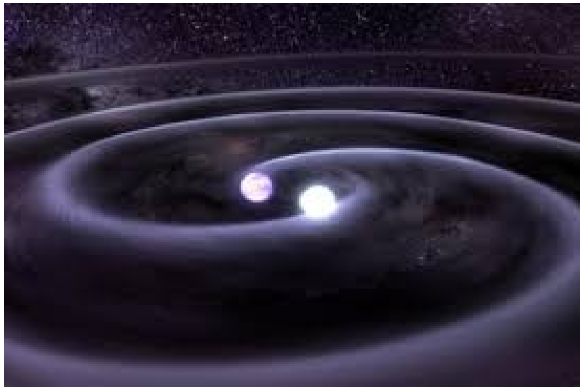
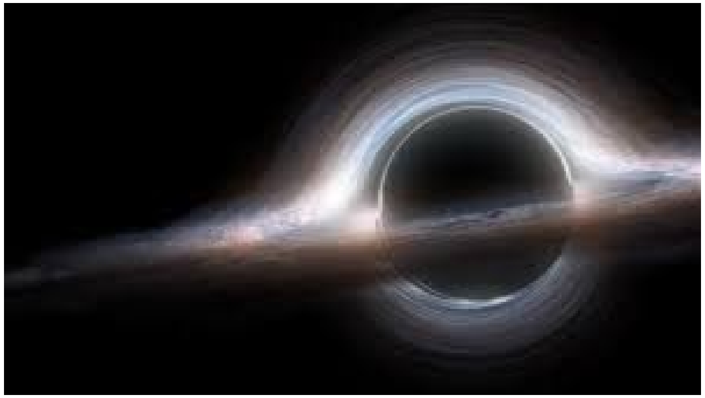
Gravitational Waves





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Main points of the lecture

- What are the GWs (history, description)
- Formalism in GR (linearization, gauges, emission)
- Detection techniques (interferometry, LIGO)
- Recent observations (BH-BH, NS-NS)
- Other issues (speed of GWs, hyperbolic encounters)

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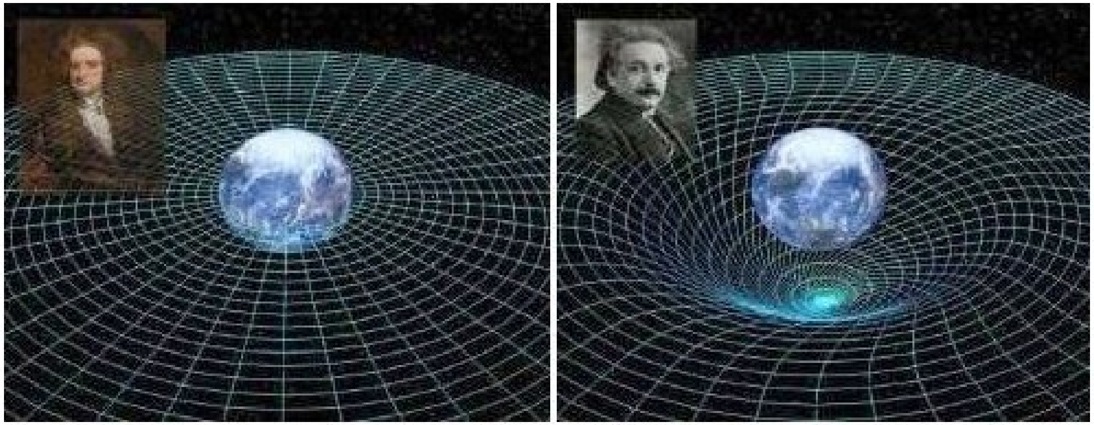
Lecture Notes on Gravitational Waves

The spacetime:

Newton vs Einstein

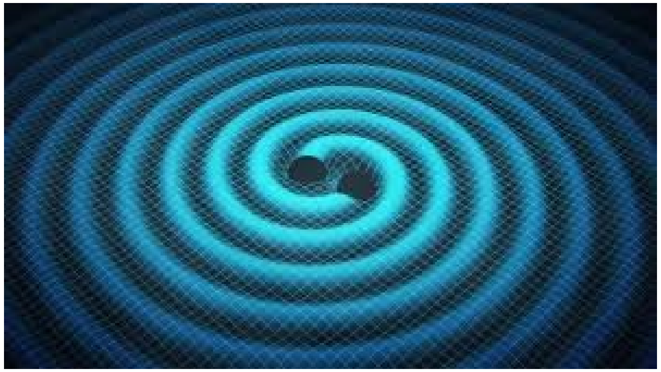


Newton's fixed space   Einstein's flexible space-time

source: NASA

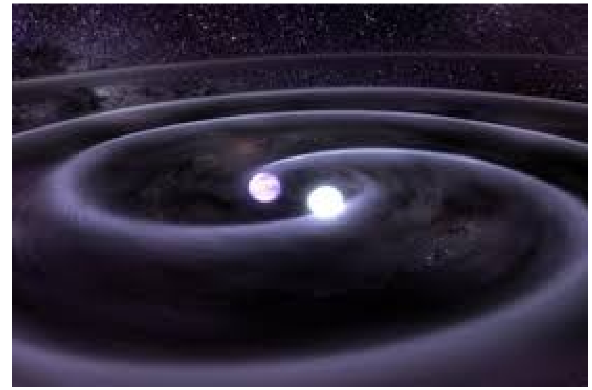
Accelerating masses

cause ripples in space-time



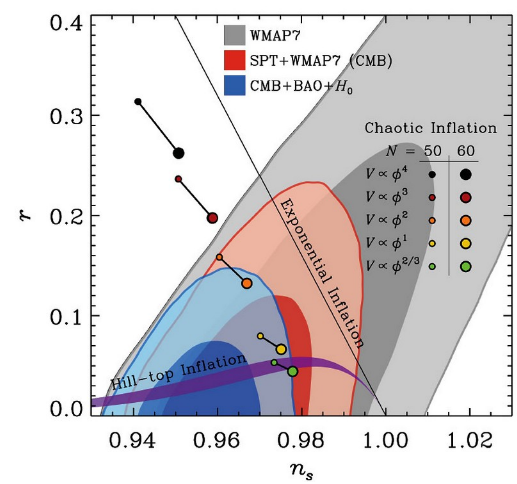
Lecture Notes on Gravitational Waves

Sources of GWs:



1) Binary systems, eg BH-BH or NS-NS etc

2) Tensor perturbations (inflation) → They affect the CMB



3) Supernovae (core collapse)

Massive star (~10-30 Msun) develops iron core which collapses in T~100ms. Proto neutron star forms → EoS stiffens → bounce → GWs

Lecture Notes on Gravitational Waves

Brief history:

In 1915-16 Einstein formulated General Relativity

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Soon after, he conjectured the existence of wave solutions, but was uncertain due to gauge artifacts.

Letter to Schwarzschild in 1916:

"Since then [November 14] I have handled Newton's case differently, of course, according to the final theory [the theory of General Relativity]. Thus there are no gravitational waves analogous to light waves. This probably is also related to the one-sidedness of the sign of the scalar T, incidentally [this implies the nonexistence of a "gravitational dipole"] [6].

Lecture Notes on Gravitational Waves

Later Einstein found 3 types of waves, but Eddington showed two of them were spurious due to the choice of frame...

In 1936 he tried to publish a paper in Physical Review that GWs do not exist (!) and the referee (Robertson of the FRW metric fame) rejected it. So, Einstein sent an angry letter to the editor:

July 27, 1936

Dear Sir.

"We (Mr. Rosen and I) had sent you our manuscript for publication and had not authorized you to show it to specialists before it is printed. I see no reason to address the—in any case erroneous—comments of your anonymous expert. On the basis of this incident I prefer to publish the paper elsewhere."

Respectfully

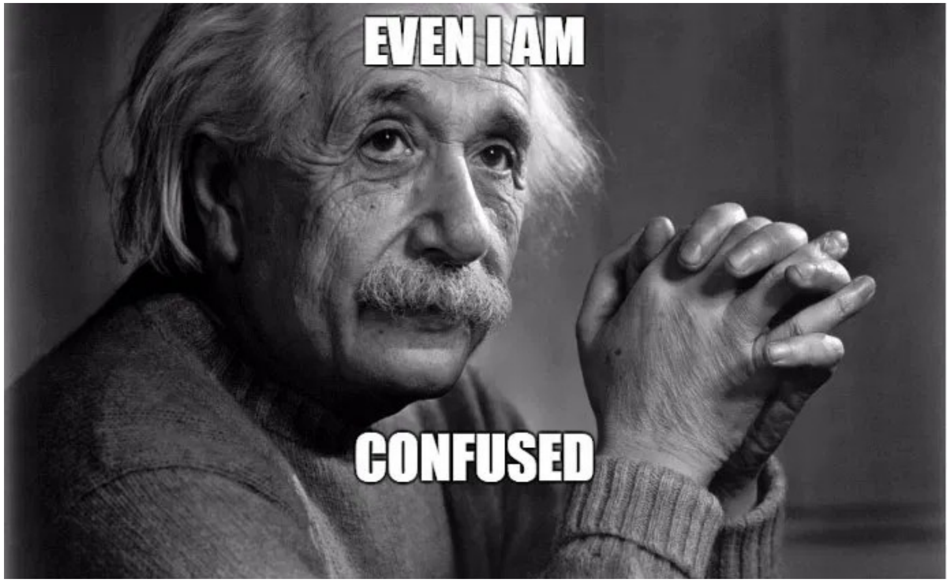
Einstein

P.S. Mr. Rosen, who has left for the Soviet Union, has authorized me to represent him in this matter.

Lecture Notes on Gravitational Waves

Later Einstein changed his mind again and now believed in GWs after realizing the error in his calculations. He then changed the title and published the paper as “On gravitational waves”.

“Note—The second part of this article was considerably altered by me after the departure to Russia of Mr. Rosen as we had misinterpreted the results of our formula. I want to thank my colleague Professor Robertson for their friendly help in clarifying the original error. I also thank Mr. Hoffmann your kind assistance in translation.”



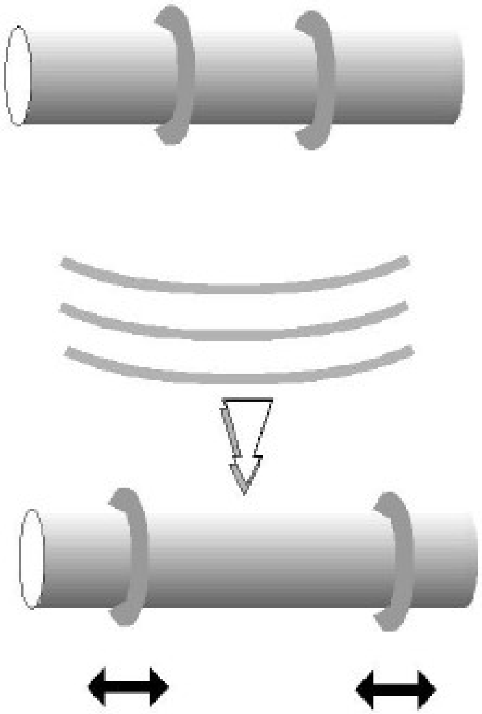
Lecture Notes on Gravitational Waves

Argument settled forever in 1957 by Feynman:

In a letter to Victor Weisskopf, Feynman recalls the 1957 conference in Chapel Hill and says, "I was surprised to find that a whole day of the conference was spent on this issue and that 'experts' were confused. That's what happens when one is considering energy conservation tensors, etc. instead of questioning, can waves do work?" [19].

Feynman's argument that GWs are real:

They displace the beads, thus producing heat (due to friction)!



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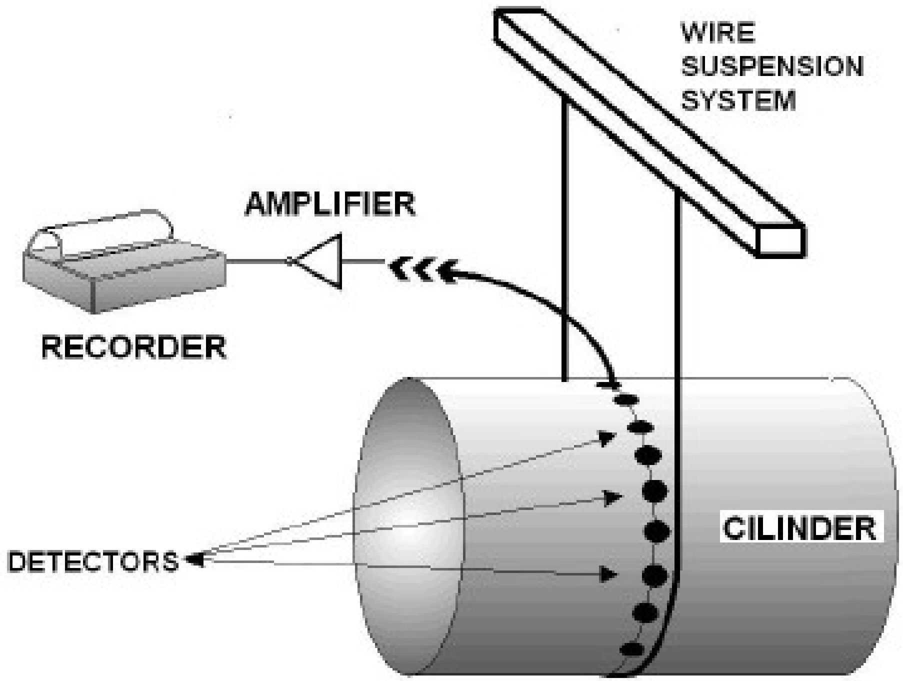
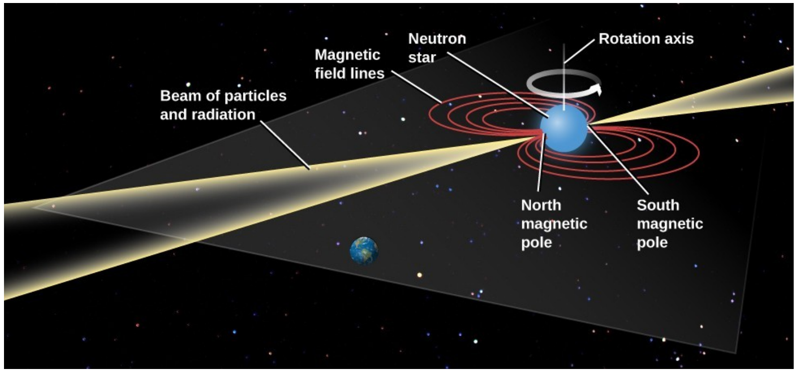
First detector in 1960 by Joseph Weber

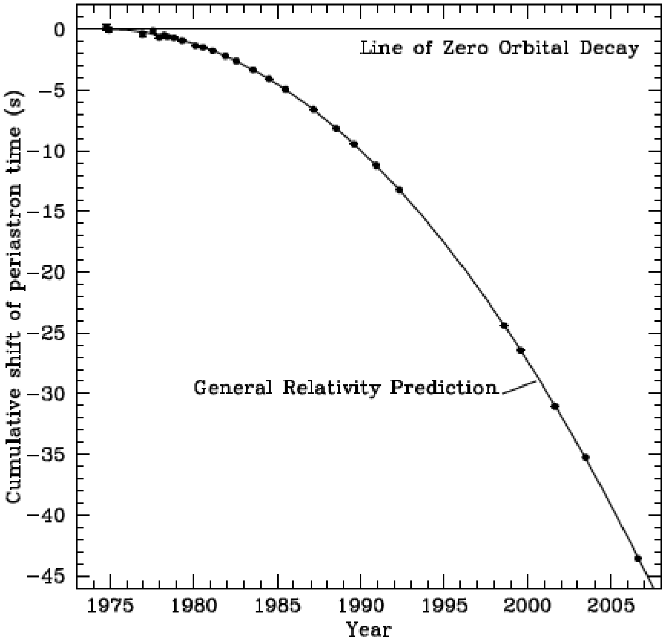


Lecture Notes on Gravitational Waves

A pulsar is a highly magnetized rotating neutron star that emits beams of EM radiation out of its magnetic poles. They are very precise clocks! Eg J0437-4715 has a period of 0.005757451936712637 secs with error of 1.7×10^−17 secs!!



In 1974 Hulse and Taylor found that a pair of binary pulsars was inspiralling in perfect agreement with GR!



Lecture Notes on Gravitational Waves

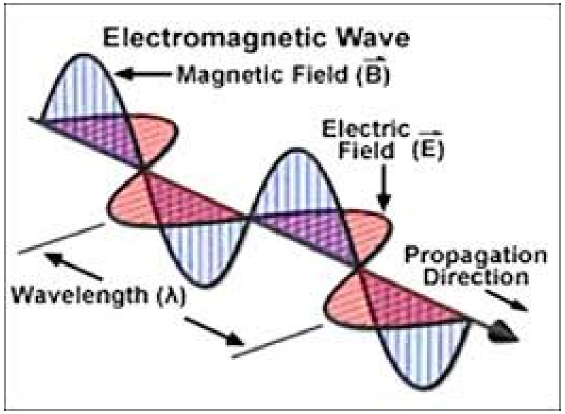
Better way to detect GWs is with interferometry! In 2002 LIGO started operating until 2010.

AdvLIGO started in 2015.
More details later on...

Lecture Notes on Gravitational Waves

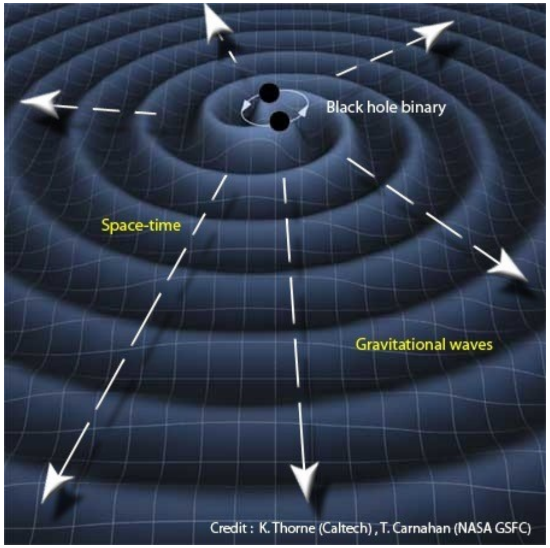
Difference between GWs and EM waves:

i) EM waves travel through space, GWs are ripples in spacetime itself



ii) EM waves can be absorbed, GWs cannot

iii) GWs are weakly interacting,
EM waves strongly interact with charges

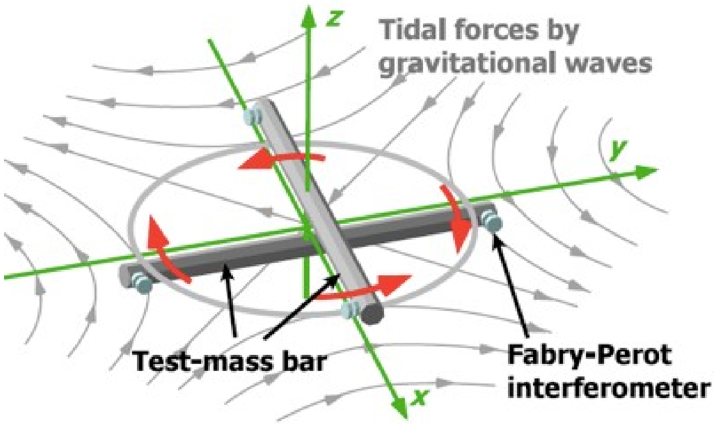


iv) GWs produced at minimum by quadrupole, EM by dipole. More later on...

