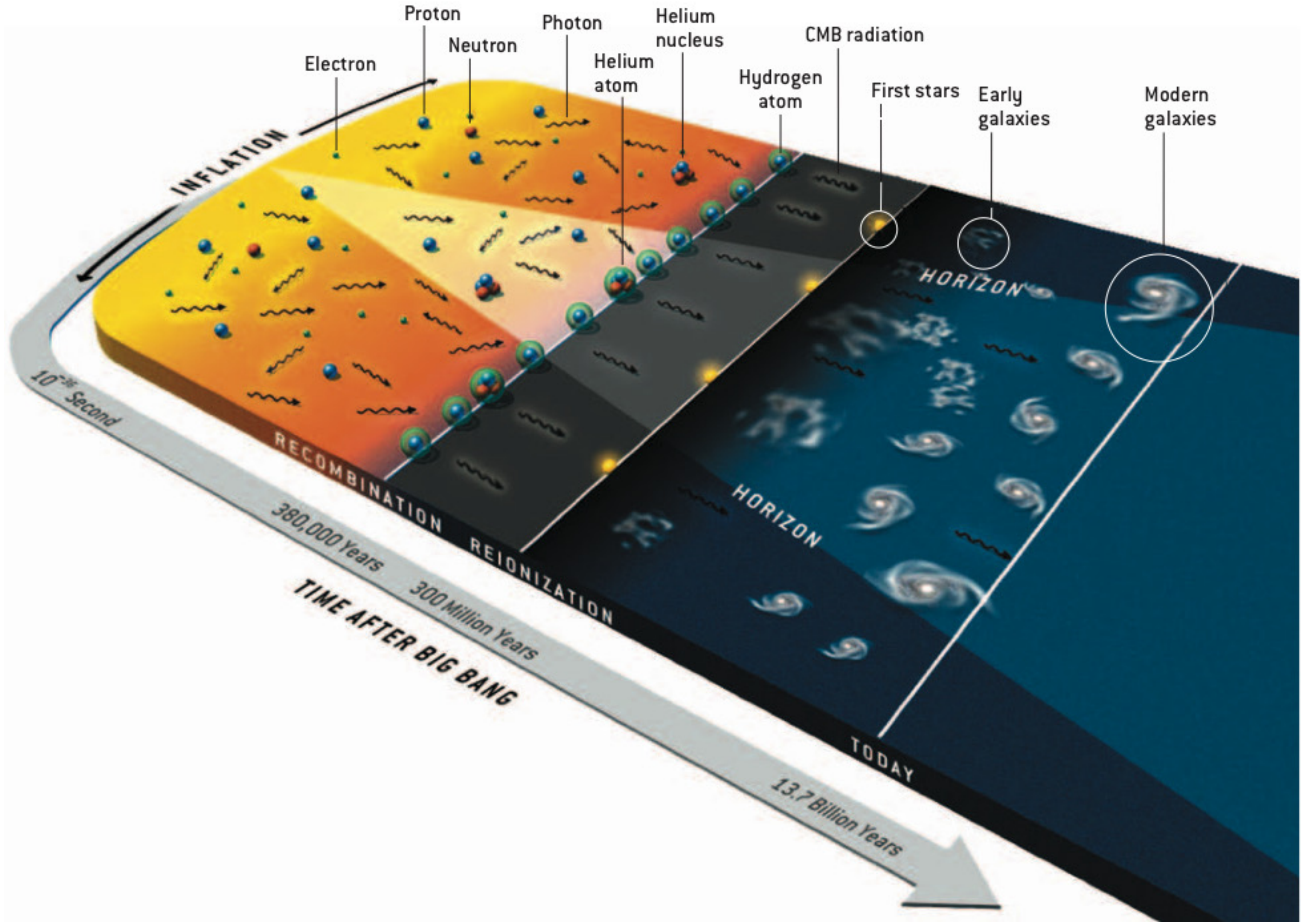
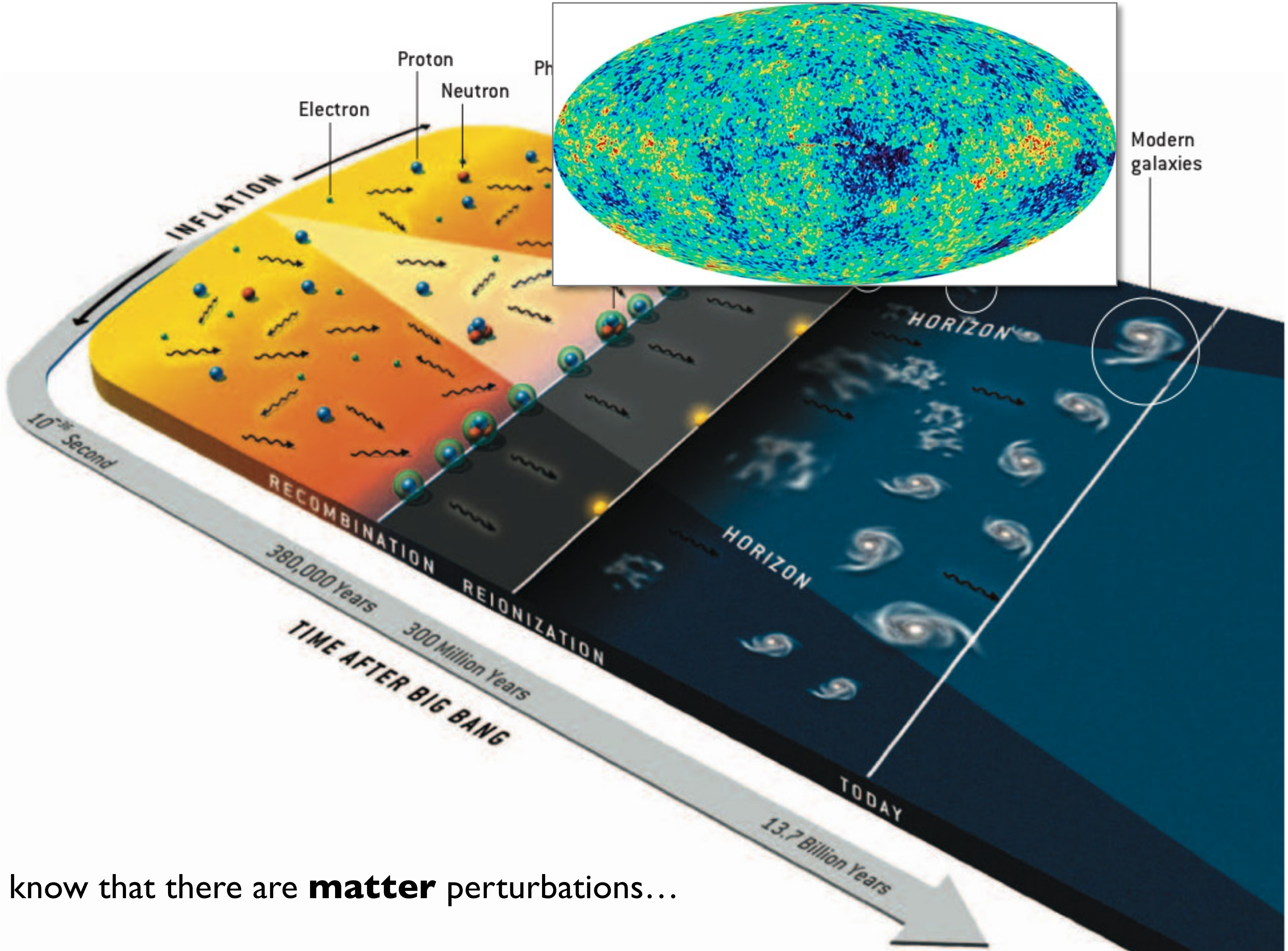
Alexander Knebe (*Universidad Autonoma de Madrid*)

