} = \frac{\Omega_{i,0}}{\Omega_{j,0}}\,(1+z)^{(n_i - n_j)}$$

with $n = 3(1+\omega)$

There are actually only three independent density parameters, since:

$$1 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{K,0} + \Omega_{\Lambda,0} \quad \text{for } z = 0$$

Interplay scale factor - redshift - time

If we know the values of the density parameters today, we can solve the Friedmann equation as a function of time, and we have a direct relation to something that we can observe (z). For standard cosmological models we have a unique mapping between a, t and z. Further, the constancy of the speed of light then also relates those to distance.

If calculations do not match observations, it is necessary to change the Ωs used to solve the equation (iteratively).

$$H^2 = H_0^2\left[\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{K,0}(1+z)^2 + \Omega_{\Lambda,0}\right]$$

$\Rightarrow$ solution is $a(t)$

$\Rightarrow$ relation to redshift $\quad a(t) = \dfrac{1}{1+z} \longrightarrow$ observable.

2.3. Solutions of Friedmann equations for simple models

2.3.1. General trends

We will analyse the solution of Friedmann equations for different parameter values.

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - Kc^2 + \frac{c^2\Lambda}{3}R^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}R$$

$\Lambda = 0$, barotropic fluid ($p = w\rho c^2$)

$\dot{R}^2 \propto R^{2-3(1+w)} - Kc^2$ as $\rho \propto R^{-3(1+w)}$

$\ddot{R} < 0$ as $\rho + \frac{3p}{c^2} > 0$ ($p = w\rho c^2$, $w > 0$)

NOTE
$2 - 3(1+w) < 0$

Today:

$$\left(\frac{\dot{R}}{R}\right)_0 > 0 \xrightarrow[R \neq 0]{} (\dot{R}^2)_0 > 0$$

$\dot{R} > 0$, $\ddot{R} < 0$

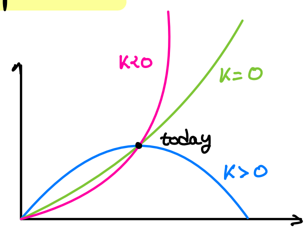
→ $R(t)$ has been monotonically increasing

$\exists\, t_c$ with $R(t_c) = 0$ ⟶ singularity

Future

- $K = 0$: $\dot{R}^2 \propto R^{2-3(1+w)} \longrightarrow R \propto t^{2/3(1+w)}$ ⇒ indefinite expansion.
  ↪ $R^{3(1+w)/2 \;-1} = dt$
- $K > 0$: $\dot{R}^2 \propto R^{2-3(1+w)} - |K|c^2 \longrightarrow \exists\, \dot{R}(t_s) = 0$ ⇒ recollapse

$$\left.\begin{array}{l}\dot{R}(0) = +\infty \\ \dot{R}(\infty) = -|K|c^2\end{array}\right\} \text{change of sign}$$



- $K < 0$: $\dot{R}^2 \propto R^{2-3(1+w)} + |K|c^2 \longrightarrow \dot{R}^2 > 0 \longrightarrow R(t) \uparrow$ ⇒ increased indefinite expansion

$\Lambda < 0$

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - Kc^2 - \frac{c^2|\Lambda|}{3}R^2 = \frac{8\pi G}{3}\rho_{m+\Lambda} R^2 - Kc^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) - \frac{c^2|\Lambda|}{3}R = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2} + \frac{c^2|\Lambda|}{12\pi G}\right) < 0$$

↪ Attractive force

$R(t)$ starts monotonically increasing and $\exists\, \dot{R}(t_s) = 0$

Recollapse happens even earlier due to the "attractive force".

$\Lambda > 0$ (measured)

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - Kc^2 - \frac{c^2|\Lambda|}{3}R^2 = \frac{8\pi G}{3}\rho_{m+\Lambda} R^2 - Kc^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) - \frac{c^2|\Lambda|}{3}R = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2} - \frac{c^2|\Lambda|}{12\pi G}\right) < 0$$

repulsive force

Forever expanding Universe (accelerated)

NOTE: ρ drops, but Λ remains constant

2.3.2. Special solutions for flat and open world models.

We will use the expression of the 1st Friedmann equation in terms of the density parameters.

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left[\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{K,0}(1+z)^2 + \Omega_{\Lambda,0}\right]$$

These solutions will be valid to first order because we will only take into account one of the components. However, this is useful to analyse the scaling relations at the epoch in which each component dominates the expansion of the Universe.