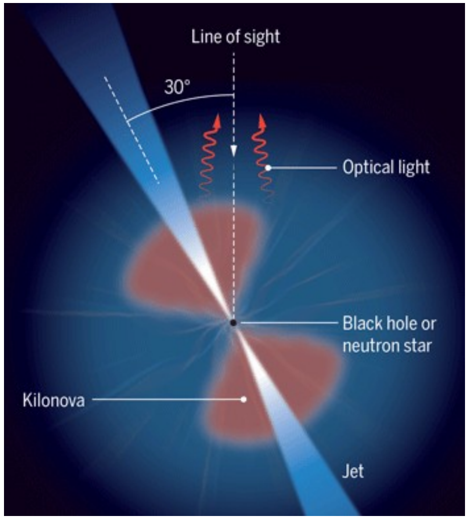
Lecture Notes on Gravitational Waves

Furthermore:

i) GWs are travelling, time-dependent tidal forces



ii) GW allow for a measurement of the luminosity distance $d_L(z)$, but not the redshift z!



iii) With EM counterpart we can construct Hubble diagram as for we do for the supernovae

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Lecture Notes on Gravitational Waves

Gravity is weak and GWs interact weakly, so we need to linearize GR

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

GR is diffeomorphism invariant

$$x^\mu \to x'^\mu(x) \quad \Rightarrow \quad g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

Small perturbation around empty space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \qquad |h_{\mu\nu}| \ll 1 \,,$$

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$

$$\Rightarrow \quad h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

Lecture Notes on Gravitational Waves

Linearize the Riemann tensor

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho})$$

Introduce "barred" h:

$$h = \eta^{\mu\nu} h_{\mu\nu}$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

$\Rightarrow$

$$\bar{h} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu} = h - 2h = -h$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$$

Use that in the full equations

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu \bar{h}_{\mu\rho} - \partial^\rho\partial_\mu \bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Lecture Notes on Gravitational Waves

Residual freedom, choose gauge (Lorentz gauge → GR eqs become decoupled wave equations)

$$\partial^\nu \bar{h}_{\mu\nu} = 0$$

Why is this possible?

$$\Box = \eta_{\mu\nu}\partial^\mu\partial^\nu = \partial_\mu\partial^\mu$$

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho) \quad \Rightarrow \quad \partial^\nu \bar{h}_{\mu\nu} \rightarrow (\partial^\nu \bar{h}_{\mu\nu})' = \partial^\nu \bar{h}_{\mu\nu} - \Box\xi_\mu$$

$$\partial^\nu \bar{h}_{\mu\nu} = f_\mu(x) \quad \Rightarrow \quad \Box\xi_\mu = f_\mu(x)$$

Final result: with sources or in vacuum

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \qquad \Box\bar{h}_{\mu\nu} = 0$$

Lecture Notes on Gravitational Waves

Use the gauge to remove spurious degrees of freedom (dof).

Question: How many *propagating* dof does GR have?

$$\Box \xi_\mu = 0 \qquad\qquad h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

Make the choices

$$\xi^0 \Rightarrow \bar{h} = 0$$

$$\xi^i(x) \Rightarrow h^{0i}(x) = 0$$

Eliminate some of the hij

$$\partial^\nu \bar{h}_{\mu\nu} = 0 \Rightarrow \partial^0 h_{00} + \partial^i h_{0i} = 0 \Rightarrow \partial^0 h_{00} = 0$$

Finally the TT gauge: $h^{0\mu} = 0\,, \quad h^i{}_i = 0\,, \quad \partial^j h_{ij} = 0$

Lecture Notes on Gravitational Waves

Solutions in vacuum are plane waves

$\Box \bar{h}_{\mu\nu} = 0$ $\Rightarrow$ $h^{TT}_{ij}(x) = e_{ij}(\mathbf{k}) e^{ikx}$ $\quad k^{\mu} = (\omega/c, \vec{k})$ and $\omega/c = |\vec{k}|$

The polarizations

$\hat{n} = \vec{k}/|\vec{k}|$

$\partial^j h_{ij} = 0$ $\Rightarrow$ $n^i h_{ij} = 0$ $\Rightarrow$ $h^{TT}_{ij}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos[\omega(t - z/c)]$

Structure of space-time

$$ds^2 = -c^2 dt^2 + dz^2 + (1 + h_+ \cos[\omega(t - z/c)])\, dx^2 + (1 - h_+ \cos[\omega(t - z/c)])\, dy^2 + 2h_\times \cos[\omega(t - z/c)] dx dy$$

Expansion in Fourier space:

$$e^{+}_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ab}$$

$$h_{ab}(t, \mathbf{x}) = \sum_{A=+,\times} \int_{-\infty}^{\infty} df \int d^2\hat{\mathbf{n}}\ \tilde{h}_A(f, \hat{\mathbf{n}}) e^{A}_{ab}(\hat{\mathbf{n}})\, e^{-2\pi i f(t - \hat{\mathbf{n}}\cdot\mathbf{x}/c)}$$

$$e^{\times}_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_{ab}$$

Lecture Notes on Gravitational Waves

Effect on masses: study the geodetic deviation for two geodesics

$$x^\mu(\tau)$$
$$x^\mu(\tau) + \xi^\mu(\tau)$$

$$\frac{D^2 \xi^\mu}{D\tau^2} = -R^\mu{}_{\nu\rho\sigma}\, \xi^\rho \, \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau}$$

$$\ddot{\xi}^i = \frac{1}{2} \ddot{h}_{ij}^{TT} \xi^j$$

The + polarization:

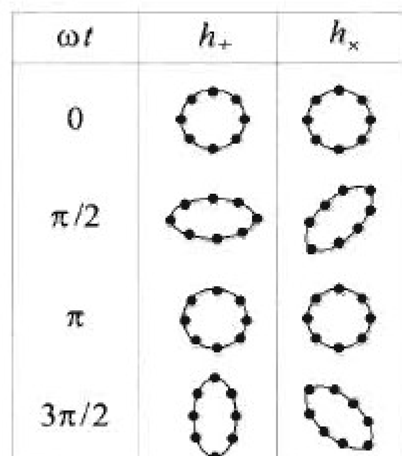
$$h_{ab}^{\mathrm{TT}} = h_+ \sin \omega t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned} \delta\ddot{x} &= -\frac{h_+}{2}(x_0 + \delta x)\omega^2 \sin(\omega t) \\ \delta\ddot{y} &= +\frac{h_+}{2}(y_0 + \delta y)\omega^2 \sin(\omega t) \end{aligned}$$

$$\begin{aligned} \delta x(t) &= +\frac{h_+}{2} x_0 \sin(\omega t) \\ \delta y(t) &= -\frac{h_+}{2} y_0 \sin(\omega t) \end{aligned}$$

The x polarization

$$\begin{aligned} \delta x(t) &= \frac{h_\times}{2} y_0 \sin(\omega t) \\ \delta y(t) &= \frac{h_\times}{2} x_0 \sin(\omega t) \end{aligned}$$



Lecture Notes on Gravitational Waves

Feynman showed that GWs do work and carry energy. Energy of a wave is E~h^2, so second order!!! Do expansion:

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \ldots$$

Rewrite the Einstein eqs and average of wavelengths:

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \Longrightarrow \bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu}\rangle + \frac{8\pi G}{c^4}\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle$$

Define the energy momentum tensor of the waves

$$t_{\mu\nu} = -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle \Longrightarrow \bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}(\bar{T}_{\mu\nu} + t_{\mu\nu})$$

Lecture Notes on Gravitational Waves

Do the expansion:

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\left[\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} - h^{\alpha\beta}\partial_\nu\partial_\beta h_{\alpha\mu} - h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu}\right.$$
$$+h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial^\beta h^\alpha_\nu\partial_\beta h_{\alpha\mu} - \partial^\beta h^\alpha_\nu\partial_\alpha h_{\beta\mu} - \partial_\beta h^{\alpha\beta}\partial_\nu h_{\alpha\mu}$$
$$+\partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu} - \partial_\beta h^{\alpha\beta}\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial^\alpha h\partial_\alpha h_{\mu\nu} + \frac{1}{2}\partial^\alpha h\partial_\nu h_{\alpha\mu}$$
$$\left.+\frac{1}{2}\partial^\alpha h\partial_\mu h_{\alpha\nu}\right].$$

The GW energy momentum tensor is

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle \quad \Longrightarrow \quad t^{00} = \frac{c^2}{16\pi G}\langle\dot{h}_+^2 + \dot{h}_\times^2\rangle$$

The energy flux and momentum carried by the waves are:

$$E_V = \int_V d^3x\, t^{00} \quad \xrightarrow{\partial_\mu t^{\mu\nu} = 0}$$

$$\frac{dE}{dt} = \frac{c^3 r^2}{32\pi G}\int d\Omega\,\langle\dot{h}^{TT}_{ij}\dot{h}^{TT}_{ij}\rangle \qquad \frac{dP^k}{dt} = -\frac{c^3}{32\pi G}r^2\int d\Omega\,\langle\dot{h}^{TT}_{ij}\partial^k h^{TT}_{ij}\rangle$$

$$\frac{dE}{dA} = \frac{c^3}{16\pi G}\int_{-\infty}^{\infty} dt\,\left(\dot{h}_+^2 + \dot{h}_\times^2\right) \qquad J^i = \frac{c^2}{32\pi G}\int d^3x\left[-\epsilon^{ikl}\dot{h}^{TT}_{ab}x^k\partial^l h^{TT}_{ab} + 2\epsilon^{ikl}h^{TT}_{ak}\dot{h}^{TT}_{al}\right]$$

Lecture Notes on Gravitational Waves

Solutions with sources via retarded Green's function

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

$$\Box_x G(x - x') = \delta^4(x - x')$$

$\Rightarrow$

$$\bar{h}_{\mu\nu}(x) = -\frac{16\pi G}{c^4} \int d^4x' \, G(x - x') T_{\mu\nu}(x')$$

$$t_{\rm ret} = t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}$$

$$G(x - x') = -\frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|} \delta(x^0_{\rm ret} - x'^0)$$

The solution can be written

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int d^3x' \, \frac{1}{|\mathbf{x} - \mathbf{x}'|} T_{\mu\nu}\left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c}, \mathbf{x}'\right)$$

Low velocity expansion

$$|\mathbf{x} - \mathbf{x}'| = r - \mathbf{x}' \cdot \hat{\mathbf{n}} + O\left(\frac{d^2}{r}\right)$$

$\Rightarrow$

$$T_{kl}\left(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}'\right) \simeq T_{kl}(t - \frac{r}{c}, \mathbf{x}')$$
$$+ \frac{x'^i n^i}{c} \partial_0 T_{kl} + \frac{1}{2c^2} x'^i x'^j n^i n^j \partial_0^2 T_{kl} + \ldots$$

Lecture Notes on Gravitational Waves

Define moments

$$M = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})\,,$$
$$M^i = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})\, x^i\,,$$
$$M^{ij} = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})\, x^i x^j\,,$$
$$M^{ijk} = \frac{1}{c^2}\int d^3x\, T^{00}(t,\mathbf{x})\, x^i x^j x^k$$

$$\left[h_{ij}^{\rm TT}(t,\mathbf{x})\right]_{\rm quad} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{M}^{kl}(t-r/c)$$

Introduce quadrupole tensor

$$M^{kl} = \left(M^{kl} - \frac{1}{3}\delta^{kl}M_{ii}\right) + \frac{1}{3}\delta^{kl}M_{ii}$$

$$\rho = \frac{1}{c^2}T^{00}$$

$$Q^{ij} \equiv M^{ij} - \frac{1}{3}\delta^{ij}M_{kk}$$
$$= \int d^3x\, \rho(t,\mathbf{x})(x^i x^j - \frac{1}{3}r^2\delta^{ij})$$

$$\left[h_{ij}^{\rm TT}(t,\mathbf{x})\right]_{\rm quad} = \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}^{\rm TT}(t-r/c)$$

Lecture Notes on Gravitational Waves

Radiated power and angular momentum

$$\left(\frac{dP}{d\Omega}\right)_{\text{quad}} = \frac{r^2c^3}{32\pi G}\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\rangle$$

$$J^i = \frac{c^2}{32\pi G}\int d^3x\left[-\epsilon^{ikl}\dot{h}_{ab}^{TT}x^k\partial^l h_{ab}^{TT} + 2\epsilon^{ikl}h_{ak}^{TT}\dot{h}_{al}^{TT}\right]$$

→

$$P_{\text{quad}} = \frac{G}{5c^5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle$$

$$\left(\frac{dJ^i}{dt}\right)_{\text{quad}} = \frac{2G}{5c^5}\epsilon^{ikl}\langle \ddot{Q}_{ka}\dddot{Q}_{la}\rangle$$

Radiation from Octupole:

$$\mathcal{O}^{klm} = M^{klm} - \frac{1}{5}\left(\delta^{kl}M^{k'k'm} + \delta^{km}M^{k'lk'} + \delta^{lm}M^{kk'k'}\right)$$

$$M^{ijk}(t) = \mu x_0^i(t)x_0^j(t)x_0^k(t)$$

$$\left(h_{ij}^{TT}\right)_{\text{oct}} = \frac{1}{r}\frac{2G}{3c^5}\Lambda_{ij,kl}(\hat{\mathbf{n}})n_m\dddot{\mathcal{O}}^{klm}$$

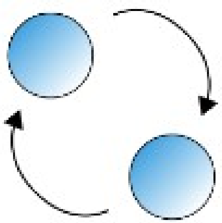
→

$$P = \frac{G}{c^5}\left[\frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle + \right.$$

$$\left. + \frac{1}{c^2}\frac{1}{189}\langle \frac{d^4\mathcal{O}_{ijk}}{dt^4}\frac{d^4\mathcal{O}_{ijk}}{dt^4}\rangle + O\left(\frac{v^4}{c^4}\right)\right]$$

Lecture Notes on Gravitational Waves

Inspiral binaries in circular orbits



$$x_0(t) = R\cos(\omega_s t + \frac{\pi}{2})$$
$$y_0(t) = R\sin(\omega_s t + \frac{\pi}{2})$$
$$z_0(t) = 0$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\omega_s^2 = \frac{Gm}{R^3}$$

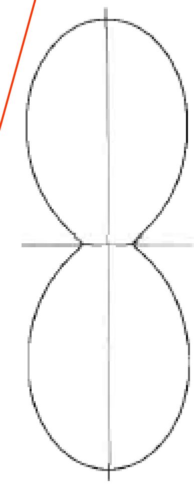
$$h_+(t;\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{\rm ret} + 2\phi),$$
$$h_\times(t;\theta,\phi)) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\theta\sin(2\omega_s t_{\rm ret} + 2\phi).$$

Power emitted

$$\left(\frac{dP}{d\Omega}\right)_{\rm quad} = \frac{r^2 c^3}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle.$$

$$\left(\frac{dP}{d\Omega}\right)_{\rm quad} = \frac{2G\mu^2 R^4 \omega_s^6}{\pi c^5} g(\theta)$$

$$g(\theta) = \left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta$$



$$P_{\rm quad} = \frac{32}{5}\frac{G\mu^2}{c^5}R^4\omega_s^6$$
$$= \frac{1}{10}\frac{G\mu^2}{c^5}R^4\omega^6$$
$$\omega = 2\omega_s$$

Frequency of the GW!