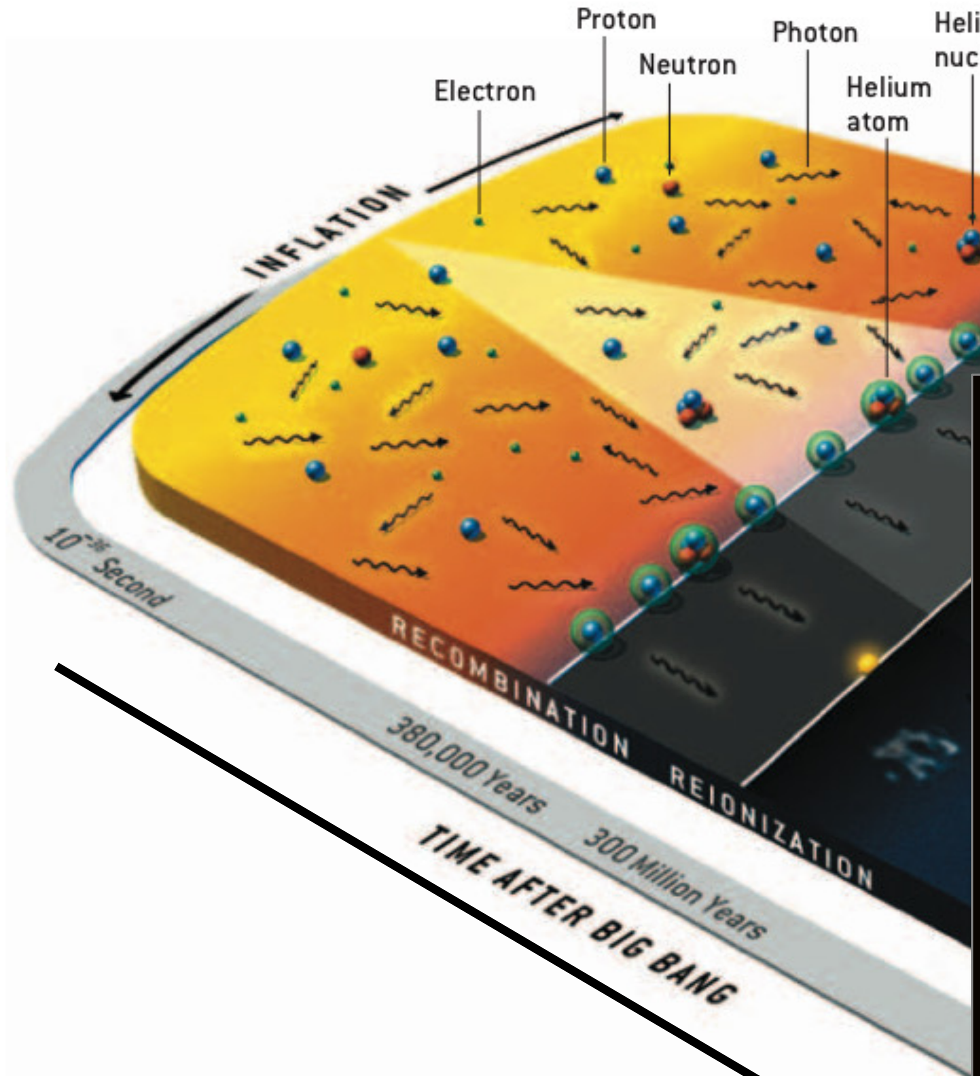

Electron
Proton
Neutron
Photon
Helium atom
Helium nucleus
Hydrogen atom
CMB radiation
First stars
Early galaxies
Modern galaxies
INFLATION
HORIZON
HORIZON
RECOMBINATION
REIONIZATION
TODAY
10^{-36} Second
380,000 Years
300 Million Years
13.7 Billion Years
TIME AFTER BIG BANG

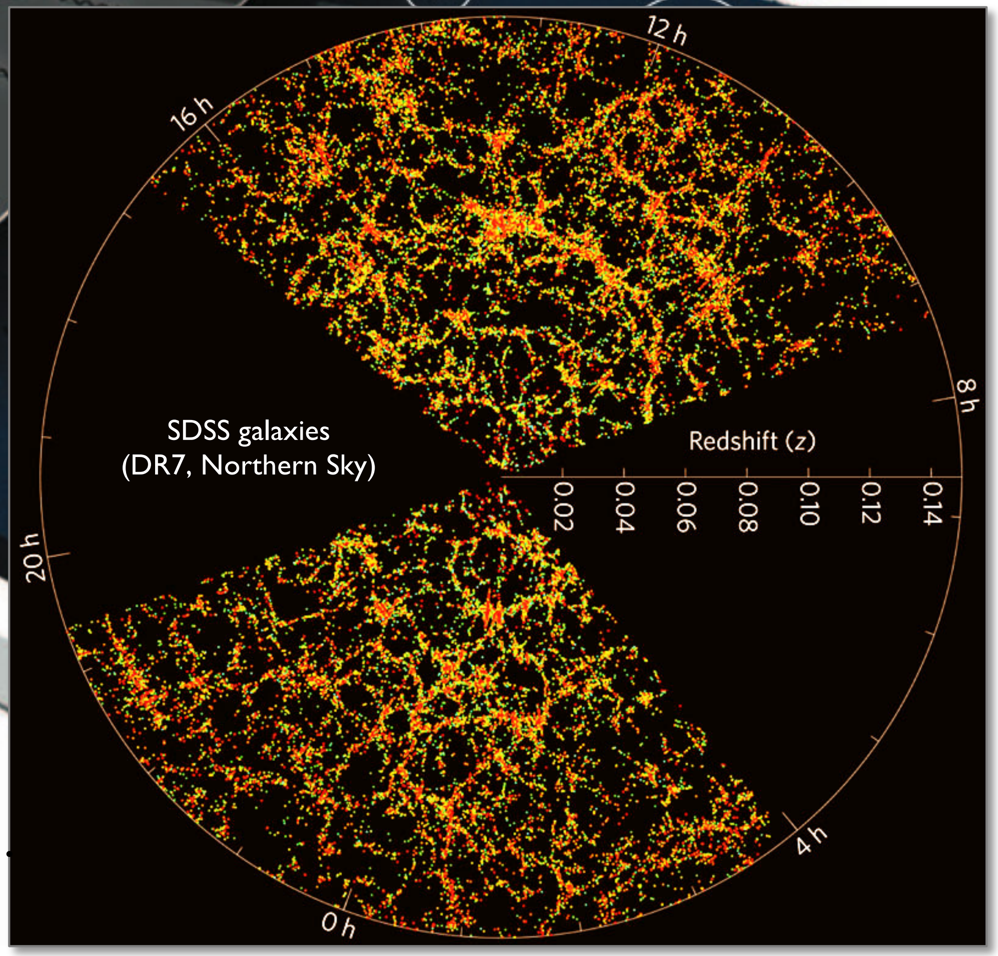


we know that there are **matter** perturbations…

Cosmological Structure Formation

what?