Flat model with $\Omega_m = 1$

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\frac{R_0}{R}\right)^3 \quad \text{(Friedmann equation)}$$

NOTE

This solution is important after matter-radiation equality

$$\frac{\dot{R}}{R} = H_0\left(\frac{R_0}{R}\right)^{3/2}$$

$$R^{3/2}\frac{\dot{R}}{R} = H_0 R_0^{3/2}$$

Matter domination

$$R^{1/2}\,dR = H_0 R_0^{3/2}\,dt \xrightarrow{\text{solving the differential equation}} \boxed{\frac{R(t)}{R_0} = \left(\frac{t}{t_0}\right)^{2/3}, \quad t_0 = \frac{2}{3H_0}}$$

Flat model with $\Omega_r = 1$

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\frac{R_0}{R}\right)^4 \quad \text{(Friedmann equation)}$$

NOTE

This solution is important before matter-radiation equality.

$$\frac{\dot{R}}{R} = H_0\left(\frac{R_0}{R}\right)^2$$

$$R^2\frac{\dot{R}}{R} = H_0 R_0^2$$

Radiation domination

$$R\,dR = H_0 R_0^2\,dt \xrightarrow{\text{solving the differential equation}} \boxed{\frac{R(t)}{R_0} = \left(\frac{t}{t_0}\right)^{1/2}, \quad t_0 = \frac{1}{2H_0}}$$

Flat model with $\Omega_\Lambda = 1$

$$\left(\frac{\dot R}{R}\right)^2 = H_0^2$$

$$\frac{\dot R}{R} = H_0$$

$$\frac{dR}{R} = H_0\, dt \longrightarrow \boxed{R(t) = e^{H_0 t}} \qquad \text{inflationary Universe}$$

General models with $\Omega_m \neq 0$, $\Omega_\Lambda = 0$ $\Omega_r = 0$

$$\left(\frac{\dot R}{R}\right)^2 = H_0^2\left(\Omega_{m,0}\left(\frac{R_0}{R}\right)^3 + \Omega_{K,0}\left(\frac{R_0}{R}\right)^2\right)$$

$$\dot R^2 = R^2 H_0^2 \left(\frac{R_0}{R}\right)^2\left(\Omega_{m,0}\left(\frac{R_0}{R}\right) + \Omega_{K,0}\right)$$

$$\dot R = H_0 R_0 \sqrt{\Omega_{m,0}\left(\frac{R_0}{R}\right) + (1-\Omega_{m,0})}$$

$$\dot R = H_0 R_0 \sqrt{\Omega_{m,0}\left(\frac{R_0}{R} - 1\right) + 1} \longrightarrow$$

Need to construct $R(t)$ using the parametric sol.

$\Omega_{m,0} > 1 \longrightarrow K < 0$

$$R = a(1-\cos\theta) \qquad a = \frac{\Omega_{m,0}}{2(\Omega_{m,0}-1)}$$

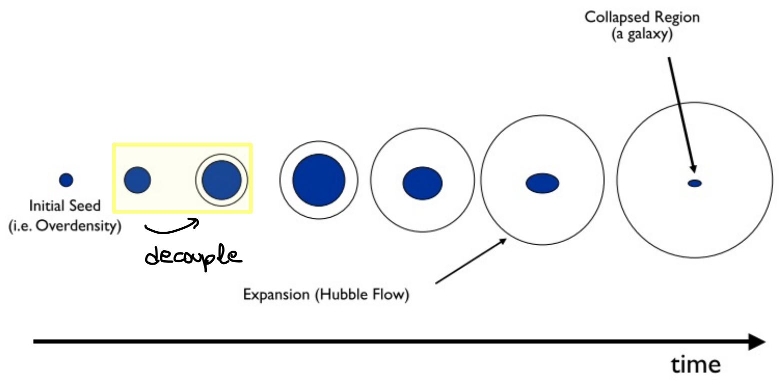
$$t = b(\theta - \cos\theta) \qquad b = \frac{\Omega_{m,0}}{2H_0(\Omega_{m,0}-1)^{3/2}}$$

$\Omega_{m,0} < 1$

$$R = A(\cosh\theta - 1) \qquad A = \frac{\Omega_{m,0}}{2(1-\Omega_{m,0})}$$

$$t = B(\sinh\theta - \theta) \qquad B = \frac{\Omega_{m,0}}{2H_0(1-\Omega_{m,0})^{3/2}}$$

The first solution ($\Omega_m > 1$) is important for structure formation.



We start with an overdensity $\delta > 1$. The background is expanding. If δ is larger than a certain value (1.68), then the density inside the volume will decouple from the expansion of the Universe and, eventually, recollapse.