Lecture Notes on Gravitational Waves

Introduce the chirp mass

$$\omega_s^2 = \frac{Gm}{R^3}$$

$$M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} .$$

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}}{c}\right)^{2/3}\frac{1+\cos^2\theta}{2}\cos(2\pi f_{\rm gw} t_{\rm ret} + 2\phi)$$

$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{\rm gw}}{c}\right)^{2/3}\cos\theta\,\sin(2\pi f_{\rm gw} t_{\rm ret} + 2\phi) ,$$

$$P = \frac{32}{5}\frac{c^5}{G}\left(\frac{GM_c\omega_{\rm gw}}{2c^3}\right)^{10/3}$$

The system is losing energy thus the frequency changes

$$\omega_s^2 = \frac{Gm}{R^3}$$

$$E_{\rm orbit} = E_{\rm kin} + E_{\rm pot} = -\frac{Gm_1 m_2}{2R} ,$$

$$\dot{R} = -\frac{2}{3} R \frac{\dot{\omega}_s}{\omega_s} = -\frac{2}{3}(\omega_s R)\frac{\dot{\omega}_s}{\omega_s^2} .$$

$$E_{\rm orbit} = -(G^2 M_c^5 \omega_{\rm gw}^2/32)^{1/3}$$

$$\dot{\omega}_{\rm gw} = \frac{12}{5} 2^{1/3}\left(\frac{GM_c}{c^3}\right)^{5/3}\omega_{\rm gw}^{11/3}$$

Lecture Notes on Gravitational Waves

Time to coalescence

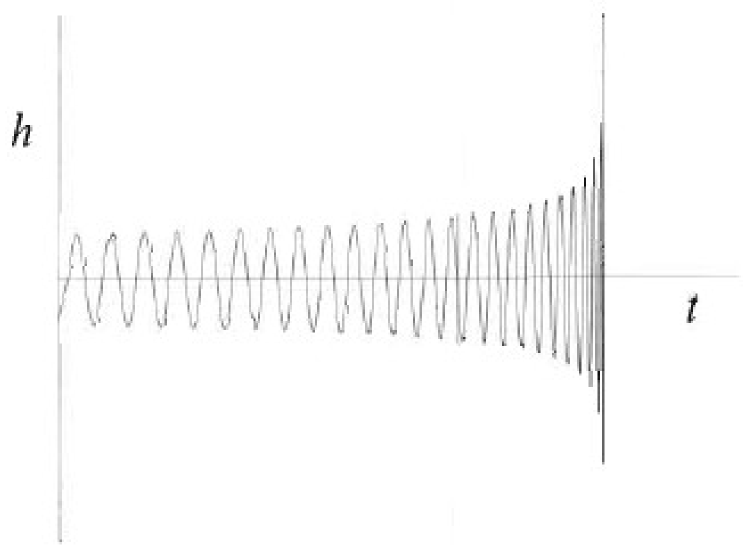
$$\dot{f}_{\rm gw} = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3} f_{\rm gw}^{11/3}$$

$$\tau \equiv t_{\rm coal} - t$$

$$f_{\rm gw}(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$

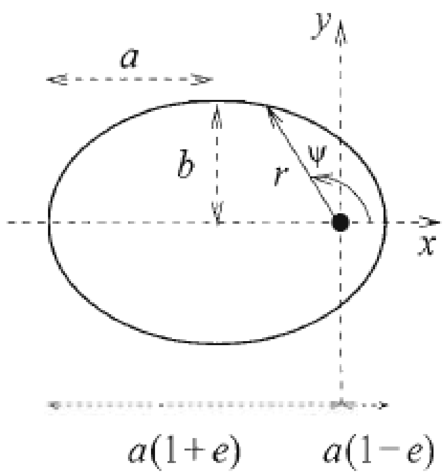
$$\simeq 134\,{\rm Hz}\left(\frac{1.21 M_\odot}{M_c}\right)^{5/8}\left(\frac{1\,s}{\tau}\right)^{3/8}$$

Change of amplitude with time



Lecture Notes on Gravitational Waves

Elliptical orbits



$$e^2 = 1 + \frac{2EL^2}{G^2 m^2 \mu^3}$$

$$\dot{\psi} = \frac{(GmR)^{1/2}}{r^2}$$

$$r = \frac{a(1-e^2)}{1+e\cos\psi}$$

$$\Rightarrow \quad M_{ab} = \mu r^2 \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{pmatrix}_{ab}$$

Radiated power

$$\begin{aligned} P(\psi) &= \frac{G}{5c^5}\left[\dddot{M}_{11}^2 + \dddot{M}_{22}^2 + 2\dddot{M}_{12}^2 - \frac{1}{3}(\dddot{M}_{11} + \dddot{M}_{22})^2\right] \\ &= \frac{2G}{15c^5}\left[\dddot{M}_{11}^2 + \dddot{M}_{22}^2 + 3\dddot{M}_{12}^2 - \dddot{M}_{11}\dddot{M}_{22}\right] \\ &= \frac{8G^4}{15c^5}\,\frac{\mu^2 m^3}{a^5(1-e^2)^5}\,(1+e\cos\psi)^4\left[12(1+e\cos\psi)^2 + e^2\sin^2\psi\right] \end{aligned}$$

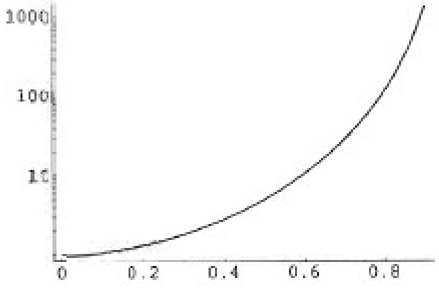
Lecture Notes on Gravitational Waves

Average over orbit

$$P \equiv \frac{1}{T}\int_0^T dt\, P(\psi)$$
$$= \frac{\omega_0}{2\pi}\int_0^{2\pi} \frac{d\psi}{\dot\psi} P(\psi)$$
$$= (1-e^2)^{3/2}\int_0^{2\pi}\frac{d\psi}{2\pi}(1+e\cos\psi)^{-2}P(\psi)$$
$$= \frac{8G^4\mu^2m^3}{15c^5a^5}(1-e^2)^{-7/2}$$
$$\times \int_0^{2\pi}\frac{d\psi}{2\pi}\left[12(1+e\cos\psi)^4+e^2(1+e\cos\psi)^2\sin^2\psi\right]$$

$$P = \frac{32G^4\mu^2m^3}{5c^5a^5}f(e)$$

$$f(e) = \frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

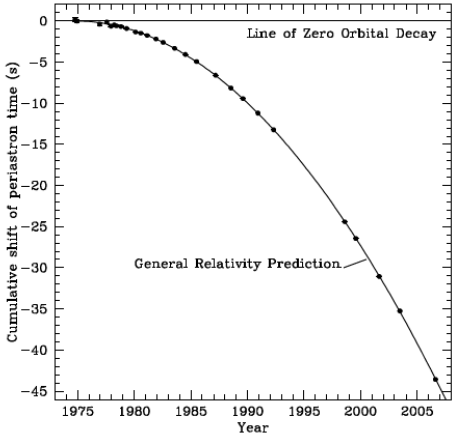


Change in period

$$a = \frac{Gm\mu}{2|E|}$$
$$\omega_0^2 = \frac{Gm}{a^3}$$

$$T = \text{const.} \times (-E)^{-3/2}$$

$$\frac{\dot T}{T} = -\frac{96}{5}\frac{G^{5/3}\mu m^{2/3}}{c^5}\left(\frac{T}{2\pi}\right)^{-8/3} f(e)$$



Lecture Notes on Gravitational Waves

Change in orbital elements

$$\frac{dE}{dt} = -\frac{32}{5}\frac{G^4\mu^2m^3}{c^5a^5}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

$$\frac{dL}{dt} = -\frac{32}{5}\frac{G^{7/2}\mu^2m^{5/2}}{c^5a^{7/2}}\frac{1}{(1-e^2)^2}\left(1+\frac{7}{8}e^2\right)$$

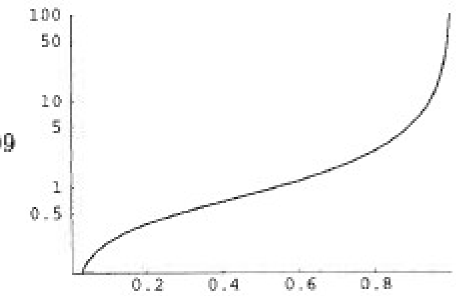
$$\Rightarrow$$

$$\frac{da}{dt} = -\frac{64}{5}\frac{G^3\mu m^2}{c^5a^3}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

$$\frac{de}{dt} = -\frac{304}{15}\frac{G^3\mu m^2}{c^5a^4}\frac{e}{(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right)$$

Orbit circularization

$$\frac{da}{de} = \frac{12}{19}a\frac{1+(73/24)e^2+(37/96)e^4}{e(1-e^2)[1+(121/304)e^2]} \Rightarrow a(e) = c_0\frac{e^{12/19}}{1-e^2}\left(1+\frac{121}{304}e^2\right)^{870/2299}$$



Time to coalescence (eg Hulse-Taylor pulsar)

$$\tau(a_0,e_0) = \frac{15}{304}\frac{c^5}{G^3m^2\mu}\int_0^{e_0} de\,\frac{a^4(e)(1-e^2)^{5/2}}{e\left(1+\frac{121}{304}e^2\right)}$$

$$\simeq 9.829\,\mathrm{Myr}\left(\frac{T_0}{1\,\mathrm{hr}}\right)^{8/3}\left(\frac{M_\odot}{m}\right)^{2/3}\left(\frac{M_\odot}{\mu}\right)F(e_0)$$

$$\Rightarrow$$

$m_1 = m_2 \simeq 1.4M_\odot$

$T_0 \simeq 7.75$ hr

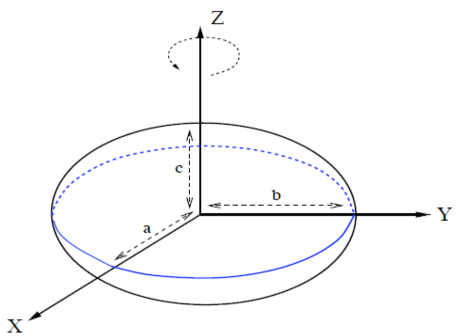
$$\Rightarrow$$

$\tau(a_0,e_0) \simeq 300$ Myr

Lecture Notes on Gravitational Waves

Do rotating spherically symmetric objects emit GWs?

NO!



$$I_{ij} = \int_V \rho \left(r^2 \delta_{ij} - x_i x_j \right) \, dx^3 \quad \Rightarrow \quad Q_{ij} = -\left(I_{ij} - \frac{1}{3}\delta_{ij} \operatorname{Tr} I \right)$$

Inertia tensor

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 + \left(\frac{x_3}{c}\right)^2 = 1 \quad \Rightarrow \quad I_{ij} = \frac{M}{5} \begin{pmatrix} b^2 + c^2 & 0 & 0 \\ 0 & c^2 + a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

Go to a rotating frame

$$x_i = R_{ij} x'_j,$$

$$R_{ij} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi = \Omega t$$

$$\Rightarrow$$

$$\begin{aligned} I_{ij} &= R_{ik} R_{jl} I'_{kl} = (R I' R^T)_{ij} \\ &= \begin{pmatrix} I_1 \cos^2\varphi + I_2 \sin^2\varphi & -\sin\varphi\cos\varphi(I_2 - I_1) & 0 \\ -\sin\varphi\cos\varphi(I_2 - I_1) & I_1 \sin^2\varphi + I_2 \cos^2\varphi & 0 \\ 0 & 0 & I_3 \end{pmatrix} \end{aligned}$$

Lecture Notes on Gravitational Waves

The quadrupole tensor is

$$Q_{ij} = -\left(I_{ij} - \frac{1}{3}\delta_{ij}\,\mathrm{Tr}\,I\right) = -I_{ij} + constant$$

$$\mathrm{Tr}\,I = I_1 + I_2 + I_3 = constant$$

$$\Rightarrow \quad Q_{ij} = \frac{I_2 - I_1}{2}\begin{pmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} + constant$$

What happens for spherically symmetric objects (a=b=c)?

$$I_1 = \frac{M}{5}(b^2 + c^2),$$

$$I_2 = \frac{M}{5}(c^2 + a^2)$$

$$\Rightarrow \quad Q_{ij} = 0$$

In general

$$\epsilon \equiv \frac{a-b}{(a+b)/2} \quad \Rightarrow \quad \frac{I_2 - I_1}{I_3} = \frac{1}{2}\epsilon\frac{a^2+b^2+2ab}{a^2+b^2} = \epsilon + O(\epsilon^3) \quad \Rightarrow$$

$$Q_{ij} = \frac{\epsilon\, I_3}{2}\begin{pmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} + constant. \quad \Rightarrow \quad L_{GW} = \frac{32G}{5c^5}\Omega^6\epsilon^2 I_3^2$$

Lecture Notes on Gravitational Waves

The Post-Newtonian expansion (PN):

Decompose in terms of v/c

$$
\begin{aligned}
g_{00} &= -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + {}^{(6)}g_{00} + \dots , \\
g_{0i} &= {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + \dots , \\
g_{ij} &= \delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + \dots ,
\end{aligned}
$$

$$
\begin{aligned}
T^{00} &= {}^{(0)}T^{00} + {}^{(2)}T^{00} + \dots , \\
T^{0i} &= {}^{(1)}T^{0i} + {}^{(3)}T^{0i} + \dots , \\
T^{ij} &= {}^{(2)}T^{ij} + {}^{(4)}T^{ij} + \dots .
\end{aligned}
$$

Expand the geodesic equation

$$
\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}
$$

$$
\begin{aligned}
\frac{d^2x^i}{dt^2} &\simeq -c^2\Gamma^i_{00} \\
&= c^2\left(\frac{1}{2}\partial^i h_{00} - \partial_0 h_0{}^i\right) = \frac{c^2}{2}\partial^i h_{00}
\end{aligned}
$$

The expansion becomes

$$
\begin{aligned}
\frac{dy_1^i}{dt} &= v_1^i , \\
\frac{dv_1^i}{dt} &= A_1^i + \frac{1}{c^2}B_1^i + \frac{1}{c^4}C_1^i + \frac{1}{c^5}D_1^i + O(6)
\end{aligned}
$$

Lecture Notes on Gravitational Waves

Where

$$A_1^i = -\frac{Gm_2}{r^2}n^i \,,$$

0PN (Newton's term)

$$B_1^i = \frac{Gm_2}{r^2}\left\{n^i\left[-v_1^2 - 2v_2^2 + 4(v_1v_2) + \frac{3}{2}(nv_2)^2 + 5\frac{Gm_1}{r} + 4\frac{Gm_2}{r}\right] + (v_1^i - v_2^i)\left[4(nv_1) - 3(nv_2)\right]\right\} ,$$

1PN

$$\begin{aligned} C_1^i = \frac{Gm_2}{r^2}\Bigg\{ n^i\Bigg[& -2v_2^4 + 4v_2^2(v_1v_2) - 2(v_1v_2)^2 + \frac{3}{2}v_1^2(nv_2)^2 \\ & + \frac{9}{2}v_2^2(nv_2)^2 - 6(v_1v_2)(nv_2)^2 - \frac{15}{8}(nv_2)^4 \\ & + \frac{Gm_1}{r}\left(-\frac{15}{4}v_1^2 + \frac{5}{4}v_2^2 - \frac{5}{2}(v_1v_2) + \frac{39}{2}(nv_1)^2 - 39(nv_1)(nv_2) + \frac{17}{2}(nv_2)^2\right) \\ & + \frac{Gm_2}{r}\left(4v_2^2 - 8(v_1v_2) + 2(nv_1)^2 - 4(nv_1)(nv_2) - 6(nv_2)^2\right)\Bigg] \\ & + (v_1^i - v_2^i)\Bigg[v_1^2(nv_2) + 4v_2^2(nv_1) - 5v_2^2(nv_2) - 4(v_1v_2)(nv_1) \\ & + 4(v_1v_2)(nv_2) - 6(nv_1)(nv_2)^2 + \frac{9}{2}(nv_2)^3 \\ & + \frac{Gm_1}{r}\left(-\frac{63}{4}(nv_1) + \frac{55}{4}(nv_2)\right) + \frac{Gm_2}{r}\left(-2(nv_1) - 2(nv_2)\right)\Bigg]\Bigg\} \\ & + \frac{G^3m_2}{r^4}n^i\left\{-\frac{57}{4}m_1^2 - 9m_2^2 - \frac{69}{2}m_1m_2\right\} \,, \end{aligned}$$

2PN

$$D_1^i = \frac{4}{5}\frac{G^2m_1m_2}{r^3}\left\{v^i\left[-v^2 + 2\frac{Gm_1}{r} - 8\frac{Gm_2}{r}\right] + n^i(nv)\left[3v^2 - 6\frac{Gm_1}{r} + \frac{52}{3}\frac{Gm_2}{r}\right]\right\}$$

2.5PN

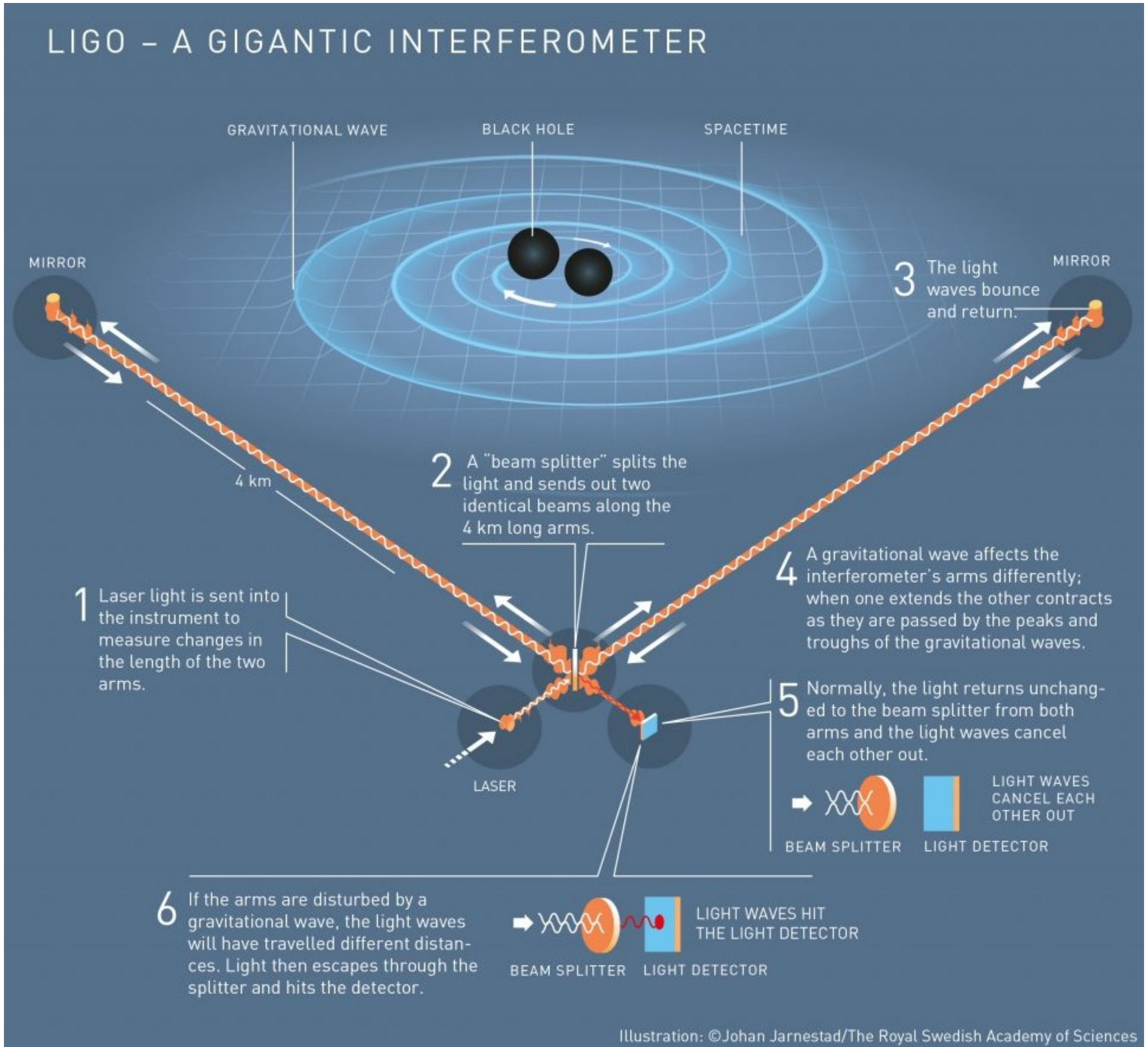
Not conservative!!!
Reason for GWs

Main points of the lecture

- What are the GWs (history, description)
- Formalism in GR (linearization, gauges, emission)
- Detection techniques (interferometry, LIGO)
- Recent observations (BH-BH, NS-NS)
- Other issues (speed of GWs, hyperbolic encounters)