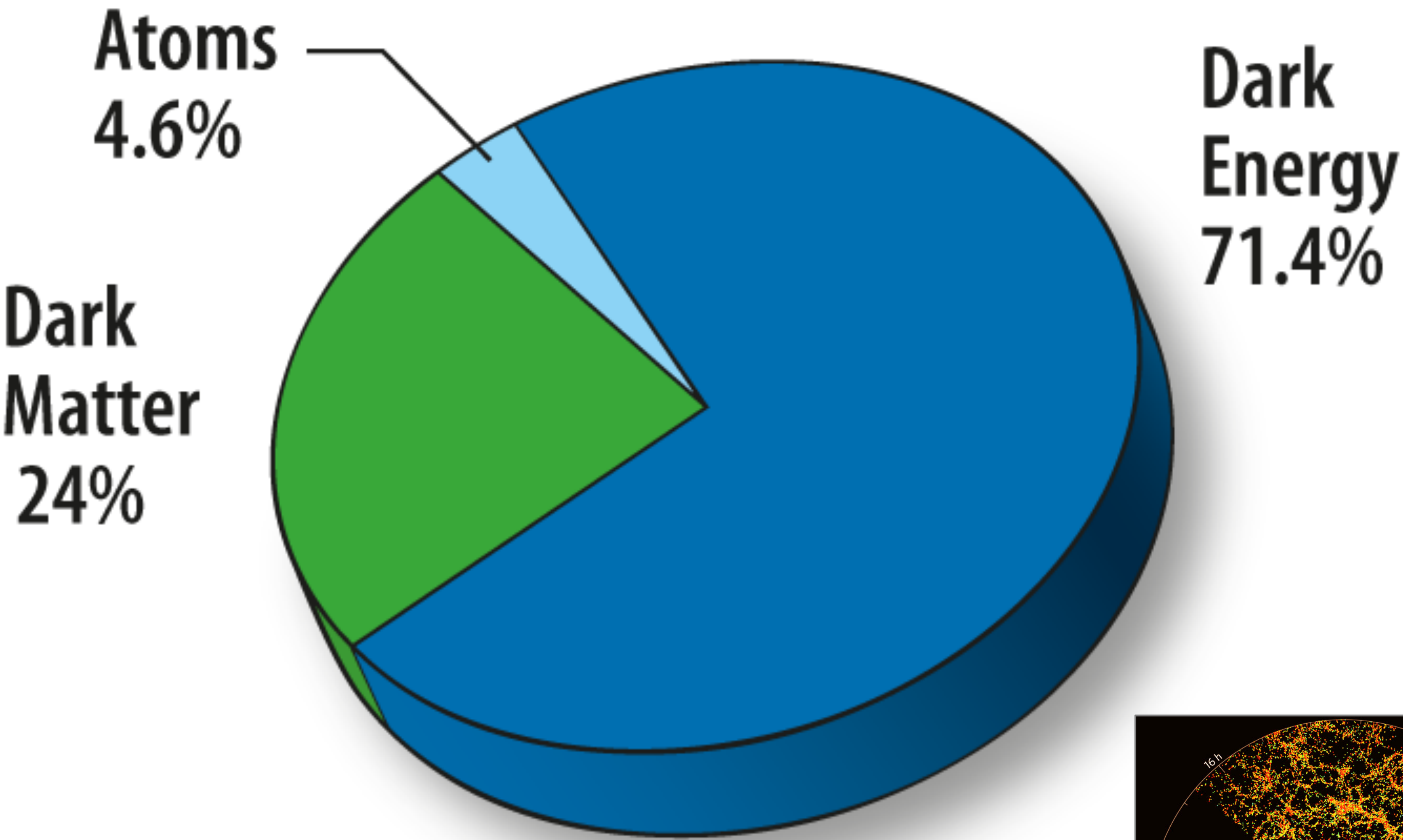


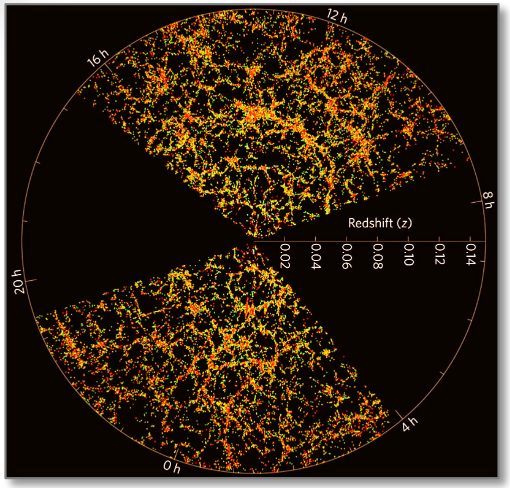
we know that there are **matter** perturbations…

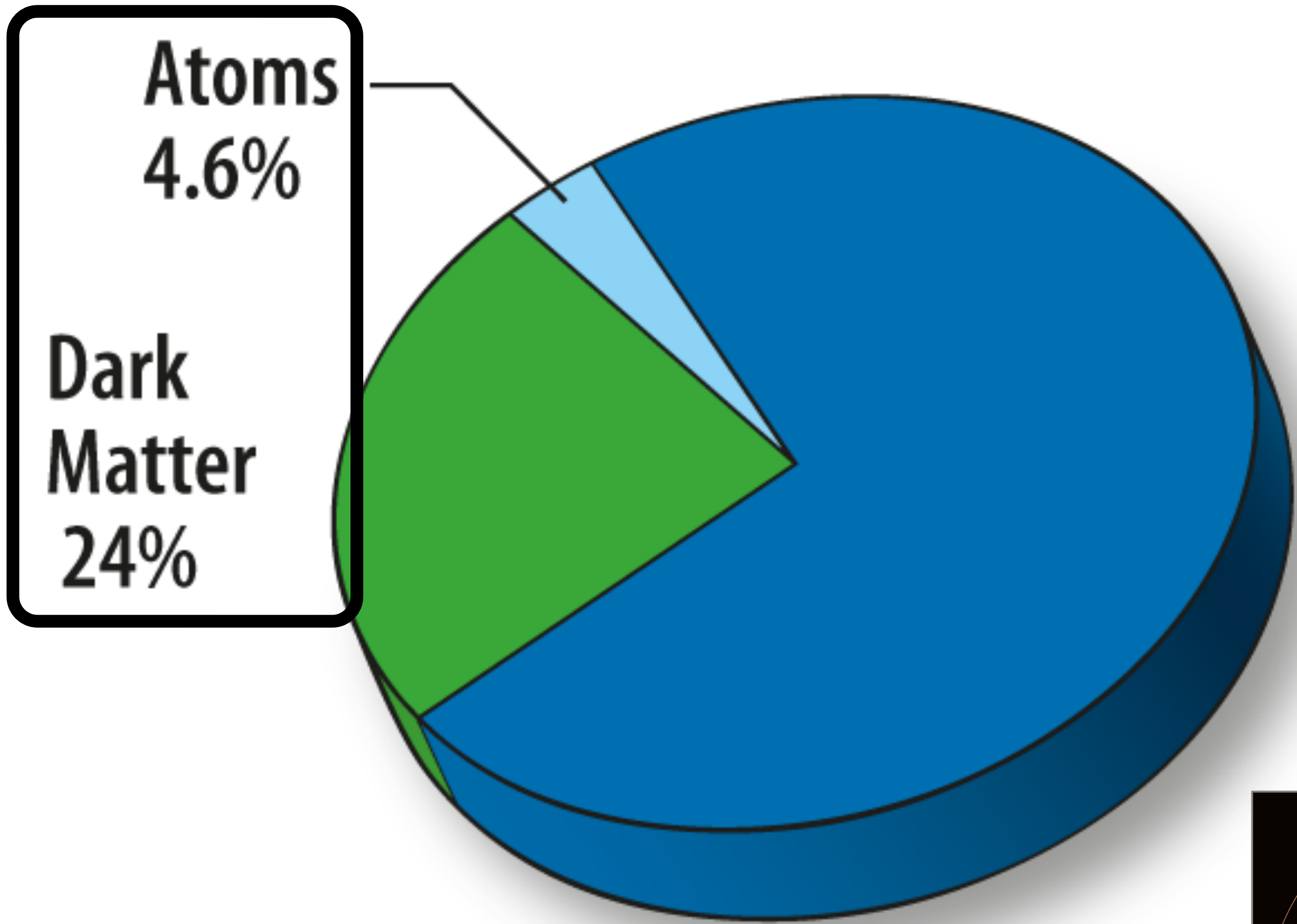
…but how do they grow?

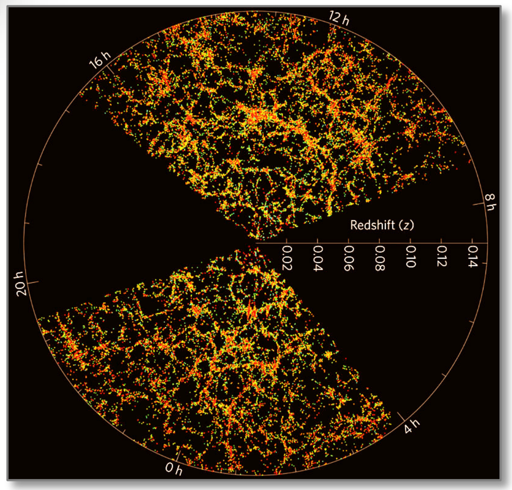
Cosmological Structure Formation

what?









throughout the whole lecture we only consider

matter perturbations well ***inside*** the Hubble radius!