We can say that this region is "a Universe inside a Universe" with $\Omega_m > 1$. Thus, we can use this solution to describe its evolution (his own $R(t)$).

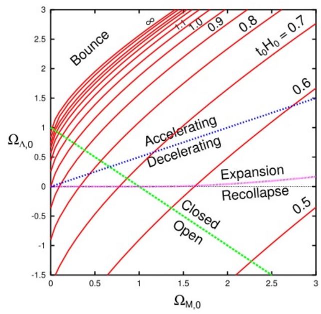
Flat model with $\Omega_m + \Omega_\Lambda = 1$, $\Omega_r = 0$ (our Universe today)

$$t(z) = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_{\Lambda,0}}} \ln\left(\sqrt{\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\left(\frac{1}{1+z}\right)^3} + \sqrt{\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\left(\frac{1}{1+z}\right)^3 + 1}\right)$$

Up to redshift 100 it is not necessary to take radiation into account (Now $\Omega_r \sim 10^{-6}$)

Can also calculate $R(z)$ from $t(z)$ finding $z(t)$ and then using $R = \frac{1}{1+z}$

General model $\Omega_m \neq 0$, $\Omega_\Lambda \neq 0$, $\Omega_r \neq 0$



There is no analytical solution for the general case.

It is necessary to integrate numerically.

2.3.3. Horizons

Definition

We always talk about $R(t)$, but it is not the size of the Universe, but the scale factor of the coordinates. There are multiple definitions for the edge of the Universe.

Cosmological horizon measure distances from which one could possibly retrieve information from the past or interact with in the future.

Distances.

We go back to our metric:

$$ds^2 = (cdt)^2 - R^2 \left[\frac{dx^2}{1-Kx^2} + x^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right]$$

Photons follow null geodesics:

$$0 = (cdt)^2 - R^2(t) \left[\frac{dx^2}{1-Kx^2} + x^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) \right]$$

We can define the distance as:

$$\boxed{d_c = f(x_E) = \int_0^{x_E} \frac{dx}{\sqrt{1-Kx^2}} = \int_{t_E}^{t_0} \frac{c\,dt}{R(t)}}$$

comoving distance

$$f(x_E) = \begin{cases} x_E & K = 0 \\ \frac{1}{\sqrt{|K|}} \arcsin(\sqrt{|K|}\, x_E) & K = 1 \\ \frac{1}{\sqrt{|K|}} \operatorname{arcsinh}(\sqrt{|K|}\, x_E) & K = -1 \end{cases}$$

$$\boxed{d_p = \frac{R(t)}{R_0} d_c = \frac{R(t)}{R_0} \int_{t_E}^{t_0} \frac{c\,dt}{R(t)}}$$

Proper distance

Past horizon:

t_0 = today $\qquad t_E$ = past

Future horizon

t_E = today $\qquad t_0 = \infty$

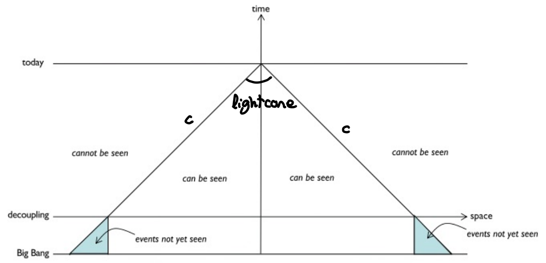
Types of horizons

Particle horizon

Max distance that a particle can have travelled since decoupling:

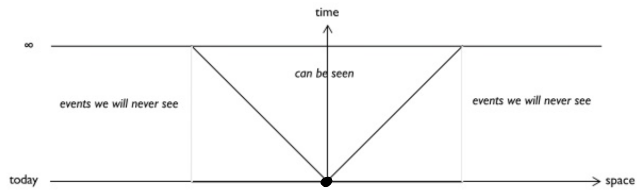
$$R_P(t) = R(t) \int_{t_{dec}}^{t} \frac{c\,dt'}{R(t')}$$

"Particle horizon" (for some textbooks)



Max distance a photon can have travelled since Big Bang (there are events we have not seen yet).

Even horizon



Max distance a particle can travel from now onwards (there are events that we will never see)

$$R_e(t) = R(t) \int_t^{\infty} \frac{c\,dt'}{R(t')}$$

Hubble radius

Distance at which recessional velocity equals speed of light

$$R_H = \frac{c}{H} \qquad R_{CH}(t) = \frac{R_0}{R}\frac{c}{H}$$

comoving