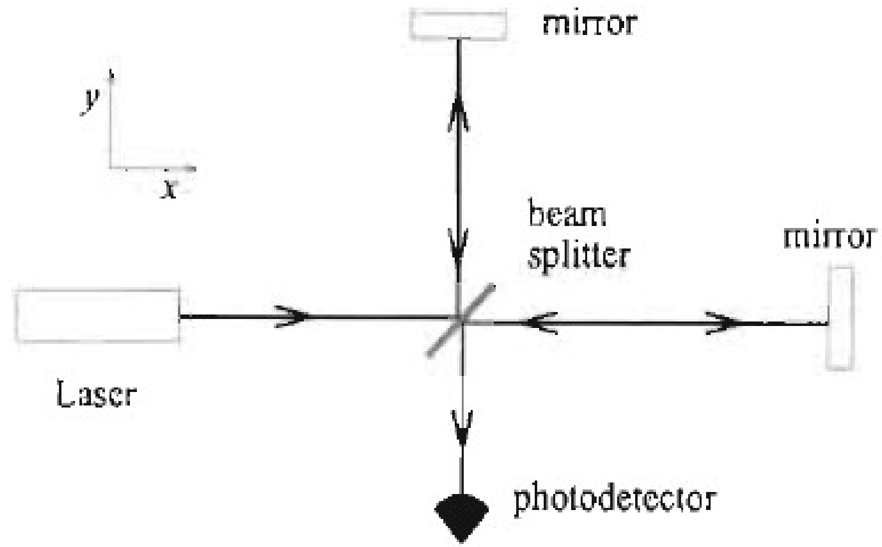
Lecture Notes on Gravitational Waves

GWs require photon-based distance measurements to be detected! We need something that travels at the speed of light (which is constant), hence:



Lecture Notes on Gravitational Waves

Laser interferometry
(Michelson 1887)



Electric field measured:

$$E_1 = -\frac{1}{2}E_0 e^{-i\omega_L t + 2ik_L L_x}$$

$$E_2 = +\frac{1}{2}E_0 e^{-i\omega_L t + 2ik_L L_y}$$

$\Rightarrow$

$$E_{\rm out} = -iE_0 e^{-i\omega_L t + ik_L(L_x+L_y)} \sin[k_L(L_y - L_x)]$$

$\Rightarrow$

$$|E_{\rm out}|^2 = E_0^2 \sin^2[k_L(L_y - L_x)]$$

$$E_{\rm out} = E_1 + E_2$$

Lecture Notes on Gravitational Waves

Connection with GWs (effect on distances)

$$h_+(t) = h_0 \cos \omega_{gw} t$$

$$ds^2 = -c^2 dt^2 + [1 + h_+(t)]dx^2 + [1 - h_+(t)]dy^2 + dz^2$$

$ds^2 = 0,$ →

$$dx = \pm\, c\,dt \left[1 - \frac{1}{2} h_+(t)\right]$$

→

$$L_x = c(t_1 - t_0) - \frac{c}{2} \int_{t_0}^{t_1} dt'\, h_+(t')$$

Similarly (on the way back):

$$L_x = c(t_2 - t_1) - \frac{c}{2} \int_{t_1}^{t_2} dt'\, h_+(t').$$

Total time and difference in phase:

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{1}{2} \int_{t_0}^{t_2} dt'\, h_+(t') = \frac{2L_x}{c} + \frac{L_x}{c} h(t_0 + L_x/c) \frac{\sin(\omega_{gw} L_x/c)}{(\omega_{gw} L_x/c)}$$

$$\Delta\phi_x(t) = h_0 \frac{\omega_L L_x}{c} \mathrm{sinc}\,(\omega_{gw} L_x/c) \cos[\omega_{gw}(t - L_x/c)]$$

→

$$P = P_0 \sin^2[\phi_0 + \Delta\phi_x(t)]$$
$$= \frac{P_0}{2} \{1 - \cos[2\phi_0 + 2\Delta\phi_x(t)]\}$$

$$\Delta\phi_{\mathrm{Mich}} \equiv \Delta\phi_x - \Delta\phi_y = 2\Delta\phi_x$$

$$(\Delta P)_{\mathrm{GW}} = \frac{P_0}{2} |\sin 2\phi_0|\, (\Delta\phi)_{\mathrm{Mich}}$$

Lecture Notes on Gravitational Waves

The detector measures the total strain but measurements given in terms of signal-to-noise

$$h(t) = D^{ij} h_{ij}(t)$$

Depends on detector geometry

Final measurement depends on the transfer function T(f)

$$\tilde{h}_{\rm out}(f) = T(f)\tilde{h}(f)$$

The output also includes the noise (more later)

$$s_{\rm out}(t) = h_{\rm out}(t) + n_{\rm out}(t)$$

$$\delta(f=0) \rightarrow \left[\int_{-T/2}^{T/2} dt\, e^{i2\pi f t} \right] \Big|_{f=0} = T$$

$$\langle \tilde{n}^*(f)\tilde{n}(f') \rangle = \delta(f-f')\frac{1}{2}S_n(f) \quad \Rightarrow \quad \langle\, |\tilde{n}(f)|^2 \,\rangle = \frac{1}{2}S_n(f)T$$

and $\langle n(t) \rangle = 0$

Lecture Notes on Gravitational Waves

Spectral noise density is variance of the noise:

$$\Delta f = \frac{1}{T}$$

$$\frac{1}{2} S_n(f) = \langle |\tilde{n}(f)|^2 \rangle \, \Delta f$$

$$\begin{aligned}\langle n^2(t)\rangle = \langle n^2(t=0)\rangle &= \int_{-\infty}^{\infty} df\, df' \langle n^*(f) n(f')\rangle \\ &= \frac{1}{2}\int_{-\infty}^{\infty} df\, S_n(f) \\ &= \int_0^{\infty} df\, S_n(f)\,.\end{aligned}$$

Spectral noise density

Signal to noise ratio (K is the filter function):

$$S = \int_{-\infty}^{\infty} dt\, \langle s(t)\rangle K(t) = \int_{-\infty}^{\infty} dt\, h(t) K(t) = \int_{-\infty}^{\infty} df\, \tilde{h}(f) \tilde{K}^*(f)$$

$$\langle n(t)\rangle = 0$$

Noise:

$$N^2 = \left[\langle \hat{s}^2(t)\rangle - \langle \hat{s}(t)\rangle^2\right]_{h=0} = \langle \hat{s}^2(t)\rangle_{h=0} = \int_{-\infty}^{\infty} dt dt'\, K(t) K(t') \langle n(t) n(t')\rangle$$

$$= \int_{-\infty}^{\infty} dt dt' K(t) K(t') \int_{-\infty}^{\infty} df df'\, e^{2\pi i f t - 2\pi i f' t'} \langle \tilde{n}^*(f) \tilde{n}(f')\rangle = \int_{-\infty}^{\infty} df\, \frac{1}{2} S_n(f) |\tilde{K}(f)|^2$$

Lecture Notes on Gravitational Waves

Final expression for the Signal to Noise:

$$\frac{S}{N} = \frac{\int_{-\infty}^{\infty} df\, \tilde{h}(f)\tilde{K}^*(f)}{\left[\int_{-\infty}^{\infty} df\, (1/2)S_n(f)|\tilde{K}(f)|^2\right]^{1/2}} \quad \Rightarrow \quad \tilde{K}(f) = \text{const.}\, \frac{\tilde{h}(f)}{S_n(f)}$$

Optimal filter

Then

$$\left(\frac{S}{N}\right)^2 = 4\int_0^{\infty} df\, \frac{|\tilde{h}(f)|^2}{S_n(f)}$$

Example 1: stochastic backgrounds

$$\langle h_{ij}(t)h^{ij}(t)\rangle = 4\int_0^{\infty} df\, S_h(f)$$

$$\rho_{\rm gw} = \frac{c^2}{32\pi G}\langle \dot{h}_{ij}\dot{h}^{ij}\rangle$$

$$\Rightarrow \quad \rho_{\rm gw} \equiv \int_{f=0}^{f=\infty} d(\log f)\, \frac{d\rho_{\rm gw}}{d\log f}\,. \quad \Rightarrow \quad \Omega_{\rm gw}(f) \equiv \frac{1}{\rho_c}\frac{d\rho_{\rm gw}}{d\log f} = \frac{4\pi^2}{3H_0^2} f^3 S_h(f)$$

Lecture Notes on Gravitational Waves

Example 2: Distance to coalescing binaries

$$\tilde{h}(f) = \left(\frac{5}{6}\right)^{1/2} \frac{1}{2\,\pi^{2/3}} \frac{c}{r} \left(\frac{G\mathcal{M}_c}{c^3}\right)^{5/6} f^{-7/6}\, e^{i\Psi}\, Q(\theta,\phi;\iota)$$

Function that depends on geometry of the system, inclination etc

$\Rightarrow$

$$\left(\frac{S}{N}\right)^2 = \frac{5}{6} \frac{1}{\pi^{4/3}} \frac{c^2}{r^2} \left(\frac{G\mathcal{M}_c}{c^3}\right)^{5/3} |Q(\theta,\phi;\iota)|^2 \int_0^{f_{\max}} df\, \frac{f^{-7/3}}{S_n(f)}$$

Averaging over inclination etc we can solve for the distance

$$d_{\mathrm{sight}} = \frac{2}{5} \left(\frac{5}{6}\right)^{1/2} \frac{c}{\pi^{2/3}} \left(\frac{G\mathcal{M}_c}{c^3}\right)^{5/6} \left[\int_0^{f_{\max}} df\, \frac{f^{-7/3}}{S_n(f)}\right]^{1/2} (S/N)^{-1}$$

Lecture Notes on Gravitational Waves

Average amplitude on Earth and length of detector

$$\Delta L = (1/2) h_0 L \quad \xrightarrow{h_0 \sim 10^{-21}} \quad \Delta L \sim 2 \times 10^{-18}\ \mathrm{m}$$

Sources on noise:

1) Shot noise: photons are discrete! They follow Poisson distribution

$$P = \frac{1}{T} N_\gamma \hbar\omega_{\mathrm{L}}$$

$$p(N;\bar{N}) = \frac{1}{N!}\bar{N}^N e^{-\bar{N}}$$

$$\Longrightarrow \quad \Delta N_\gamma = \sqrt{N_\gamma}. \quad \Longrightarrow$$

$$(\Delta P)_{\mathrm{shot}} = \frac{1}{T} N_\gamma^{1/2} \hbar\omega_{\mathrm{L}} = \left(\frac{\hbar\omega_{\mathrm{L}}}{T} P\right)^{1/2} = \left(\frac{\hbar\omega_{\mathrm{L}}}{T} P_0\right)^{1/2} |\sin\phi_0|$$

$$P = P_0 \sin^2\phi_0$$

Large N limit of Poisson → Gaussian!
Can you prove it?

Total signal to noise:

$$\frac{S}{N} = \frac{(\Delta P)_{\mathrm{GW}}}{(\Delta P)_{\mathrm{shot}}} = \left(\frac{P_0 T}{\hbar\omega_{\mathrm{L}}}\right)^{1/2} \frac{4\pi L}{\lambda_{\mathrm{L}}} h_0 |\cos\phi_0|$$

$$(\Delta P)_{\mathrm{GW}} = \frac{P_0}{2} |\sin 2\phi_0| \frac{4\pi L}{\lambda_{\mathrm{L}}} h_0$$

$$P = P_0 \sin^2[\phi_0 + \Delta\phi_x(t)] = \frac{P_0}{2}\{1 - \cos[2\phi_0 + 2\Delta\phi_x(t)]\}$$

$$\Delta\phi_{\mathrm{Mich}} \equiv \Delta\phi_x - \Delta\phi_y = 2\Delta\phi_x$$

$$(\Delta P)_{\mathrm{GW}} = \frac{P_0}{2} |\sin 2\phi_0| (\Delta\phi)_{\mathrm{Mich}}$$

Lecture Notes on Gravitational Waves

2) Radiation pressure (laser beam hitting mirror)

$$\Delta F = 2\Delta P/c = 2\sqrt{\frac{\hbar\omega_L P}{c^2 T}} \quad \Rightarrow \quad S_F^{1/2} = 2\sqrt{\frac{2\hbar\omega_L P}{c^2}}$$

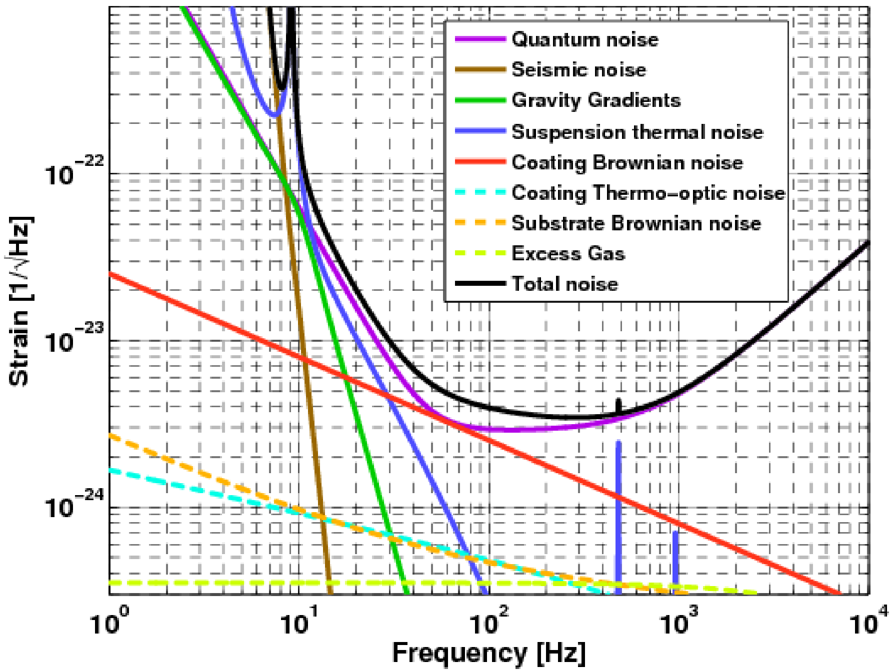
$$\langle A^2(t)\rangle = \frac{1}{2T} S_A$$

3) The quantum limit (shot noise+rad pressure)

$$S_n(f)|_{\rm opt} = S_n(f)|_{\rm shot} + S_n(f)|_{\rm rad}$$

4) Seismic noise

$$x(f) \simeq A\left(\frac{1\,{\rm Hz}}{f^\nu}\right)\,{\rm m\,Hz}^{-1/2}$$

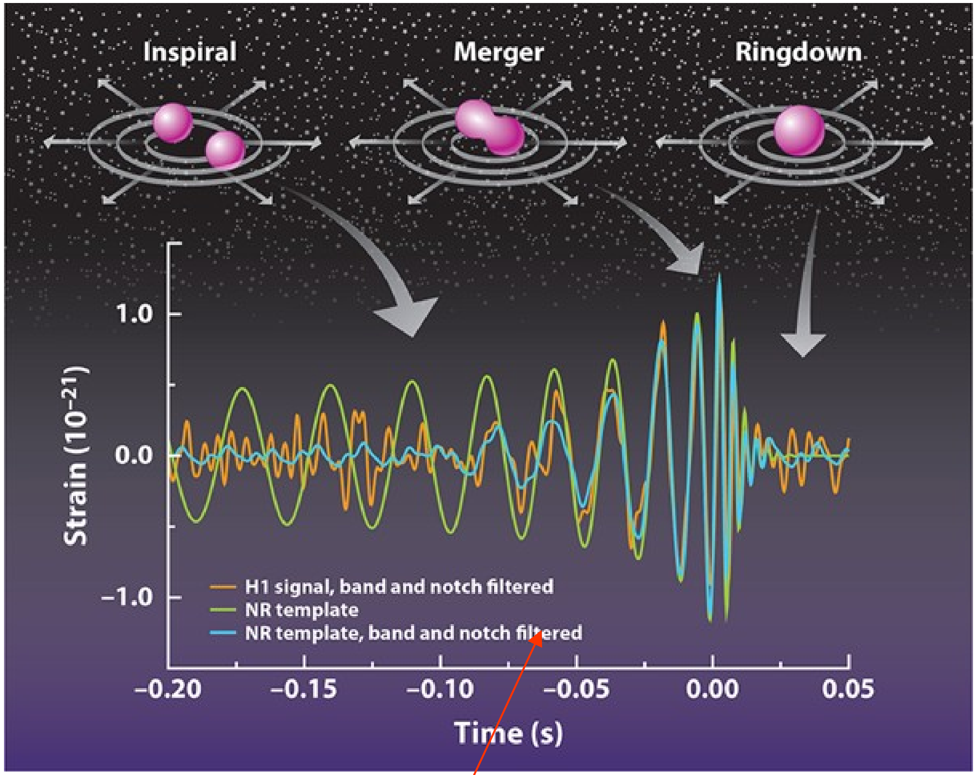


Main points of the lecture

- What are the GWs (history, description)
- Formalism in GR (linearization, gauges, emission)
- Detection techniques (interferometry, LIGO)
- Recent observations (BH-BH, NS-NS)
- Other issues (speed of GWs, hyperbolic encounters)

Lecture Notes on Gravitational Waves

LIGO detection (2015):

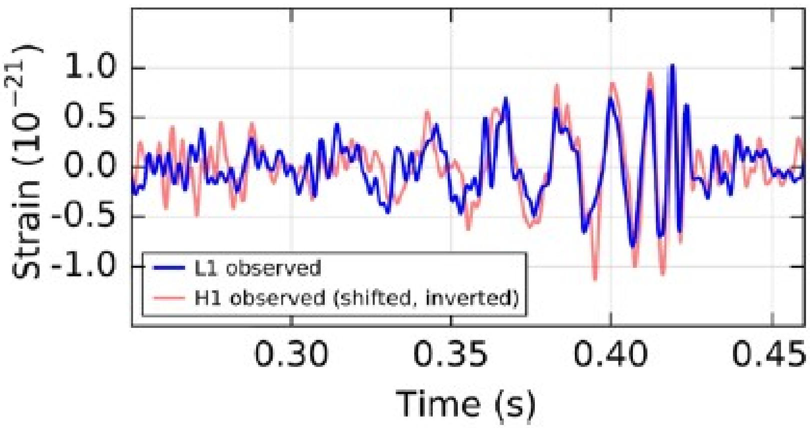


The two LIGO detectors. The signals have to appear in both of them!

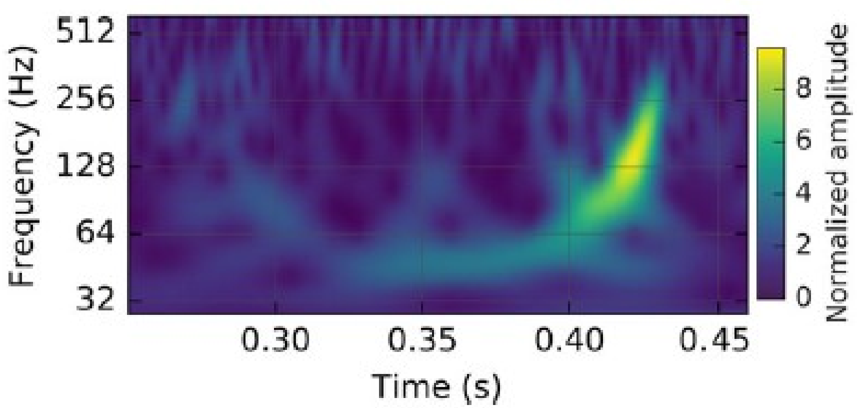
Processed to look real (noise reduction etc)
Notch → Removes noise eg at 60Hx etc.