- governing equations
- growth of matter perturbations
- statistics of perturbations
- non-linear structure formation

- **governing equations**
- growth of matter perturbations
- statistics of perturbations
- non-linear structure formation

the Universe is filled with a perfect fluid

$$T_{\mu\nu} = -pg^{\mu\nu} + (\rho c^2 + p)u^\mu u^\nu$$

the Universe is filled with a perfect fluid...

$$T_{\mu\nu} = -pg^{\mu\nu} + (\rho c^2 + p)u^\mu u^\nu$$

...which we treat non-relativisticly

- Poisson's equation

$$\Delta\Psi = 4\pi G(\rho + \frac{3p}{c^2})$$

- continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left((\rho + \frac{p}{c^2})\vec{v}\right) = 0$$

- conservation of momentum

$$(\rho + \frac{p}{c^2})\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -(\rho + \frac{p}{c^2})\nabla\Psi - \nabla p$$

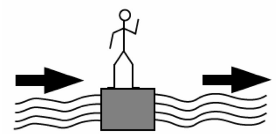
- equation of state

$$p = p(\rho, S)$$

two distinct approaches to following structure formation

- Eulerian viewpoint:

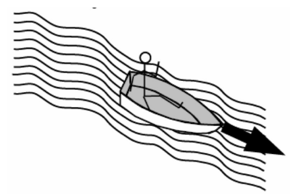
$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left((\rho + \frac{p}{c^2})\vec{v}\right) = 0$$



(we measure the flow…)

- Lagrange viewpoint:

$$\vec{x}(t) = \vec{q} + D(t)\vec{S}(\vec{q})$$

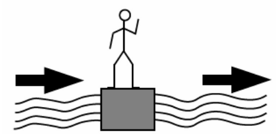


(we follow mass elements…)

two distinct approaches to following structure formation

• Eulerian viewpoint: → ***preferred approach for the time being...***

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left((\rho + \frac{p}{c^2})\vec{v} \right) = 0$$



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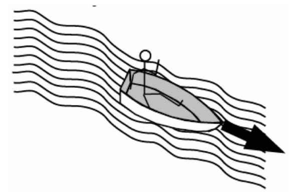

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cosmology!?

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we want to solve these equations for a **non-relativistic fluid** in…

…an **expanding Universe**, and characterized by

…**small perturbations** about a homogeneous and isotropic background

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Note:
- ∇p can nevertheless be large
- baryonic matter fulfills $p << \rho c^2$, too
- we further assume adiabatic perturbations: $\nabla p = c_s^2 \nabla\rho$

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- Poisson's equation

✓ non-relativistic fluid

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comoving coordinates!

- comoving coordinates

$$\vec{r} = a\vec{x}$$

x	comoving coordinate

- comoving coordinates

$$\vec{r} = a\vec{x}$$

$$\vec{v} = \vec{u} + \frac{\dot{a}}{a}\vec{r} \qquad \text{with} \qquad \vec{u} = a\dot{\vec{x}}$$

x	comoving coordinate
u	peculiar velocity field

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$$\Psi = \Phi - \frac{1}{2}a\ddot{a}x^2$$

x	comoving coordinate
u	peculiar velocity field
Φ	peculiar potential*

*we will see below that Φ is sourced by matter ***perturbations***

- comoving coordinates

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$$\nabla = \frac{1}{a}\nabla_x$$

$$\left.\frac{\partial}{\partial t}\right|_{\vec{r}} = \left.\frac{\partial}{\partial t}\right|_{\vec{x}} - \frac{\dot{a}}{a}(\vec{x}\cdot\nabla_x)$$ convective time derivative

x	comoving coordinate
u	peculiar velocity field
Φ	peculiar potential

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$f = f(\vec{r},t)$

$g = g(\vec{x},t)$

$df = dg$

1. $df = \left.\frac{\partial f(\vec{r},t)}{\partial \vec{r}}\right|_t \cdot \overrightarrow{dr} + \left.\frac{\partial f(\vec{r},t)}{\partial t}\right|_{\vec{r}} dt$

$\overrightarrow{dr} = a\overrightarrow{dx} + \vec{x}da \qquad = \nabla f(\vec{r},t)\cdot(a\overrightarrow{dx} + \vec{x}da) + \left.\frac{\partial f(\vec{r},t)}{\partial t}\right|_{\vec{r}} dt$

$da = \dot{a}dt \qquad = \left[a\nabla f(\vec{r},t)\right]\cdot\overrightarrow{dx} + \left[\dot{a}\vec{x}\cdot(\nabla f(\vec{r},t)) + \left.\frac{\partial f(\vec{r},t)}{\partial t}\right|_{\vec{r}}\right]dt$

2. $dg = \left.\frac{\partial g(\vec{x},t)}{\partial \vec{x}}\right|_t \cdot \overrightarrow{dx} + \left.\frac{\partial g(\vec{x},t)}{\partial t}\right|_{\vec{x}} dt$

$= \nabla_x g(\vec{x},t)\cdot\overrightarrow{dx} + \left.\frac{\partial g(\vec{x},t)}{\partial t}\right|_{\vec{x}} dt$

3. $df = dg$

$$\underline{\left[a\nabla f(\vec{r},t)\right]}\cdot\overrightarrow{dx} + \left[\dot{a}\vec{x}\cdot(\nabla f(\vec{r},t)) + \left.\frac{\partial f(\vec{r},t)}{\partial t}\right|_{\vec{r}}\right]dt = \underline{\nabla_x g(\vec{x},t)}\cdot\overrightarrow{dx} + \left.\frac{\partial g(\vec{x},t)}{\partial t}\right|_{\vec{x}} dt$$

✓ non-relativistic fluid

- Poisson's equation

$$\Delta\Psi = 4\pi G\rho$$

- continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\vec{v}\right) = 0$$

- conservation of momentum

$$\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla\Psi - \frac{\nabla p}{\rho}$$

- adiabatic perturbations

$$\nabla p = c_s^2\nabla\rho$$

we want to solve these equations in…

…an **expanding Universe**, and characterized by

transformation to comoving coordinates →

…**small perturbations** about a homogeneous and isotropic background

$$\vec{r} = a\vec{x}$$

$$\vec{v} = \vec{u} + \frac{\dot{a}}{a}\vec{r} \qquad \text{with} \qquad \vec{u} = a\dot{\vec{x}}$$

$$\Psi = \Phi - \frac{1}{2}a\ddot{a}x^2$$

$$\nabla = \frac{1}{a}\nabla_x$$

$$\left.\frac{\partial}{\partial t}\right|_{\vec{r}} = \left.\frac{\partial}{\partial t}\right|_{\vec{x}} - \frac{\dot{a}}{a}(\vec{x}\cdot\nabla_x)$$

- transformation to comoving coordinates in detail…
  - comoving Poisson equation $\Delta\Psi = 4\pi G\rho$

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- transformation to comoving coordinates in detail…
  - comoving Poisson equation $\Delta_x\Phi = 4\pi G a^2(\rho - \bar{\rho})$

<u>proof:</u>

$$\Psi = \Phi - \frac{1}{2}a\ddot{a}x^2 \qquad \Delta_x\Phi = \Delta_x(\Psi + \frac{1}{2}a\ddot{a}\,|\vec{x}\,|^2)$$

$$= \Delta_x\Psi + \frac{1}{2}a\ddot{a}\Delta_x\,|\vec{x}\,|^2$$

$$\Delta = \frac{1}{a^2}\Delta_x \qquad \Delta_x\,|\vec{x}\,|^2 = 6 \qquad = a^2 4\pi G\rho + \frac{1}{2}a\ddot{a}6$$

2nd Friedmann equation: $\ddot{a} = -\frac{4\pi G}{3}\bar{\rho}a \qquad = 4\pi G a^2\rho + 3a(-\frac{4\pi G}{3}\bar{\rho}a)$

$$= 4\pi G a^2(\rho - \bar{\rho}) \quad \textit{q.e.d.}$$

this is **not** the comoving density but rather the physical!

- transformation to comoving coordinates in detail…
  - comoving Poisson equation $\Delta_x \Phi = 4\pi G a^2(\rho - \overline{\rho})$

$$\vec{r} = a\vec{x}$$

$$\vec{v} = \vec{u} + \frac{\dot{a}}{a}\vec{r} \qquad \text{with} \quad \vec{u} = a\dot{\vec{x}}$$

$$\Psi = \Phi - \frac{1}{2} a\ddot{a}x^2$$

$$\nabla = \frac{1}{a}\nabla_x$$

$$\left.\frac{\partial}{\partial t}\right|_{\vec{r}} = \left.\frac{\partial}{\partial t}\right|_{\vec{x}} - \frac{\dot{a}}{a}(\vec{x}\cdot\nabla_x)$$

Notes:

- Friedmann equations are for the "background" $\overline{\rho}$
- the comoving potential Φ is responsible for the growth of perturbations
- there is no solution to Poisson's equation in infinite space unless the source function averages to zero
- inclusion of Λ-term will not change result (it would be compensated by the appearance in the 2nd Friedmann equation)

- transformation to comoving coordinates in detail…

  - comoving continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\vec{v}\right)=0$$

$$\vec{r}=a\vec{x}$$

$$\vec{v}=\vec{u}+\frac{\dot{a}}{a}\vec{r} \quad \text{with} \quad \vec{u}=a\dot{\vec{x}}$$

$$\Psi=\Phi-\frac{1}{2}a\ddot{a}x^2$$

$$\nabla=\frac{1}{a}\nabla_x$$

$$\left.\frac{\partial}{\partial t}\right|_{\vec{r}}=\left.\frac{\partial}{\partial t}\right|_{\vec{x}}-\frac{\dot{a}}{a}(\vec{x}\cdot\nabla_x)$$

$$\vec{r} = a\vec{x}$$

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- transformation to comoving coordinates in detail…
  - comoving continuity equation $\frac{\partial \rho}{\partial t} + \frac{1}{a}\nabla_x \cdot (\rho\vec{u}) + 3\frac{\dot{a}}{a}\rho = 0$

---

proof:

$$\underbrace{\frac{\partial \rho}{\partial t}}_{A} + \underbrace{\nabla \cdot (\rho\vec{v})}_{B} = 0$$

A

$$\left.\frac{\partial \rho(\vec{r},t)}{\partial t}\right|_{\vec{r}} = \left.\frac{\partial \rho(\vec{x},t)}{\partial t}\right|_{\vec{x}} - \frac{\dot{a}}{a}\vec{x}\cdot[\nabla_x \rho(\vec{x},t)]$$

B

$$\vec{v} = \vec{u} + \frac{\dot{a}}{a}\vec{r} \quad \text{with} \quad \vec{u} = a\dot{\vec{x}}$$

$$\begin{aligned}
\nabla \cdot (\rho\vec{v}) &= \rho[\nabla \cdot \vec{v}] + \vec{v}[\nabla\rho] \\
&= \rho[\nabla \cdot (\frac{\dot{a}}{a}\vec{r} + \vec{u})] + (\frac{\dot{a}}{a}\vec{r} + \vec{u})\cdot[\nabla\rho] \\
&= \rho[\frac{\dot{a}}{a}3 + \nabla\cdot\vec{u}] + \frac{\dot{a}}{a}\vec{r}\cdot\nabla\rho + \vec{u}\cdot\nabla\rho \\
&= \frac{\dot{a}}{a}3\rho + \frac{1}{a}\rho\nabla_x\cdot\vec{u} + \frac{\dot{a}}{a}\vec{x}\nabla_x\rho + \frac{1}{a}\vec{u}\cdot\nabla_x\rho \\
&= \frac{\dot{a}}{a}(3\rho + \vec{x}\cdot\nabla_x\rho) + \frac{1}{a}\nabla_x\cdot(\rho\vec{u})
\end{aligned}$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{a}\nabla_x \cdot (\rho\vec{u}) + 3\frac{\dot{a}}{a}\rho = 0$$

q.e.d.

$$\vec{r} = a\vec{x}$$

$$\vec{v} = \vec{u} + \frac{\dot{a}}{a}\vec{r} \quad \text{with} \quad \vec{u} = a\dot{\vec{x}}$$

$$\Psi = \Phi - \frac{1}{2} a\ddot{a}x^2$$

$$\nabla = \frac{1}{a}\nabla_x$$

$$\left.\frac{\partial}{\partial t}\right|_{\vec{r}} = \left.\frac{\partial}{\partial t}\right|_{\vec{x}} - \frac{\dot{a}}{a}(\vec{x}\cdot\nabla_x)$$

- transformation to comoving coordinates in detail…
  - comoving continuity equation $$\frac{\partial\rho}{\partial t} + \frac{1}{a}\nabla_x\cdot(\rho\vec{u}) + 3\underline{\underline{\frac{\dot{a}}{a}\rho}} = 0$$

Notes:

- it contains an additional drag term due to the cosmic expansion

- transformation to comoving coordinates in detail…
  - comoving conservation of momentum $$\left(\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right)=-\nabla\Psi-\frac{\nabla p}{\rho}$$

- transformation to comoving coordinates in detail…
  - comoving conservation of momentum $$\frac{\partial \vec{u}}{\partial t}+\frac{1}{a}(\vec{u}\cdot\nabla_x)\vec{u}+\frac{\dot{a}}{a}\vec{u}=-\frac{1}{a}\nabla_x\Phi-\frac{1}{a}\frac{\nabla_x p}{\rho}$$

<u>proof:</u> $$\underbrace{\frac{\partial \vec{v}}{\partial t}}_{\text{A}}+\underbrace{(\vec{v}\cdot\nabla)\vec{v}}_{\text{B}}=-\underbrace{\nabla\Psi}_{\text{C}}-\underbrace{\frac{\nabla p}{\rho}}_{\text{D}}$$

A

$$\frac{\partial \vec{v}}{\partial t}=\frac{\ddot{a}}{a}\vec{r}-\frac{\dot{a}^2}{a^2}\vec{r}+\left.\frac{\partial \vec{u}}{\partial t}\right|_{\vec{x}}-\frac{\dot{a}}{a}(\vec{x}\cdot\nabla_x)\vec{u}$$
$$=\ddot{a}\vec{x}-\frac{\dot{a}^2}{a}\vec{x}+\left.\frac{\partial \vec{u}}{\partial t}\right|_{\vec{x}}-\frac{\dot{a}}{a}(\vec{x}\cdot\nabla_x)\vec{u}$$

B

$$(\vec{v}\cdot\nabla)\vec{v}=[(\frac{\dot{a}}{a}\vec{r}+\vec{u})\cdot\nabla](\frac{\dot{a}}{a}\vec{r}+\vec{u})$$
$$=(\frac{\dot{a}}{a}\vec{r}\cdot\nabla+\vec{u}\cdot\nabla)(\frac{\dot{a}}{a}\vec{r}+\vec{u})$$
$$=\frac{\dot{a}}{a}(\vec{r}\cdot\nabla)(\frac{\dot{a}}{a}\vec{r}+\vec{u})+(\vec{u}\cdot\nabla)(\frac{\dot{a}}{a}\vec{r}+\vec{u})$$
$$=\frac{\dot{a}^2}{a^2}\vec{r}+\frac{\dot{a}}{a}(\vec{r}\cdot\nabla)\vec{u}+\frac{\dot{a}}{a}(\vec{u}\cdot\nabla)\vec{r}+(\vec{u}\cdot\nabla)\vec{u}$$
$$=\frac{\dot{a}^2}{a^2}\vec{r}+\frac{\dot{a}}{a}(\vec{r}\cdot\nabla)\vec{u}+\frac{\dot{a}}{a}\vec{u}+(\vec{u}\cdot\nabla)\vec{u}$$
$$=\frac{\dot{a}^2}{a}\vec{x}+\frac{\dot{a}}{a}(\vec{x}\cdot\nabla_x)\vec{u}+\frac{\dot{a}}{a}\vec{u}+(\vec{u}\cdot\nabla)\vec{u}$$