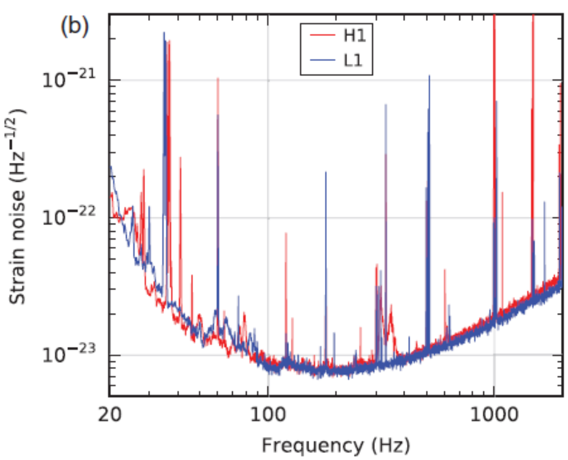
Lecture Notes on Gravitational Waves

The signal (strain) from both LIGO detectors (S/N~24):



The signal in time – frequency and strain-frequency domain:



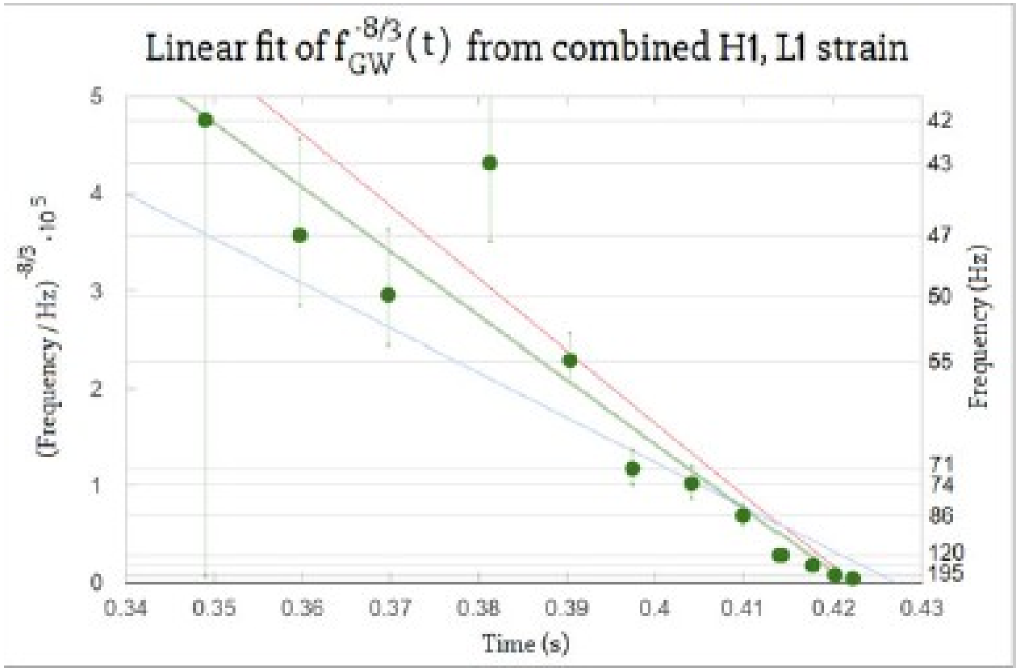


Lecture Notes on Gravitational Waves

Fit to the data:

$$\mathcal{M} = \frac{c^3}{G}\left(\left(\frac{5}{96}\right)^3 \pi^{-8} (f_{\rm GW})^{-11} \left(\dot{f}_{\rm GW}\right)^3\right)^{1/5} \Rightarrow f_{\rm GW}^{-8/3}(t) = \frac{(8\pi)^{8/3}}{5}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}(t_c - t)$$

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$



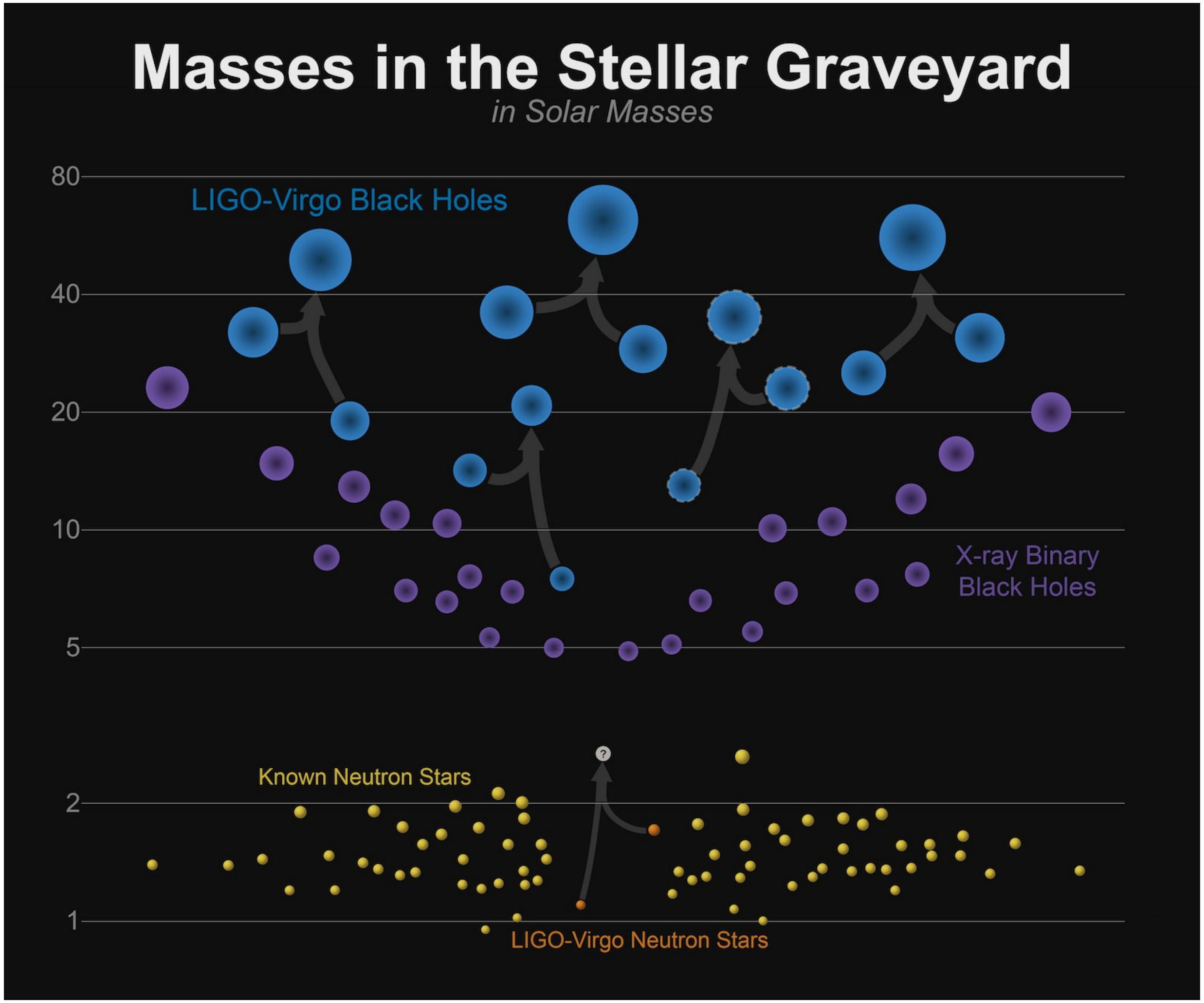
$$\Rightarrow \mathcal{M} = 30\,{\rm M}_\odot$$

Primary black hole mass	$36^{+5}_{-4} M_\odot$
Secondary black hole mass	$29^{+4}_{-4} M_\odot$
Final black hole mass	$62^{+4}_{-4} M_\odot$
Final black hole spin	$0.67^{+0.05}_{-0.07}$
Luminosity distance	410^{+160}_{-180} Mpc
Source redshift z	$0.09^{+0.03}_{-0.04}$

$$d_L \sim 45{\rm Gpc}\left(\frac{\rm Hz}{f_{\rm GW}|_{\rm max}}\right)\left(\frac{10^{-21}}{h|_{\rm max}}\right)$$

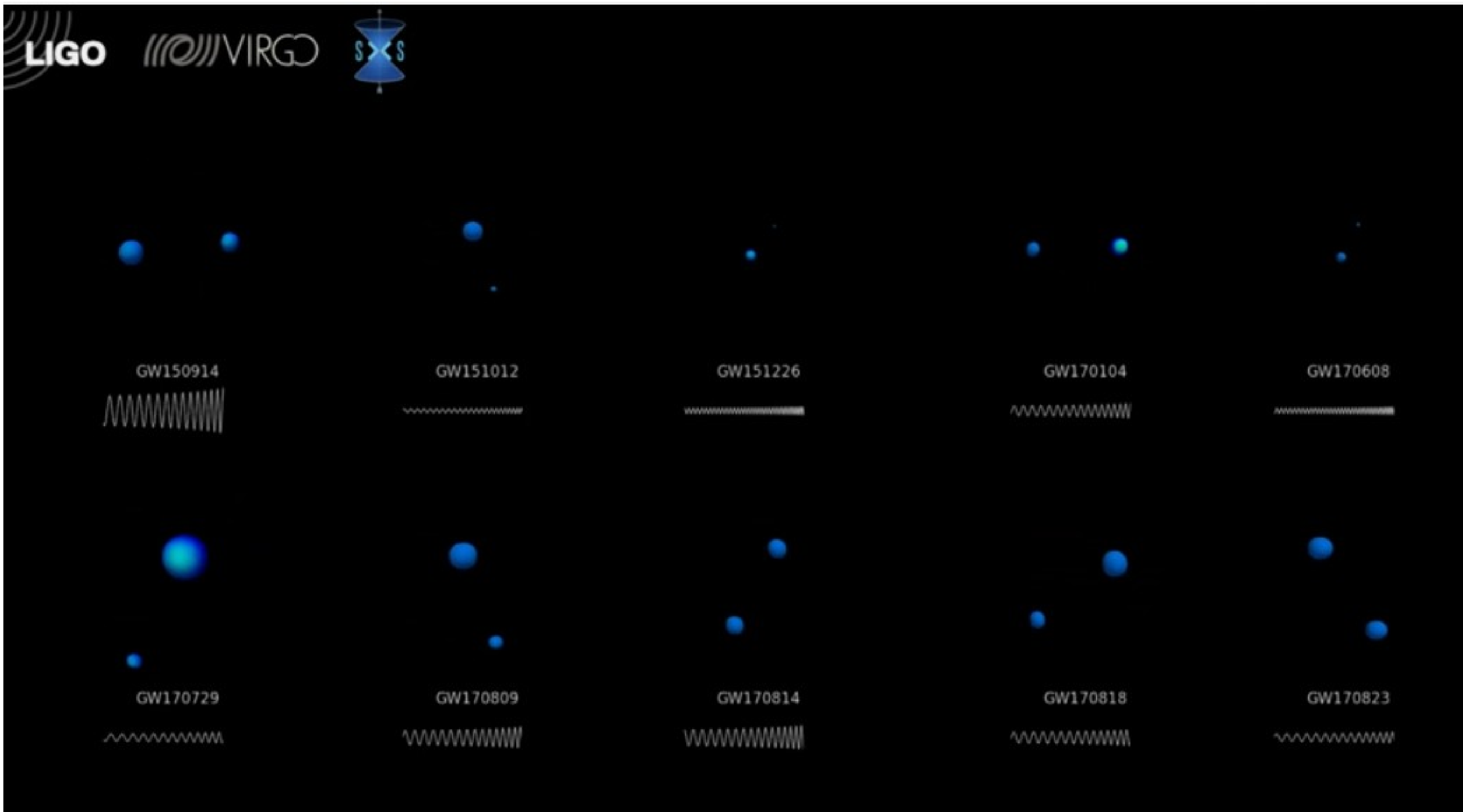
Lecture Notes on Gravitational Waves

Overview and comparison of all GW observations



Lecture Notes on Gravitational Waves

Overview and comparison of all GW observations (update Dec. 2018)



Lecture Notes on Gravitational Waves

Overview and comparison of all GW observations (update Dec. 2018)

ArXiv: 1811.12907

Event	$m_1/\mathrm{M}_\odot$	$m_2/\mathrm{M}_\odot$	$\mathcal{M}/\mathrm{M}_\odot$	χ_{eff}	$M_{\mathrm{f}}/\mathrm{M}_\odot$	a_{f}	$E_{\mathrm{rad}}/(\mathrm{M}_\odot c^2)$	$\ell_{\mathrm{peak}}/(\mathrm{erg\,s^{-1}})$	d_L/Mpc	z	$\Delta\Omega/\mathrm{deg}^2$
GW150914	$35.6^{+4.8}_{-3.0}$	$30.6^{+3.0}_{-4.4}$	$28.6^{+1.6}_{-1.5}$	$-0.01^{+0.12}_{-0.13}$	$63.1^{+3.3}_{-3.0}$	$0.69^{+0.05}_{-0.04}$	$3.1^{+0.4}_{-0.4}$	$3.6^{+0.4}_{-0.4} \times 10^{56}$	430^{+150}_{-170}	$0.09^{+0.03}_{-0.03}$	179
GW151012	$23.3^{+14.0}_{-5.5}$	$13.6^{+4.1}_{-4.8}$	$15.2^{+2.0}_{-1.1}$	$0.04^{+0.28}_{-0.19}$	$35.7^{+9.9}_{-3.8}$	$0.67^{+0.13}_{-0.11}$	$1.5^{+0.5}_{-0.5}$	$3.2^{+0.8}_{-1.7} \times 10^{56}$	1060^{+540}_{-480}	$0.21^{+0.09}_{-0.09}$	1555
GW151226	$13.7^{+8.8}_{-3.2}$	$7.7^{+2.2}_{-2.6}$	$8.9^{+0.3}_{-0.3}$	$0.18^{+0.20}_{-0.12}$	$20.5^{+6.4}_{-1.5}$	$0.74^{+0.07}_{-0.05}$	$1.0^{+0.1}_{-0.2}$	$3.4^{+0.7}_{-1.7} \times 10^{56}$	440^{+180}_{-190}	$0.09^{+0.04}_{-0.04}$	1033
GW170104	$31.0^{+7.2}_{-5.6}$	$20.1^{+4.9}_{-4.5}$	$21.5^{+2.1}_{-1.7}$	$-0.04^{+0.17}_{-0.20}$	$49.1^{+5.2}_{-3.9}$	$0.66^{+0.08}_{-0.10}$	$2.2^{+0.5}_{-0.5}$	$3.3^{+0.6}_{-0.9} \times 10^{56}$	960^{+430}_{-410}	$0.19^{+0.07}_{-0.08}$	924
GW170608	$10.9^{+5.3}_{-1.7}$	$7.6^{+1.3}_{-2.1}$	$7.9^{+0.2}_{-0.2}$	$0.03^{+0.19}_{-0.07}$	$17.8^{+3.2}_{-0.7}$	$0.69^{+0.04}_{-0.04}$	$0.9^{+0.0}_{-0.1}$	$3.5^{+0.4}_{-1.3} \times 10^{56}$	320^{+120}_{-110}	$0.07^{+0.02}_{-0.02}$	396
GW170729	$50.6^{+16.6}_{-10.2}$	$34.3^{+9.1}_{-10.1}$	$35.7^{+6.5}_{-4.7}$	$0.36^{+0.21}_{-0.25}$	$80.3^{+14.6}_{-10.2}$	$0.81^{+0.07}_{-0.13}$	$4.8^{+1.7}_{-1.7}$	$4.2^{+0.9}_{-1.5} \times 10^{56}$	2750^{+1350}_{-1320}	$0.48^{+0.19}_{-0.20}$	1033
GW170809	$35.2^{+8.3}_{-6.0}$	$23.8^{+5.2}_{-5.1}$	$25.0^{+2.1}_{-1.6}$	$0.07^{+0.16}_{-0.16}$	$56.4^{+5.2}_{-3.7}$	$0.70^{+0.08}_{-0.09}$	$2.7^{+0.6}_{-0.6}$	$3.5^{+0.6}_{-0.9} \times 10^{56}$	990^{+320}_{-380}	$0.20^{+0.05}_{-0.07}$	340
GW170814	$30.7^{+5.7}_{-3.0}$	$25.3^{+2.9}_{-4.1}$	$24.2^{+1.4}_{-1.1}$	$0.07^{+0.12}_{-0.11}$	$53.4^{+3.2}_{-2.4}$	$0.72^{+0.07}_{-0.05}$	$2.7^{+0.4}_{-0.3}$	$3.7^{+0.4}_{-0.5} \times 10^{56}$	580^{+160}_{-210}	$0.12^{+0.03}_{-0.04}$	87
GW170817	$1.46^{+0.12}_{-0.10}$	$1.27^{+0.09}_{-0.09}$	$1.186^{+0.001}_{-0.001}$	$0.00^{+0.02}_{-0.01}$	≤ 2.8	≤ 0.89	≥ 0.04	$\geq 0.1 \times 10^{56}$	40^{+10}_{-10}	$0.01^{+0.00}_{-0.00}$	16
GW170818	$35.5^{+7.5}_{-4.7}$	$26.8^{+4.3}_{-5.2}$	$26.7^{+2.1}_{-1.7}$	$-0.09^{+0.18}_{-0.21}$	$59.8^{+4.8}_{-3.8}$	$0.67^{+0.07}_{-0.08}$	$2.7^{+0.5}_{-0.5}$	$3.4^{+0.5}_{-0.7} \times 10^{56}$	1020^{+430}_{-360}	$0.20^{+0.07}_{-0.07}$	39
GW170823	$39.6^{+10.0}_{-6.6}$	$29.4^{+6.3}_{-7.1}$	$29.3^{+4.2}_{-3.2}$	$0.08^{+0.20}_{-0.22}$	$65.6^{+9.4}_{-6.6}$	$0.71^{+0.08}_{-0.10}$	$3.3^{+0.9}_{-0.8}$	$3.6^{+0.6}_{-0.9} \times 10^{56}$	1850^{+840}_{-840}	$0.34^{+0.13}_{-0.14}$	1651

TABLE III. Selected source parameters of the eleven confident detections. We report median values with 90% credible intervals that include statistical errors, and systematic errors from averaging the results of two waveform models for BBHs. For GW170817 credible intervals and statistical errors are shown for IMRPhenomPv2NRT with low spin prior, while the sky area was computed from TaylorF2 samples. The redshift for NGC 4993 from [87] and its associated uncertainties were used to calculate source frame masses for GW170817. For BBH events the redshift was calculated from the luminosity distance and assumed cosmology as discussed in Appendix B. The columns show source frame component masses m_i and chirp mass $\mathcal{M}$, dimensionless effective aligned spin χ_{eff}, final source frame mass M_f, final spin a_f, radiated energy E_{rad}, peak luminosity l_{peak}, luminosity distance d_L, redshift z and sky localization $\Delta\Omega$. The sky localization is the area of the 90% credible region. For GW170817 we give conservative bounds on parameters of the final remnant discussed in Sec. V E.