$(\vec{r}\cdot\nabla)\vec{r}=\vec{r}$

$(\vec{u}\cdot\nabla)\vec{r}=\vec{u}$

C

$$\nabla\Psi=\frac{1}{a}\nabla_x(\Phi-\frac{1}{2}a\ddot{a}\vec{x}^2)$$
$$=\frac{1}{a}\nabla_x\Phi-\frac{1}{2}\ddot{a}\nabla_x\vec{x}^2$$
$$=\frac{1}{a}\nabla_x\Phi-\ddot{a}\vec{x}$$

D

$$\nabla p=\frac{1}{a}\nabla_x p$$

A+B=-C-D

$\Rightarrow$ *q.e.d.*

- transformation to comoving coordinates in detail…
  - comoving conservation of momentum $\frac{\partial\vec{u}}{\partial t}+\frac{1}{a}(\vec{u}\cdot\nabla_x)\vec{u}+\underline{\underline{\frac{\dot{a}}{a}\vec{u}}}=-\frac{1}{a}\nabla_x\Phi-\frac{1}{a}\frac{\nabla_x p}{\rho}$

Notes:

- it also contains an additional drag term due to the cosmic expansion

- Poisson's equation

✓ non-relativistic fluid
✓ comoving coordinates

$$\Delta_x \Phi = 4\pi G a^2 (\rho - \overline{\rho})$$

- continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{a}\nabla_x \cdot (\rho \vec{u}) + 3\frac{\dot{a}}{a}\rho = 0$$

- conservation of momentum

$$\frac{\partial \vec{u}}{\partial t} + \frac{1}{a}(\vec{u}\cdot\nabla_x)\vec{u} + \frac{\dot{a}}{a}\vec{u} = -\frac{1}{a}\nabla_x \Phi - \frac{1}{a}\frac{\nabla_x p}{\rho}$$

- adiabatic perturbations

$$\nabla_x p = c_s^2 \nabla_x \rho$$

we want to solve these equations for…

…**small perturbations** about a homogeneous and isotropic background

✓ non-relativistic fluid
✓ comoving coordinates

- Poisson's equation

$$\Delta_x \Phi = 4\pi G a^2 \boxed{(\rho - \bar{\rho})}$$

perturbations are the source

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density contrast: $\delta = \dfrac{\rho - \overline{\rho}}{\overline{\rho}}$

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- adiabatic perturbations

$$\nabla_x p = c_s^2 \nabla_x \rho$$

we want to solve these equations for…

$$\rho = \bar{\rho}(1+\delta), \quad \delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}$$

…**small perturbations** about a homogeneous and isotropic background

(and dropping subscript x)

- introducing density contrast in detail…
  - Poisson's equation $\Delta_x \Phi = 4\pi G a^2 (\rho - \overline{\rho})$

- introducing density contrast in detail…
  - Poisson's equation $\Delta\Phi = 4\pi G a^2 \overline{\rho}\delta$

---

proof:

$$\Delta_x \Phi = 4\pi G a^2 (\rho - \overline{\rho})$$

$\rho - \overline{\rho} = \overline{\rho}\delta$

$$= 4\pi G a^2 \overline{\rho}\delta$$

- introducing density contrast in detail…

  - Poisson's equation $$\Delta\Phi = 4\pi G a^2 \overline{\rho}\delta$$

    we also drop the subscript x from now on...

---

proof:

$$\Delta_x\Phi = 4\pi G a^2 (\rho - \overline{\rho})$$

$\rho - \overline{\rho} = \overline{\rho}\delta$ $$= 4\pi G a^2 \overline{\rho}\delta$$

- introducing density contrast in detail…
  - continuity equation $$\frac{\partial\rho}{\partial t} + \frac{1}{a}\nabla_x \cdot (\rho\vec{u}) + 3\frac{\dot{a}}{a}\rho = 0$$

- introducing density contrast in detail…
  - continuity equation $$\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla\cdot[(1+\delta)\vec{u}] = 0$$

<u>proof:</u>

$$\underbrace{\frac{\partial\rho}{\partial t}}_{\text{A}} + \underbrace{\frac{1}{a}\nabla_x\cdot(\rho\vec{u})}_{\text{B}} + \underbrace{3\frac{\dot{a}}{a}\rho}_{\text{C}} = 0$$

A

$$\frac{\partial\rho}{\partial t} = \frac{\partial(\bar{\rho}(1+\delta))}{\partial t}$$
$$= \dot{\bar{\rho}}(1+\delta) + \bar{\rho}\dot{\delta}$$
$$\bar{\rho}a^3 = const. \qquad = -3\frac{\dot{a}}{a}\bar{\rho}(1+\delta) + \bar{\rho}\dot{\delta}$$

we are interested in matter!
(cf. FRW lecture)

B

$$\frac{1}{a}\nabla\cdot(\rho\vec{u}) = \frac{1}{a}\bar{\rho}\nabla\cdot[(1+\delta)\vec{u}]$$

C

$$3\frac{\dot{a}}{a}\rho = 3\frac{\dot{a}}{a}\bar{\rho}(1+\delta)$$

- introducing density contrast in detail…
  - conservation of momentum $$\frac{\partial \vec{u}}{\partial t} + \frac{1}{a}(\vec{u} \cdot \nabla_x)\vec{u} + \frac{\dot{a}}{a}\vec{u} = -\frac{1}{a}\nabla_x \Phi - \frac{1}{a}\frac{\nabla_x p}{\rho}$$

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  - conservation of momentum $$\frac{\partial\vec{u}}{\partial t}+\frac{1}{a}(\vec{u}\cdot\nabla)\vec{u}+\frac{\dot{a}}{a}\vec{u}=-\frac{1}{a}\nabla\Phi-\frac{c_s^2}{a}\frac{\nabla\delta}{1+\delta}$$

<u>proof:</u>

$$\frac{\partial\vec{u}}{\partial t}+\frac{1}{a}(\vec{u}\cdot\nabla_x)\vec{u}+\frac{\dot{a}}{a}\vec{u}=-\frac{1}{a}\nabla_x\Phi-\underbrace{\frac{1}{a}\frac{\nabla_x p}{\rho}}_{\text{A}}$$

A

$$\frac{\nabla p}{\rho}=\frac{c_s^2\nabla\rho}{\rho}=\frac{c_s^2\nabla\left(\overline{\rho}(1+\delta)\right)}{\rho}=\frac{c_s^2\left[(1+\delta)\nabla\overline{\rho}+\overline{\rho}\nabla(1+\delta)\right]}{\rho}=\frac{c_s^2\overline{\rho}\nabla(1+\delta)}{\rho}=\frac{c_s^2\overline{\rho}\nabla\delta}{\rho}=\frac{c_s^2\nabla\delta}{1+\delta}$$

adiabatic perturbations

$\nabla p = c_s^2\ \nabla\rho$

- Poisson's equation

$$\Delta\Phi = 4\pi G a^2 \overline{\rho}\delta$$

✓ non-relativistic fluid
✓ comoving coordinates
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we want to solve these equations for…

linearization $\delta << 1\,,\ \frac{(\vec{u}\cdot\nabla)}{a} << H = \frac{\dot{a}}{a}$

…**small perturbations** about a homogeneous and isotropic background

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$$\nabla p = c_s^2\,\nabla\rho$$

- Poisson's equation – **careful** (multiple components)

$$\Delta\Phi = 4\pi G a^2 \bar{\rho}_{tot} \left(\frac{\bar{\rho}}{\bar{\rho}_{tot}} \delta + \frac{\bar{\rho}_X}{\bar{\rho}_{tot}} \delta_X + \frac{\bar{\rho}_Y}{\bar{\rho}_{tot}} \delta_Y + \cdots \right)$$

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$$\nabla p = c_s^2 \nabla \rho$$

these equations remain *individually* for the decoupled component of interest!

the coupling between different component is primarily via gravity

- Poisson's equation – **careful** (multiple components)

$$\Delta\Phi = 4\pi G a^2 \bar{\rho}_{tot} \left(\frac{\bar{\rho}}{\bar{\rho}_{tot}} \delta + \frac{\bar{\rho}_X}{\bar{\rho}_{tot}} \delta_X + \frac{\bar{\rho}_Y}{\bar{\rho}_{tot}} \delta_Y + \cdots \right)$$

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- adiabatic perturbations

$$\nabla p = c_s^2\,\nabla\rho$$

these equations remain for the decoupled component of interest!

the only quantity common to all possible components is the grav. potential!

decoupled matter well inside the horizon

- Poisson's equation

$$\Delta\Phi = 4\pi G a^2 \overline{\rho}\delta$$

- continuity equation

$$\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla \cdot \vec{u} = 0$$

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combine and eliminate u, ∇p, and Φ →

- combination in detail…

continuity equation: $$0 = \frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot \vec{u}$$

momentum equation: $$0 = \frac{\partial \vec{u}}{\partial t} + \frac{\dot{a}}{a} \vec{u} + \frac{1}{a} \nabla \Phi + \frac{c_s^2}{a} \nabla \delta$$

- combination in detail…

$$\left.\begin{aligned} A = 0 &= \frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla\cdot\vec{u} \\ B = 0 &= \frac{\partial \vec{u}}{\partial t} + \frac{\dot{a}}{a}\vec{u} + \frac{1}{a}\nabla\Phi + \frac{c_s^2}{a}\nabla\delta \end{aligned}\right\} \quad 0 = \frac{\partial A}{\partial t} - \frac{1}{a}\nabla\cdot B$$

- combination in detail…

$$\left.\begin{aligned} A = 0 &= \frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla\cdot\vec{u} \\ B = 0 &= \frac{\partial\vec{u}}{\partial t} + \frac{\dot{a}}{a}\vec{u} + \frac{1}{a}\nabla\Phi + \frac{c_s^2}{a}\nabla\delta \end{aligned}\right\} \quad 0 = \frac{\partial A}{\partial t} - \frac{1}{a}\nabla\cdot B$$

$$\begin{aligned} \frac{\partial A}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla\cdot\vec{u}\right) \\ &= \frac{\partial^2\delta}{\partial t^2} - \frac{\dot{a}}{a^2}\nabla\cdot\vec{u} + \frac{1}{a}\frac{\partial}{\partial t}\nabla\cdot\vec{u} \\ &= \frac{\partial^2\delta}{\partial t^2} + \frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} + \frac{1}{a}\frac{\partial}{\partial t}\nabla\cdot\vec{u} \end{aligned}$$

continuity equation

$$\begin{aligned} \frac{1}{a}\nabla\cdot B &= \frac{1}{a}\nabla\cdot\left(\frac{\partial\vec{u}}{\partial t} + \frac{\dot{a}}{a}\vec{u} + \frac{1}{a}\nabla\Phi + \frac{c_s^2}{a}\nabla\delta\right) \\ &= \frac{1}{a}\left(\frac{\partial}{\partial t}\nabla\cdot\vec{u} + \frac{\dot{a}}{a}\nabla\cdot\vec{u} + \frac{1}{a}\Delta\Phi + \frac{c_s^2}{a}\Delta\delta\right) \\ &= \frac{1}{a}\left(\frac{\partial}{\partial t}\nabla\cdot\vec{u} - \dot{a}\frac{\partial\delta}{\partial t} + \frac{1}{a}\Delta\Phi + \frac{c_s^2}{a}\Delta\delta\right) \end{aligned}$$

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$$\left.\begin{aligned} A = 0 &= \frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla\cdot\vec{u} \\ B = 0 &= \frac{\partial\vec{u}}{\partial t} + \frac{\dot{a}}{a}\vec{u} + \frac{1}{a}\nabla\Phi + \frac{c_s^2}{a}\nabla\delta \end{aligned}\right\} \quad 0 = \frac{\partial A}{\partial t} - \frac{1}{a}\nabla\cdot B \Longrightarrow$$

$$\begin{aligned} \frac{\partial A}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla\cdot\vec{u}\right) \\ &= \frac{\partial^2\delta}{\partial t^2} - \frac{\dot{a}}{a^2}\nabla\cdot\vec{u} + \frac{1}{a}\frac{\partial}{\partial t}\nabla\cdot\vec{u} \\ &= \frac{\partial^2\delta}{\partial t^2} + \frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} + \frac{1}{a}\frac{\partial}{\partial t}\nabla\cdot\vec{u} \end{aligned}$$

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- evolution of density contrast $\delta(x, t)$

$$\boxed{\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} - 4\pi G\bar{\rho}\delta - \frac{c_s^2}{a^2}\Delta\delta = 0}$$

✓ non-relativistic fluid
✓ comoving coordinates
✓ small density contrast

- valid for arbitrary cosmologies
- valid for collisionless ($c_s = 0$) and collisional matter (c_s != 0)
- cosmological expansion acts as damping term

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→ *describes* ***matter perturbations*** well ***inside*** the Hubble radius!

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- valid for arbitrary cosmologies
- valid for collisionless ($c_s = 0$) and collisional matter ($c_s \mathrel{!}= 0$)
- cosmological expansion acts as damping term

→ *describes **matter perturbations*** well ***inside*** the Hubble radius!

→ *additional components enter only into* 4πG-term!*

*this equation describes the evolution of the perturbations of a single, specific component which nevertheless could be coupled gravitationally to other components...

- governing equations
- **growth of matter perturbations**
- statistics of perturbations
- non-linear structure formation

- evolution of density contrast $\delta(x, t)$

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - 4\pi G\bar{\rho}\delta - \frac{c_s^2}{a^2}\Delta\delta = 0$$

  - valid for arbitrary cosmologies
  - valid for collisionless ($c_s = 0$) and collisional matter (c_s != 0)
  - cosmological expansion acts as damping term

- Ansatz for solution $\delta(x, t)$: decomposition* into waves

$$\delta(\vec{x},t) = \sum_{\vec{k}} \delta_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

*we are dealing with a linear equation...

- evolution of density contrast $\delta(x, t)$

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - 4\pi G\bar{\rho}\delta - \frac{c_s^2}{a^2}\Delta\delta = 0$$

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  - valid for collisionless ($c_s = 0$) and collisional matter (c_s != 0)
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- Ansatz for solution $\delta(x, t)$: single wave

$$\delta\left(\vec{x},t\right) = \delta_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

- evolution of density contrast $\delta(x, t)$

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - 4\pi G\bar{\rho}\delta - \frac{c_s^2}{a^2}\Delta\delta = 0$$

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- Ansatz for solution $\delta(x, t)$: single wave

$$\delta\left(\vec{x},t\right) = \delta_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

$$\xrightarrow{\Delta\delta = -k^2\delta}$$

$$\delta(\vec{x},t) = \delta_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

- evolution of density contrast $\delta_k(t)$

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_k}{\partial t} + \left(\frac{c_s^2}{a^2}k^2 - 4\pi G\bar{\rho}\right)\delta_k = 0$$

- valid for arbitrary cosmologies
- valid for collisionless ($c_s = 0$) and collisional matter (c_s != 0)
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- valid for arbitrary cosmologies
- valid for collisionless ($c_s = 0$) and collisional matter ($c_s \mathrel{!=} 0$)
- cosmological expansion acts as damping term

$$m\frac{d^2x(t)}{dt^2} + c\frac{dx(t)}{dt} + kx(t) = 0$$

damped harmonic oscillator

$$\delta(\vec{x},t) = \delta_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