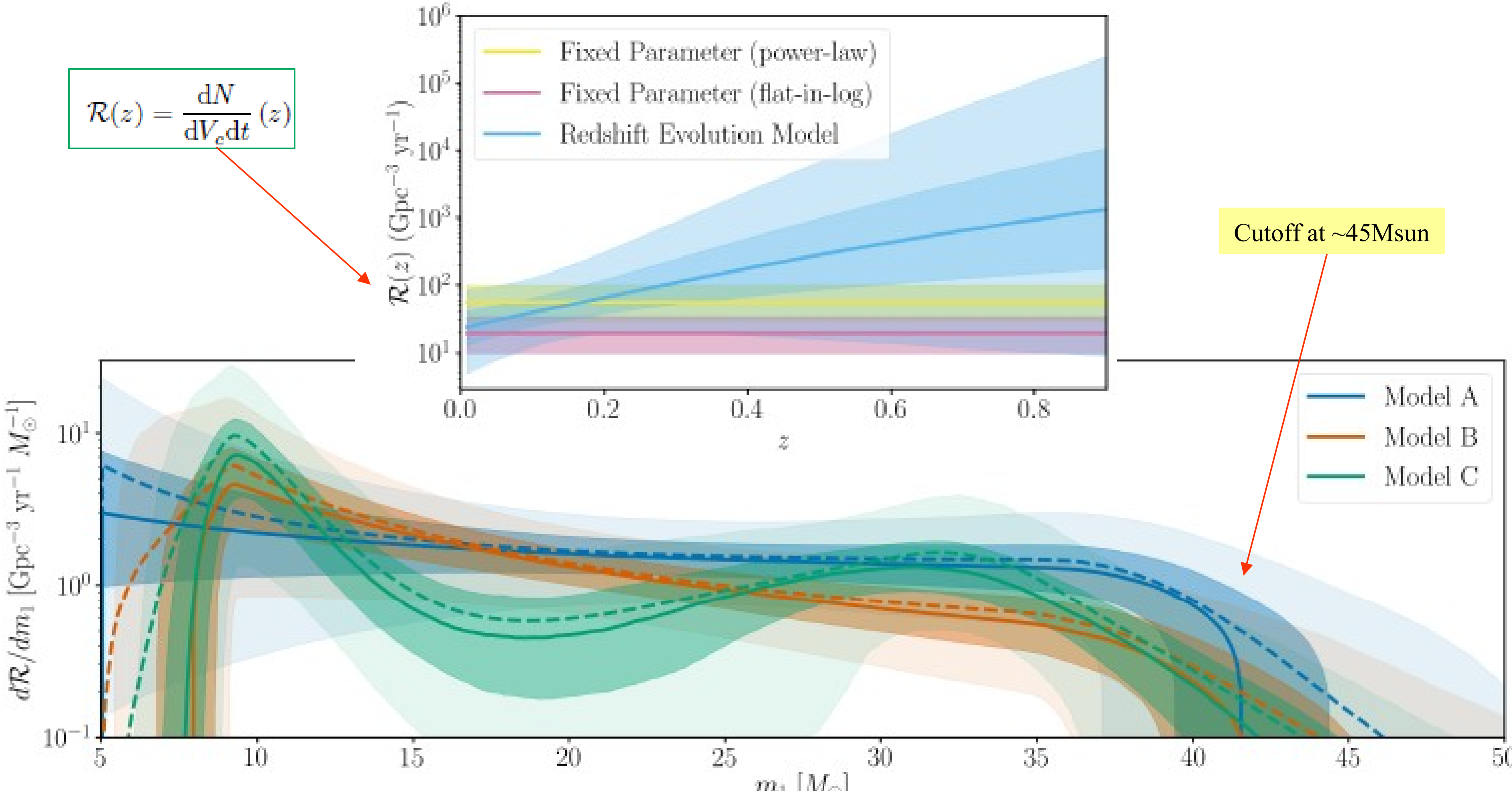
Lecture Notes on Gravitational Waves

Merger rate of events (up to Dec. 2018), as a function of redshift and mass

ArXiv: 1811.12940

$$\mathcal{R}(z) = \frac{dN}{dV_c dt}(z)$$

Cutoff at ~45Msun



Lecture Notes on Gravitational Waves

Overview and comparison of all GW observations (update Oct. 2020)

arXiv: 2010.14527

	M $(M_\odot)$	$\mathcal{M}$ $(M_\odot)$	m_1 $(M_\odot)$	m_2 $(M_\odot)$	$\chi_{\rm eff}$	D_L (Gpc)	z	M_f $(M_\odot)$	χ_f	$\Delta\Omega$ (deg^2)	SNR
GW190408_181802	$43.0^{+4.2}_{-3.0}$	$18.3^{+1.9}_{-1.2}$	$24.6^{+5.1}_{-3.4}$	$18.4^{+3.3}_{-3.6}$	$-0.03^{+0.14}_{-0.19}$	$1.55^{+0.40}_{-0.60}$	$0.29^{+0.06}_{-0.10}$	$41.1^{+3.9}_{-2.8}$	$0.67^{+0.06}_{-0.07}$	150	$15.3^{+0.2}_{-0.3}$
GW190412	$38.4^{+3.8}_{-3.7}$	$13.3^{+0.4}_{-0.3}$	$30.1^{+4.7}_{-5.1}$	$8.3^{+1.6}_{-0.9}$	$0.25^{+0.08}_{-0.11}$	$0.74^{+0.14}_{-0.17}$	$0.15^{+0.03}_{-0.03}$	$37.3^{+3.9}_{-3.8}$	$0.67^{+0.05}_{-0.06}$	21	$18.9^{+0.2}_{-0.3}$
GW190413_052954	$58.6^{+13.3}_{-9.7}$	$24.6^{+5.5}_{-4.1}$	$34.7^{+12.6}_{-8.1}$	$23.7^{+7.3}_{-6.7}$	$-0.01^{+0.29}_{-0.34}$	$3.55^{+2.27}_{-1.66}$	$0.59^{+0.29}_{-0.24}$	$56.0^{+12.5}_{-9.2}$	$0.68^{+0.12}_{-0.13}$	1500	$8.9^{+0.4}_{-0.7}$
GW190413_134308	$78.8^{+17.4}_{-11.9}$	$33.0^{+8.2}_{-5.4}$	$47.5^{+13.5}_{-10.7}$	$31.8^{+11.7}_{-10.8}$	$-0.03^{+0.25}_{-0.29}$	$4.45^{+2.48}_{-2.12}$	$0.71^{+0.31}_{-0.30}$	$75.5^{+16.4}_{-11.4}$	$0.68^{+0.10}_{-0.12}$	730	$10.0^{+0.4}_{-0.5}$
GW190421_213856	$72.9^{+13.4}_{-9.2}$	$31.2^{+5.9}_{-4.2}$	$41.3^{+10.4}_{-6.9}$	$31.9^{+8.0}_{-8.8}$	$-0.06^{+0.22}_{-0.27}$	$2.88^{+1.37}_{-1.38}$	$0.49^{+0.19}_{-0.21}$	$69.7^{+12.5}_{-8.7}$	$0.67^{+0.10}_{-0.11}$	1200	$10.7^{+0.2}_{-0.4}$
GW190424_180648	$72.6^{+13.3}_{-10.7}$	$31.0^{+5.8}_{-4.6}$	$40.5^{+11.1}_{-7.3}$	$31.8^{+7.6}_{-7.7}$	$0.13^{+0.22}_{-0.22}$	$2.20^{+1.58}_{-1.16}$	$0.39^{+0.23}_{-0.19}$	$68.9^{+12.4}_{-10.1}$	$0.74^{+0.09}_{-0.09}$	28000	$10.4^{+0.2}_{-0.4}$
GW190425	$3.4^{+0.3}_{-0.1}$	$1.44^{+0.02}_{-0.02}$	$2.0^{+0.6}_{-0.3}$	$1.4^{+0.3}_{-0.3}$	$0.06^{+0.11}_{-0.05}$	$0.16^{+0.07}_{-0.07}$	$0.03^{+0.01}_{-0.02}$	–	–	10000	$12.4^{+0.3}_{-0.4}$
GW190426_152155	$7.2^{+3.5}_{-1.5}$	$2.41^{+0.08}_{-0.08}$	$5.7^{+3.9}_{-2.3}$	$1.5^{+0.8}_{-0.5}$	$-0.03^{+0.32}_{-0.30}$	$0.37^{+0.18}_{-0.16}$	$0.08^{+0.04}_{-0.03}$	–	–	1300	$8.7^{+0.5}_{-0.6}$
GW190503_185404	$71.7^{+9.4}_{-8.3}$	$30.2^{+4.2}_{-4.2}$	$43.3^{+9.2}_{-8.1}$	$28.4^{+7.7}_{-8.0}$	$-0.03^{+0.20}_{-0.26}$	$1.45^{+0.69}_{-0.63}$	$0.27^{+0.11}_{-0.11}$	$68.6^{+8.8}_{-7.7}$	$0.66^{+0.09}_{-0.12}$	94	$12.4^{+0.2}_{-0.3}$
GW190512_180714	$35.9^{+3.8}_{-3.5}$	$14.6^{+1.3}_{-1.0}$	$23.3^{+5.3}_{-5.8}$	$12.6^{+3.6}_{-2.5}$	$0.03^{+0.12}_{-0.13}$	$1.43^{+0.55}_{-0.55}$	$0.27^{+0.09}_{-0.10}$	$34.5^{+3.8}_{-3.5}$	$0.65^{+0.07}_{-0.07}$	220	$12.2^{+0.2}_{-0.4}$
GW190513_205428	$53.9^{+8.6}_{-5.9}$	$21.6^{+3.8}_{-1.9}$	$35.7^{+9.5}_{-9.2}$	$18.0^{+7.7}_{-4.1}$	$0.11^{+0.28}_{-0.17}$	$2.06^{+0.88}_{-0.80}$	$0.37^{+0.13}_{-0.13}$	$51.6^{+8.2}_{-5.8}$	$0.68^{+0.14}_{-0.12}$	520	$12.9^{+0.3}_{-0.4}$
GW190514_065416	$67.2^{+18.7}_{-10.8}$	$28.5^{+7.9}_{-4.8}$	$39.0^{+14.7}_{-8.2}$	$28.4^{+9.3}_{-8.8}$	$-0.19^{+0.29}_{-0.32}$	$4.13^{+2.65}_{-2.17}$	$0.67^{+0.33}_{-0.31}$	$64.5^{+17.9}_{-10.4}$	$0.63^{+0.11}_{-0.15}$	3000	$8.2^{+0.3}_{-0.6}$
GW190517_055101	$63.5^{+9.6}_{-9.6}$	$26.6^{+4.0}_{-4.0}$	$37.4^{+11.7}_{-7.6}$	$25.3^{+7.0}_{-7.3}$	$0.52^{+0.19}_{-0.19}$	$1.86^{+1.62}_{-0.84}$	$0.34^{+0.24}_{-0.14}$	$59.3^{+9.1}_{-8.9}$	$0.87^{+0.05}_{-0.07}$	470	$10.7^{+0.4}_{-0.6}$
GW190519_153544	$106.6^{+13.5}_{-14.8}$	$44.5^{+6.4}_{-7.1}$	$66.0^{+10.7}_{-12.0}$	$40.5^{+11.0}_{-11.1}$	$0.31^{+0.20}_{-0.22}$	$2.53^{+1.83}_{-0.92}$	$0.44^{+0.25}_{-0.14}$	$101.0^{+12.4}_{-13.8}$	$0.79^{+0.07}_{-0.13}$	860	$15.6^{+0.2}_{-0.3}$
GW190521	$163.9^{+39.2}_{-23.5}$	$69.2^{+17.0}_{-10.6}$	$95.3^{+28.7}_{-18.9}$	$69.0^{+22.7}_{-23.1}$	$0.03^{+0.32}_{-0.39}$	$3.92^{+2.19}_{-1.95}$	$0.64^{+0.28}_{-0.28}$	$156.3^{+36.8}_{-22.4}$	$0.71^{+0.12}_{-0.16}$	1000	$14.2^{+0.3}_{-0.3}$
GW190521_074359	$74.7^{+7.0}_{-4.8}$	$32.1^{+3.2}_{-2.5}$	$42.2^{+5.9}_{-4.8}$	$32.8^{+5.4}_{-6.4}$	$0.09^{+0.10}_{-0.13}$	$1.24^{+0.40}_{-0.57}$	$0.24^{+0.07}_{-0.10}$	$71.0^{+6.5}_{-4.4}$	$0.72^{+0.05}_{-0.07}$	550	$25.8^{+0.1}_{-0.2}$
GW190527_092055	$59.1^{+21.3}_{-9.8}$	$24.3^{+9.1}_{-4.2}$	$36.5^{+16.4}_{-9.0}$	$22.6^{+10.5}_{-8.1}$	$0.11^{+0.28}_{-0.28}$	$2.49^{+2.48}_{-1.24}$	$0.44^{+0.34}_{-0.20}$	$56.4^{+20.2}_{-9.3}$	$0.71^{+0.12}_{-0.16}$	3700	$8.1^{+0.3}_{-0.9}$
GW190602_175927	$116.3^{+19.0}_{-15.6}$	$49.1^{+9.1}_{-8.5}$	$69.1^{+15.7}_{-13.0}$	$47.8^{+14.3}_{-17.4}$	$0.07^{+0.25}_{-0.24}$	$2.69^{+1.79}_{-1.12}$	$0.47^{+0.25}_{-0.17}$	$110.9^{+17.7}_{-14.9}$	$0.70^{+0.10}_{-0.14}$	690	$12.8^{+0.2}_{-0.3}$
GW190620_030421	$92.1^{+18.5}_{-13.1}$	$38.3^{+8.3}_{-6.5}$	$57.1^{+16.0}_{-12.7}$	$35.5^{+12.2}_{-12.3}$	$0.33^{+0.22}_{-0.25}$	$2.81^{+1.68}_{-1.31}$	$0.49^{+0.23}_{-0.20}$	$87.2^{+16.8}_{-12.1}$	$0.79^{+0.08}_{-0.15}$	7200	$12.1^{+0.3}_{-0.4}$
GW190630_185205	$59.1^{+4.6}_{-4.8}$	$24.9^{+2.1}_{-2.1}$	$35.1^{+6.9}_{-5.6}$	$23.7^{+5.2}_{-5.1}$	$0.10^{+0.12}_{-0.13}$	$0.89^{+0.56}_{-0.37}$	$0.18^{+0.10}_{-0.07}$	$56.4^{+4.4}_{-4.6}$	$0.70^{+0.05}_{-0.07}$	1200	$15.6^{+0.2}_{-0.3}$
GW190701_203306	$94.3^{+12.1}_{-9.5}$	$40.3^{+5.4}_{-4.9}$	$53.9^{+11.8}_{-8.0}$	$40.8^{+8.7}_{-12.0}$	$-0.07^{+0.23}_{-0.29}$	$2.06^{+0.76}_{-0.73}$	$0.37^{+0.11}_{-0.12}$	$90.2^{+11.3}_{-8.9}$	$0.66^{+0.09}_{-0.13}$	46	$11.3^{+0.2}_{-0.3}$

Overview and comparison of all GW observations (update Oct. 2020)

GW190706_222641	$104.1^{+20.2}_{-13.9}$	$42.7^{+10.0}_{-7.0}$	$67.0^{+14.6}_{-16.2}$	$38.2^{+14.6}_{-13.3}$	$0.28^{+0.26}_{-0.29}$	$4.42^{+2.59}_{-1.93}$	$0.71^{+0.32}_{-0.27}$	$99.0^{+18.3}_{-13.5}$	$0.78^{+0.09}_{-0.18}$	650	$12.6^{+0.2}_{-0.4}$
GW190707_093326	$20.1^{+1.9}_{-1.3}$	$8.5^{+0.6}_{-0.5}$	$11.6^{+3.3}_{-1.7}$	$8.4^{+1.4}_{-1.7}$	$-0.05^{+0.10}_{-0.08}$	$0.77^{+0.38}_{-0.37}$	$0.16^{+0.07}_{-0.07}$	$19.2^{+1.9}_{-1.3}$	$0.66^{+0.03}_{-0.04}$	1300	$13.3^{+0.2}_{-0.4}$
GW190708_232457	$30.9^{+2.5}_{-1.8}$	$13.2^{+0.9}_{-0.6}$	$17.6^{+4.7}_{-2.3}$	$13.2^{+2.0}_{-2.7}$	$0.02^{+0.10}_{-0.08}$	$0.88^{+0.33}_{-0.39}$	$0.18^{+0.06}_{-0.07}$	$29.5^{+2.5}_{-1.8}$	$0.69^{+0.04}_{-0.04}$	14000	$13.1^{+0.2}_{-0.3}$
GW190719_215514	$57.8^{+18.3}_{-10.7}$	$23.5^{+6.5}_{-4.0}$	$36.5^{+18.0}_{-10.3}$	$20.8^{+9.0}_{-7.2}$	$0.32^{+0.29}_{-0.31}$	$3.94^{+2.59}_{-2.00}$	$0.64^{+0.33}_{-0.29}$	$54.9^{+17.3}_{-10.2}$	$0.78^{+0.11}_{-0.17}$	2900	$8.3^{+0.3}_{-0.8}$
GW190720_000836	$21.5^{+4.3}_{-2.3}$	$8.9^{+0.5}_{-0.8}$	$13.4^{+6.7}_{-3.0}$	$7.8^{+2.3}_{-2.2}$	$0.18^{+0.14}_{-0.12}$	$0.79^{+0.69}_{-0.32}$	$0.16^{+0.12}_{-0.06}$	$20.4^{+4.5}_{-2.2}$	$0.72^{+0.06}_{-0.05}$	460	$11.0^{+0.3}_{-0.7}$
GW190727_060333	$67.1^{+11.7}_{-8.0}$	$28.6^{+5.3}_{-3.7}$	$38.0^{+9.5}_{-6.2}$	$29.4^{+7.1}_{-8.4}$	$0.11^{+0.26}_{-0.25}$	$3.30^{+1.54}_{-1.50}$	$0.55^{+0.21}_{-0.22}$	$63.8^{+10.9}_{-7.5}$	$0.73^{+0.10}_{-0.10}$	830	$11.9^{+0.3}_{-0.5}$
GW190728_064510	$20.6^{+4.5}_{-1.3}$	$8.6^{+0.5}_{-0.3}$	$12.3^{+7.2}_{-2.2}$	$8.1^{+1.7}_{-2.6}$	$0.12^{+0.20}_{-0.07}$	$0.87^{+0.26}_{-0.37}$	$0.18^{+0.05}_{-0.07}$	$19.6^{+4.7}_{-1.3}$	$0.71^{+0.04}_{-0.04}$	400	$13.0^{+0.2}_{-0.4}$
GW190731_140936	$70.1^{+15.8}_{-11.3}$	$29.5^{+7.1}_{-5.2}$	$41.5^{+12.2}_{-9.0}$	$28.8^{+9.7}_{-9.5}$	$0.06^{+0.24}_{-0.24}$	$3.30^{+2.39}_{-1.72}$	$0.55^{+0.31}_{-0.26}$	$67.0^{+14.6}_{-10.8}$	$0.70^{+0.10}_{-0.13}$	3400	$8.7^{+0.2}_{-0.5}$
GW190803_022701	$64.5^{+12.6}_{-9.0}$	$27.3^{+5.7}_{-4.1}$	$37.3^{+10.6}_{-7.0}$	$27.3^{+7.8}_{-8.2}$	$-0.03^{+0.24}_{-0.27}$	$3.27^{+1.95}_{-1.58}$	$0.55^{+0.26}_{-0.24}$	$61.7^{+11.8}_{-8.5}$	$0.68^{+0.10}_{-0.11}$	1500	$8.6^{+0.3}_{-0.5}$
GW190814	$25.8^{+1.0}_{-0.9}$	$6.09^{+0.06}_{-0.06}$	$23.2^{+1.1}_{-1.0}$	$2.59^{+0.08}_{-0.09}$	$0.00^{+0.06}_{-0.06}$	$0.24^{+0.04}_{-0.05}$	$0.05^{+0.009}_{-0.010}$	$25.6^{+1.1}_{-0.9}$	$0.28^{+0.02}_{-0.02}$	19	$24.9^{+0.1}_{-0.2}$
GW190828_063405	$58.0^{+7.7}_{-4.8}$	$25.0^{+3.4}_{-2.1}$	$32.1^{+5.8}_{-4.0}$	$26.2^{+4.6}_{-4.8}$	$0.19^{+0.15}_{-0.16}$	$2.13^{+0.66}_{-0.93}$	$0.38^{+0.10}_{-0.15}$	$54.9^{+7.2}_{-4.3}$	$0.75^{+0.06}_{-0.07}$	520	$16.2^{+0.2}_{-0.3}$
GW190828_065509	$34.4^{+5.4}_{-4.4}$	$13.3^{+1.2}_{-1.0}$	$24.1^{+7.0}_{-7.2}$	$10.2^{+3.6}_{-2.1}$	$0.08^{+0.16}_{-0.16}$	$1.60^{+0.62}_{-0.60}$	$0.30^{+0.10}_{-0.10}$	$33.1^{+5.5}_{-4.5}$	$0.65^{+0.08}_{-0.08}$	660	$10.0^{+0.3}_{-0.5}$
GW190909_114149	$75.0^{+55.9}_{-17.6}$	$30.9^{+17.2}_{-7.5}$	$45.8^{+52.7}_{-13.3}$	$28.3^{+13.4}_{-12.7}$	$-0.06^{+0.37}_{-0.36}$	$3.77^{+3.27}_{-2.22}$	$0.62^{+0.41}_{-0.33}$	$72.0^{+54.9}_{-16.8}$	$0.66^{+0.15}_{-0.20}$	4700	$8.1^{+0.4}_{-0.6}$
GW190910_112807	$79.6^{+9.3}_{-9.1}$	$34.3^{+4.1}_{-4.1}$	$43.9^{+7.6}_{-6.1}$	$35.6^{+6.3}_{-7.2}$	$0.02^{+0.18}_{-0.18}$	$1.46^{+1.03}_{-0.58}$	$0.28^{+0.16}_{-0.10}$	$75.8^{+8.5}_{-8.6}$	$0.70^{+0.08}_{-0.07}$	11000	$14.1^{+0.2}_{-0.3}$
GW190915_235702	$59.9^{+7.5}_{-6.4}$	$25.3^{+3.2}_{-2.7}$	$35.3^{+9.5}_{-6.4}$	$24.4^{+5.6}_{-6.1}$	$0.02^{+0.20}_{-0.25}$	$1.62^{+0.71}_{-0.61}$	$0.30^{+0.11}_{-0.10}$	$57.2^{+7.1}_{-6.0}$	$0.70^{+0.09}_{-0.11}$	400	$13.6^{+0.2}_{-0.3}$
GW190924_021846	$13.9^{+5.1}_{-1.0}$	$5.8^{+0.2}_{-0.2}$	$8.9^{+7.0}_{-2.0}$	$5.0^{+1.4}_{-1.9}$	$0.03^{+0.30}_{-0.09}$	$0.57^{+0.22}_{-0.22}$	$0.12^{+0.04}_{-0.04}$	$13.3^{+5.2}_{-1.0}$	$0.67^{+0.05}_{-0.05}$	360	$11.5^{+0.3}_{-0.4}$
GW190929_012149	$104.3^{+34.9}_{-25.2}$	$35.8^{+14.9}_{-8.2}$	$80.8^{+33.0}_{-33.2}$	$24.1^{+19.3}_{-10.6}$	$0.01^{+0.34}_{-0.33}$	$2.13^{+3.65}_{-1.05}$	$0.38^{+0.49}_{-0.17}$	$101.5^{+33.6}_{-25.3}$	$0.66^{+0.20}_{-0.31}$	2200	$10.1^{+0.6}_{-0.8}$
GW190930_133541	$20.3^{+8.9}_{-1.5}$	$8.5^{+0.5}_{-0.5}$	$12.3^{+12.4}_{-2.3}$	$7.8^{+1.7}_{-3.3}$	$0.14^{+0.31}_{-0.15}$	$0.76^{+0.36}_{-0.32}$	$0.15^{+0.06}_{-0.06}$	$19.4^{+9.2}_{-1.5}$	$0.72^{+0.07}_{-0.06}$	1700	$9.5^{+0.3}_{-0.5}$

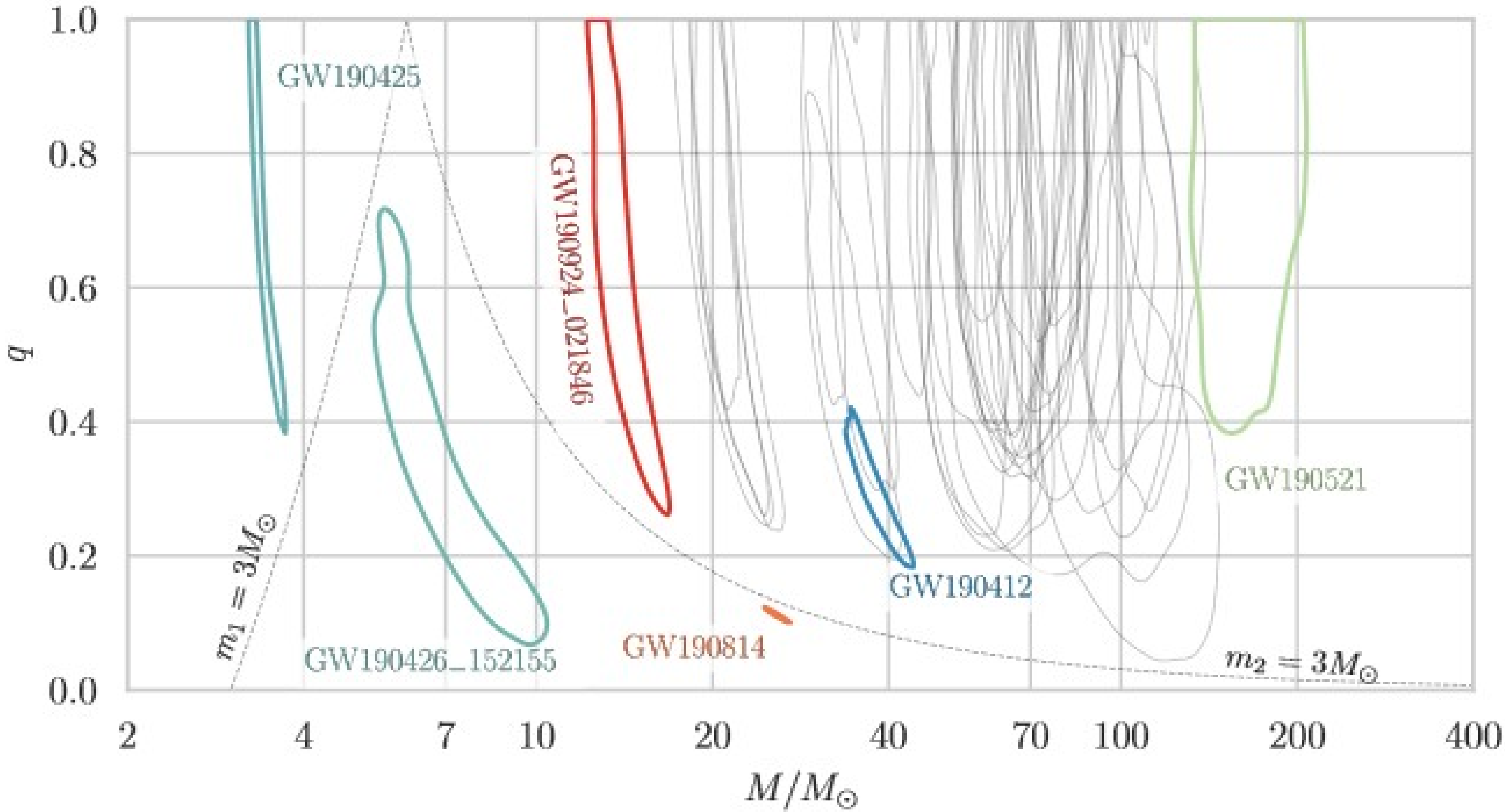
TABLE VI. Median and 90% symmetric credible intervals on selected source parameters. The columns show source total mass M, chirp mass $\mathcal{M}$ and component masses m_i, dimensionless effective inspiral spin $\chi_{\rm eff}$, luminosity distance $D_{\rm L}$, redshift z, final mass $M_{\rm f}$, final spin $\chi_{\rm f}$, and sky localization $\Delta\Omega$. The sky localization is the area of the 90% credible region. For GW190425 we show the results using the high-spin prior ($|\vec{\chi}_i| \leq 0.89$). We also report the network matched filter SNR for all events. These SNRs are from LALInference IMRPhenomPv2 runs since RIFT does not produce the SNRs automatically, except for GW190425 and GW190426_152155 which use the SNRs from fiducial runs, and GW190412, GW190521, and GW190814, which use IMRPhenomPv3HM SNRs. For GW190521 we report results averaged over three waveform families, in contrast to the results highlighting one waveform family in [34].

arXiv: 2010.14527

Lecture Notes on Gravitational Waves

Overview and comparison of all GW observations (update Oct. 2020)

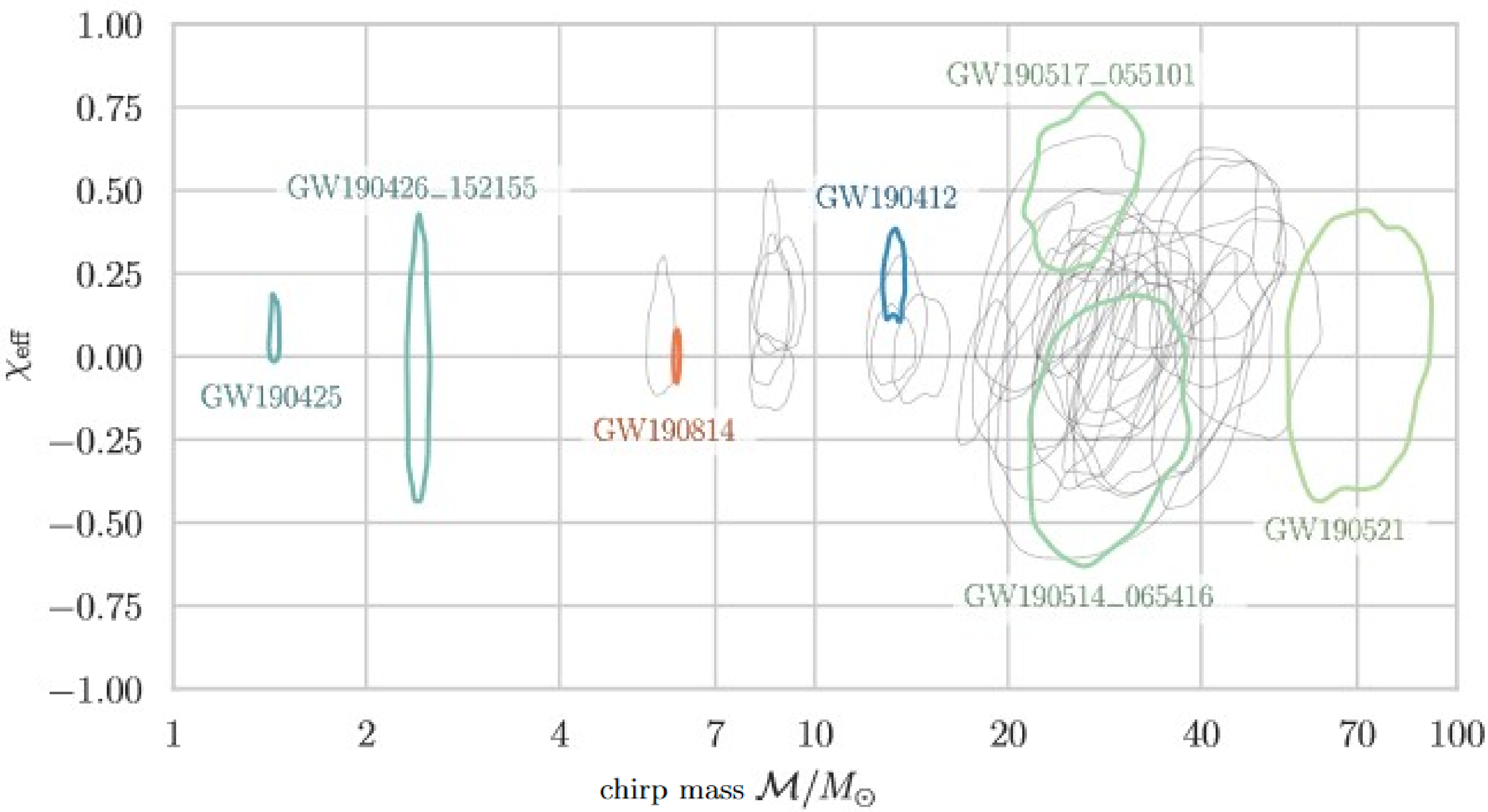
arXiv: 2010.14527



Lecture Notes on Gravitational Waves

Overview and comparison of all GW observations (update Oct. 2020)

arXiv: 2010.14527

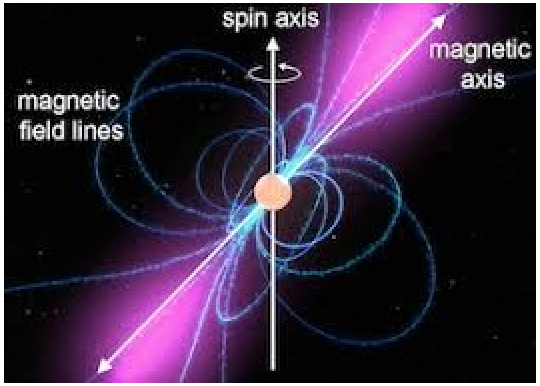


Lecture Notes on Gravitational Waves

Neutron star- neutron star binary: there's an optical counterpart!

Neutron stars are collapsed stars, supported by neutron degeneracy pressure. Masses <1.4 M_{sun}

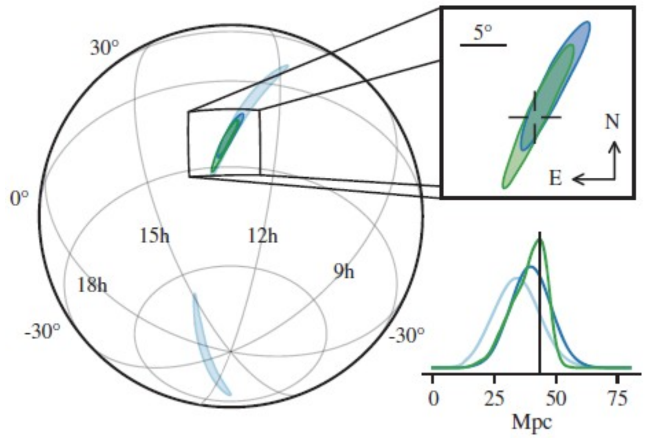
Usually emit radiation in pulses (→ pulsars)



LIGO saw event GW170817 linked to GRB170817A, detected by Fermi

Lecture Notes on Gravitational Waves

Detected by 2 LIGOs and Virgo → triangulation!

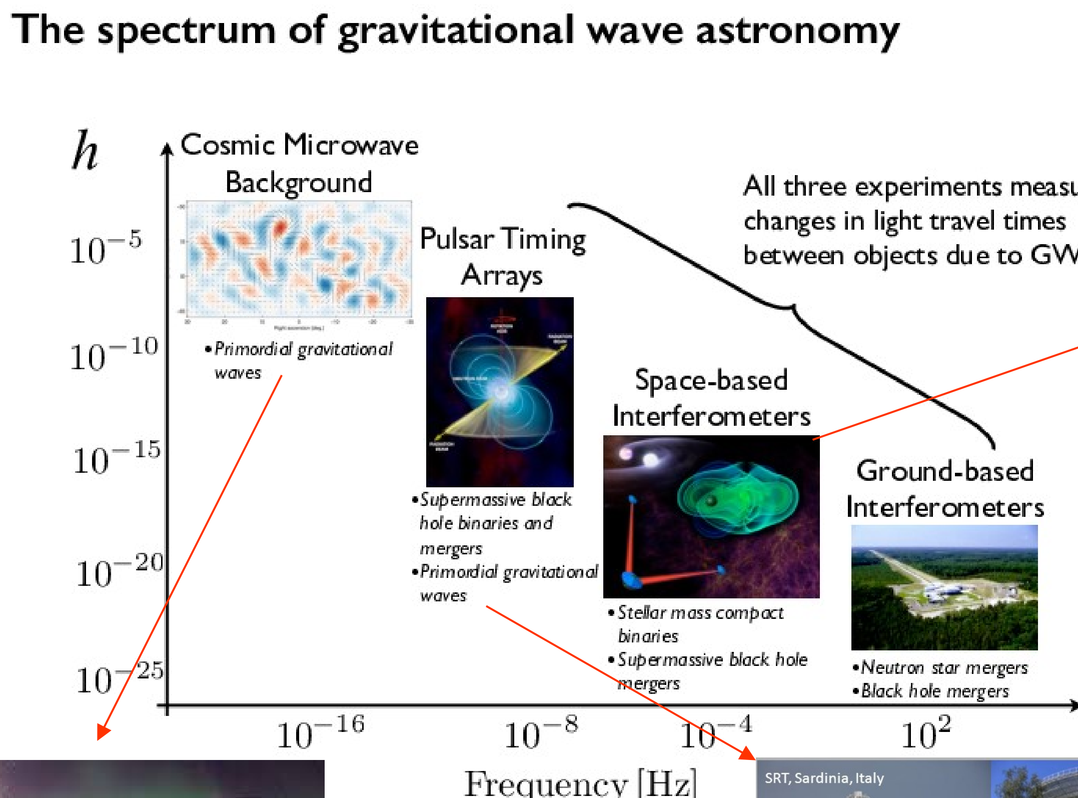


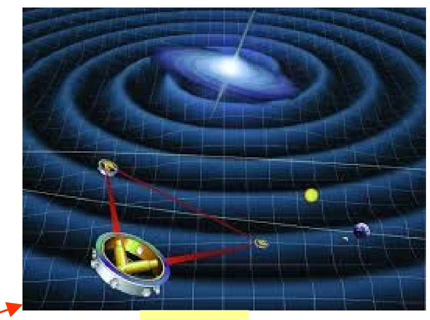
Spin of the objects is important in this case:

	Low-spin priors ($|\chi| \leq 0.05$)	High-spin priors ($|\chi| \leq 0.89$)
Primary mass m_1	1.36–1.60 $M_\odot$	1.36–2.26 $M_\odot$
Secondary mass m_2	1.17–1.36 $M_\odot$	0.86–1.36 $M_\odot$
Chirp mass $\mathcal{M}$	$1.188^{+0.004}_{-0.002} M_\odot$	$1.188^{+0.004}_{-0.002} M_\odot$
Mass ratio m_2/m_1	0.7–1.0	0.4–1.0
Total mass $m_{\rm tot}$	$2.74^{+0.04}_{-0.01} M_\odot$	$2.82^{+0.47}_{-0.09} M_\odot$
Radiated energy $E_{\rm rad}$	$> 0.025 M_\odot c^2$	$> 0.025 M_\odot c^2$
Luminosity distance $D_{\rm L}$	40^{+8}_{-14} Mpc	40^{+8}_{-14} Mpc
Viewing angle Θ	$\lesssim 55°$	$\lesssim 56°$
Using NGC 4993 location	$\lesssim 28°$	$\lesssim 28°$
Combined dimensionless tidal deformability $\tilde{\Lambda}$	$\lesssim 800$	$\lesssim 700$
Dimensionless tidal deformability $\Lambda(1.4 M_\odot)$	$\lesssim 800$	$\lesssim 1400$

Lecture Notes on Gravitational Waves

Other GW experiments/detectors:





LISA

BICEP 2

Main points of the lecture

- What are the GWs (history, description)
- Formalism in GR (linearization, gauges, emission)
- Detection techniques (interferometry, LIGO)
- Recent observations (BH-BH, NS-NS)
- Other issues (speed of GWs, hyperbolic encounters)

The speed of Gravitational Waves

GRB170817A was observed ~1.7 s after GW170817 →

1) Constraints on speed of GWs and modifications of gravity!

$$-3 \cdot 10^{-15} \leq c_g/c - 1 \leq 7 \cdot 10^{-16}$$

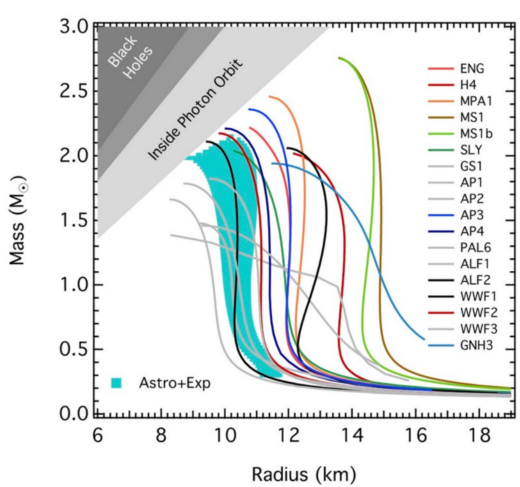
$$c_g^2 = 1 + \alpha_T$$

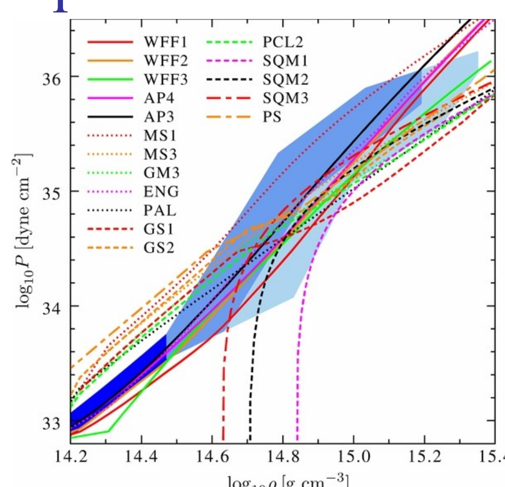
$$\ddot{h}_{ij} + (3 + \alpha_M) H \dot{h}_{ij} + (1 + \alpha_T) k^2 h_{ij} = 0$$

2) Optical counterpart → redshift → cosmological constraints

$$H_0 = 70^{+12}_{-8}\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$$

3) Possible constraints on the equation of state of neutron stars

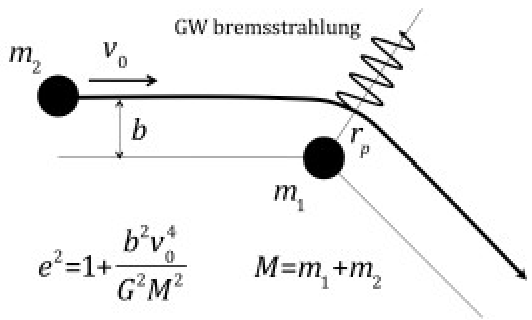




GW emission from hyperbolic PBH encounters

1) PBHs (see Inflation lecture) may scatter in clusters (aka hyperbolic encounters)

1711.09702



$$r(\varphi) = \frac{b\,\sin\varphi_0}{\cos(\varphi-\varphi_0)-\cos\varphi_0} = \frac{a\,(e^2-1)}{1+e\cos(\varphi-\varphi_0)}$$

$$\varphi_0 = \arccos\left(-\frac{1}{e}\right)$$

$$r_{\rm min} = a\,(e-1) = b\sqrt{\frac{e-1}{e+1}} > R_s \equiv \frac{2GM}{c^2}$$

2) Amplitude and power emitted

$$Q_{ij} = \mu\, r^2(\varphi)\begin{pmatrix} 3\cos^2\varphi - 1 & 3\cos\varphi\sin\varphi & 0 \\ 3\cos\varphi\sin\varphi & 3\sin^2\varphi - 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Longrightarrow$$

$$P = \frac{dE}{dt} = -\frac{G}{45c^5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle = \frac{32G\mu^2 v_0^6}{45c^5\,b^2}\,f(\varphi,e)$$

$$h_c = \frac{2G}{R\,c^4}\langle \ddot{Q}_{ij}\ddot{Q}^{ij}\rangle^{1/2}_{i,j=1,2} = \frac{2G\mu\,v_0^2}{R\,c^4}\,g(\varphi,e)$$

$$f(\varphi,e) = \frac{3\,(1+e\cos(\varphi-\varphi_0))^4}{8(e^2-1)^4}\Big[24+13\,e^2 + 48\,e\cos(\varphi-\varphi_0) + 11\,e^2\cos 2(\varphi-\varphi_0)\Big]$$

$$g(\varphi,e) = \frac{\sqrt{2}}{e^2-1}\Big[36+59\,e^2+10\,e^4 + (108+47\,e^2)\,e\cos(\varphi-\varphi_0) + 59\,e^2\cos 2(\varphi-\varphi_0) + 9\,e^3\cos 3(\varphi-\varphi_0)\Big]^{1/2}$$

GW emission from hyperbolic PBH encounters

3) Frequency domain and power spectrum

$$\Delta E = \int_{-\infty}^{\infty} P(t)\, dt = \frac{1}{\pi}\int_0^{\infty} P(\omega)\, d\omega$$
$$= -\frac{8}{15}\frac{G^{7/2}}{c^5}\frac{M^{1/2} m_1^2 m_2^2}{r_{min}^{7/2}} f(e)$$

$$P(\omega) = \frac{G}{45c^5}\sum_{i,j} |\widehat{\dddot{Q}_{ij}}|^2$$
$$= \frac{G}{45c^5}\,\omega^6 \sum_{i,j} |\widehat{Q_{ij}}|^2,$$

4) The quadrupole tensor is given by

$$Q_{ij} = \frac{1}{2}a^2\mu \begin{pmatrix} (3-e^2)\cosh 2\xi - 8e\cosh\xi & 3\sqrt{e^2-1}(2e\sinh\xi - \sinh 2\xi) & 0 \\ 3\sqrt{e^2-1}(2e\sinh\xi - \sinh 2\xi) & (2e^2-3)\cosh 2\xi + 4e\cosh\xi & 0 \\ 0 & 0 & 4e\cosh\xi - e^2\cosh 2\xi \end{pmatrix}$$

$$t(\xi) = \nu_0(e\sinh\xi - \xi)\,,$$
$$r(\xi) = a(e\cosh\xi - 1)\,.$$

$$\nu_0 = \sqrt{a^3/GM}.$$

GW emission from hyperbolic PBH encounters

5) The power spectrum:

$$P(\omega) = \frac{G^3\mu^2M^2}{a^2c^5}\left(\frac{\pi^2}{180}\,\nu^4\sum_{i,j}|\widehat{C_{ij}}|^2\right) = \frac{G^3\mu^2M^2}{a^2c^5}\,\frac{16\pi^2}{180}\,\nu^4 F_e(\nu),$$

$$\begin{aligned}F_e(\nu) &= \left|\frac{3(e^2-1)}{e}H^{(1)\prime}_{i\nu}(i\nu e) + \frac{e^2-3}{e^2}\,\frac{i}{\nu}\,H^{(1)}_{i\nu}(i\nu e)\right|^2 \\ &+ \left|\frac{3(e^2-1)}{e}H^{(1)\prime}_{i\nu}(i\nu e) + \frac{2e^2-3}{e^2}\,\frac{i}{\nu}\,H^{(1)}_{i\nu}(i\nu e)\right|^2 \\ &+ \left|\frac{i}{\nu}H^{(1)}_{i\nu}(i\nu e)\right|^2 + \frac{18\,(e^2-1)}{e^2}\times \\ &\times \left|\frac{(e^2-1)}{e}\,i\,H^{(1)}_{i\nu}(i\nu e) + \frac{1}{\nu}\,H^{(1)\prime}_{i\nu}(i\nu e)\right|^2\end{aligned}$$

Hankel function

6) Total power and peak frequency

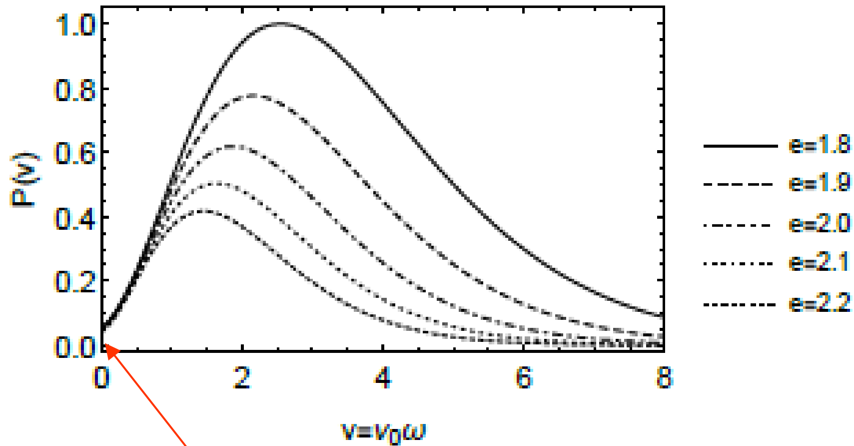
$$\Delta E = \int_{-\infty}^{+\infty} P(t)dt = \int_0^{+\infty}\frac{P(\omega)}{\pi}d\omega = \left(\frac{G^{7/2}\mu^2M^{5/2}}{c^5a^{7/2}}\right)\frac{16\,\pi}{180}\int_0^{+\infty}\nu^4F_e(\nu)d\nu$$

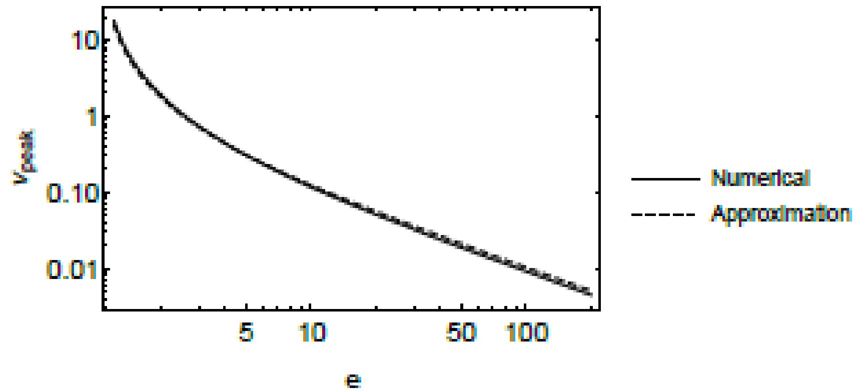
$$\begin{aligned}\nu^4F_e(\nu) &\simeq \frac{12\,F_y(\nu)}{\pi\,y\,(y^2+1)^2}\,e^{-2\nu z(y)}\,, \\ F_y(\nu) &= \nu\left(1-y^2-3\nu y^3+4y^4+9\nu y^5+6\nu^2y^6\right) \\ z(y) &= y-\arctan y\,, \qquad y\equiv\sqrt{e^2-1}\end{aligned}$$

$$\nu_{\rm max}(e) = \sqrt{\frac{e+1}{(e-1)^3}}\,, \quad \omega_{\rm max}(e) = \frac{v_0}{b}\left(\frac{e+1}{e-1}\right)$$

GW emission from hyperbolic PBH encounters

7) Peak frequency is important (detectable by LIGO)





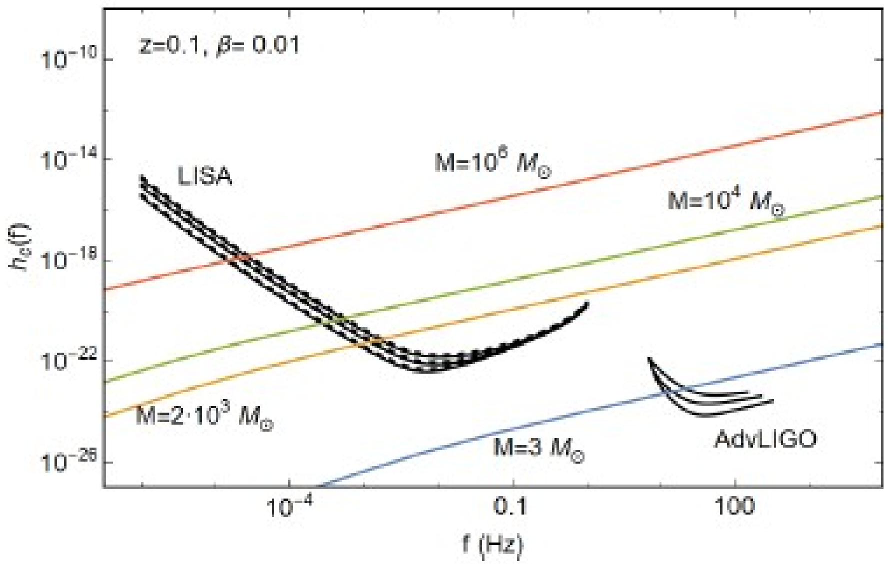
8) GW memory effect! After scattering (ω⟶0) spacetime remembers event…

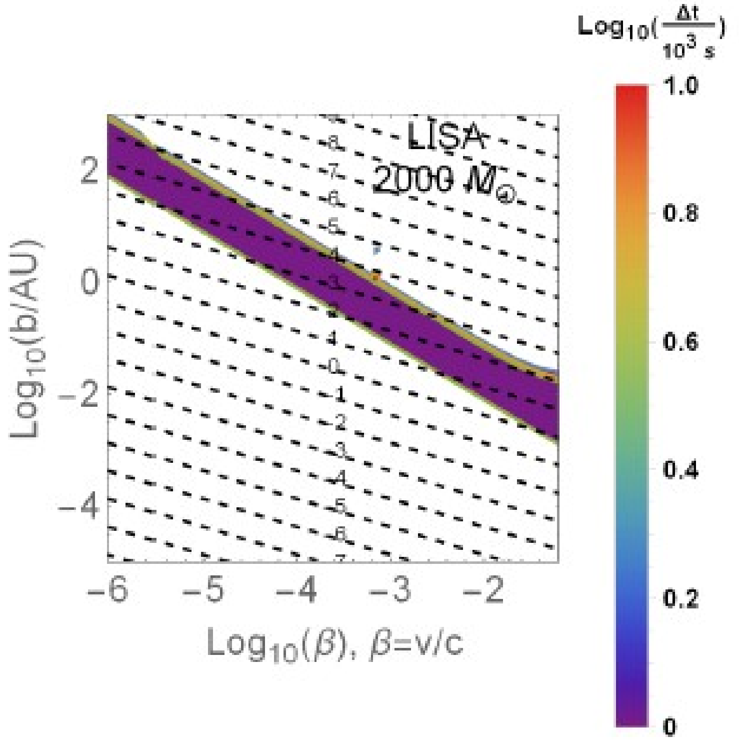
$$P(\omega = 0) = \frac{G^3 \mu^2 M^2}{a^2 c^5} \, \frac{32\left(e^2 - 1\right)}{5e^4}$$

GW emission from hyperbolic PBH encounters

9) Possibility of detection by LISA-LIGO:

i) LISA+LIGO are sensitive in specific frequencies-strains.
ii) These are known as sensitivity curves (see below).
iii) PBH by hyperbolic encounters gives unique predictions for strain+frequency. Also unique stain for detector.
iv) The scattering will be seen as a unique (not periodic even like in the binaries) event, aka a glitch.





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