- evolution of density contrast $\delta_k(t)$

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_k}{\partial t} + \left(\frac{c_s^2}{a^2}\underline{\underline{k^2}} - 4\pi G\bar{\rho}\right)\delta_k = 0$$

- valid for arbitrary cosmologies
- valid for collisionless ($c_s = 0$) and collisional matter (c_s != 0)
- cosmological expansion acts as damping term

we need to solve this
for every wave (as characterized by its individual k) separately...

$$\delta\left(\vec{x},t\right) = \delta_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

- evolution of density contrast $\delta_k(t)$

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_k}{\partial t} + \left(\frac{c_s^2}{a^2}k^2 - 4\pi G\bar{\rho}\right)\delta_k = 0$$

- valid for arbitrary cosmologies
- valid for collisionless ($c_s = 0$) and collisional matter (c_s != 0)
- cosmological expansion acts as damping term
- '()' reflects balance between pressure support* and gravity

*note, we allowed for baryonic matter/pressure *gradients*…

$$\delta(\vec{x},t) = \delta_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

- evolution of density contrast $\delta_k(t)$

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_k}{\partial t} + \left(\frac{c_s^2}{a^2}k^2 - 4\pi G\bar{\rho}\right)\delta_k = 0$$

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- '()' reflects balance between pressure support and gravity

$\frac{k^2}{a^2} < \frac{4\pi G\bar{\rho}}{c_s^2}$ => '()' < 0 => gravitational collapse

$\frac{k^2}{a^2} > \frac{4\pi G\bar{\rho}}{c_s^2}$ => '()' > 0 => oscillations (w/ decreasing amplitude due to damping term)

$$\delta(\vec{x},t) = \delta_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

- evolution of density contrast $\delta_k(t)$

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_k}{\partial t} + \left(\frac{c_s^2}{a^2}k^2 - 4\pi G\bar{\rho}\right)\delta_k = 0$$

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- cosmological expansion acts as damping term
- '()' reflects balance between pressure support and gravity

$\frac{k^2}{a^2} < \frac{4\pi G\bar{\rho}}{c_s^2}$ => '()' < 0 => gravitational collapse

$\frac{k^2}{a^2} > \frac{4\pi G\bar{\rho}}{c_s^2}$ => '()' > 0 => oscillations (w/ decreasing amplitude due to damping term)

- Jeans limits: $\lambda_J = c_s\sqrt{\frac{\pi}{G\bar{\rho}}}$ $\quad M_{J,w} = \frac{4\pi}{3}\left(\frac{\lambda_J}{2}\right)^3 \bar{\rho}_w$

$k = \frac{2\pi a}{\lambda}$

$$\delta\left(\vec{x},t\right) = \delta_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

- evolution of density contrast $\delta_k(t)$

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_k}{\partial t} + \left(\frac{c_s^2}{a^2}k^2 - 4\pi G\bar{\rho}\right)\delta_k = 0$$

- valid for arbitrary cosmologies
- valid for collisionless ($c_s = 0$) and collisional matter (c_s != 0)
- cosmological expansion acts as damping term
- '()' reflects balance between pressure support and gravity

$\frac{k^2}{a^2} < \frac{4\pi G\bar{\rho}}{c_s^2}$ => '()' < 0 => gravitational collapse

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Jeans *length* depends on all gravitating components

Jeans *mass* defined for certain component

$$\delta(\vec{x},t) = \delta_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

- evolution of density contrast $\delta_k(t)$

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_k}{\partial t} + \left(\frac{c_s^2}{a^2}k^2 - 4\pi G\bar{\rho}\right)\delta_k = 0$$

• valid for arbitrary cosmologies

• valid for colli…

…ological expansion acts as damping term

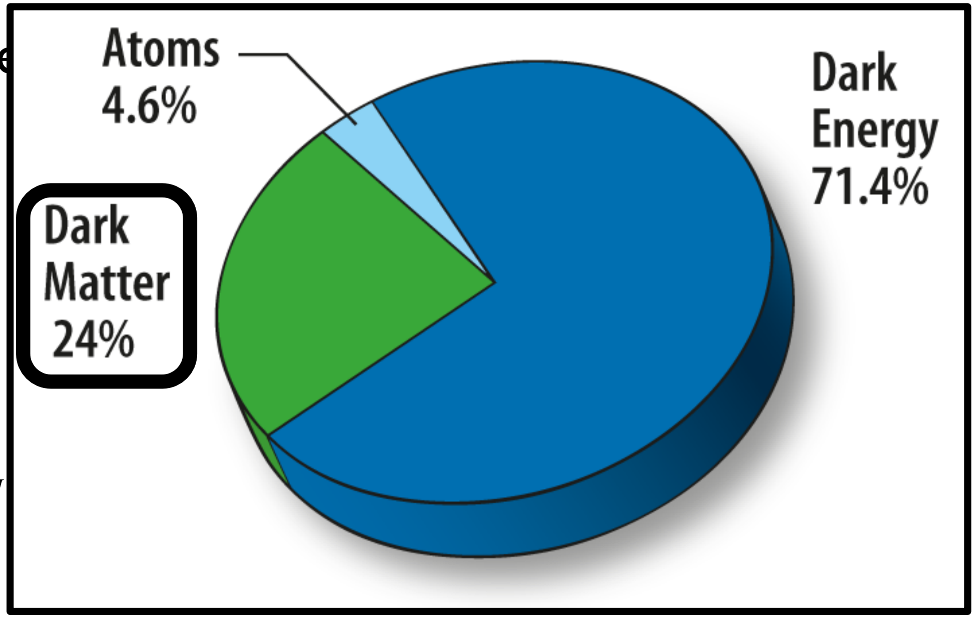
dominant non-relativistic component is dark matter (i.e. $c_s = 0$)

• '()' reflects balance betwe…

$$\frac{k^2}{a^2} < \frac{4\pi G\bar{\rho}}{c_s^2} \quad => \quad \text{'()'} < 0$$

$$\frac{k^2}{a^2} > \frac{4\pi G\bar{\rho}}{c_s^2} \quad => \quad \text{'()'} > 0$$

- Jeans limits: $\lambda_J = c_s\sqrt{\frac{\pi}{G\bar{\rho}}}$ $\quad M_J$



- evolution of density contrast $\delta(t)$ for *dark matter*

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} - 4\pi G\bar{\rho}\delta = 0$$

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$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} - 4\pi G\bar{\rho}\delta = 0$$

'–' ≙ no oscillations!

- evolution of density contrast $\delta(t)$ for *dark matter*

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - 4\pi G\bar{\rho}\delta = 0$$

other formulations possible

- evolution of density contrast $\delta(t)$ for *dark matter*

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - 4\pi G\overline{\rho}\delta = 0$$

$$\downarrow \quad 4\pi G\overline{\rho} = 4\pi G\Omega_m\rho_{crit} = 4\pi G\Omega_m \frac{3H^2}{8\pi G} = \frac{3}{2}\Omega_m H^2$$

$$\frac{\partial^2\delta}{\partial t^2} + 2H\frac{\partial\delta}{\partial t} - \frac{3}{2}\Omega_m H^2\delta = 0$$

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$$\Big\downarrow \; t = t(a)$$

$$a^2\frac{\partial^2 \delta}{\partial a^2} + a(2-q)\frac{\partial \delta}{\partial a} - \frac{3}{2}\Omega_m\delta = 0 \qquad q = -\frac{\ddot{a}a}{\dot{a}^2}$$

other formulations possible

- evolution of density contrast $\delta(t)$ for *dark matter*

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other formulations possible:

the choice is yours....

- evolution of density contrast $\delta(t)$ for *dark matter*

$$\frac{\partial^2\delta}{\partial t^2}+2\frac{\dot a}{a}\frac{\partial\delta}{\partial t}-4\pi G\overline{\rho}\delta=0$$

solutions:

$$\delta(a)=\frac{5}{2}\Omega_{m,0}\frac{H}{H_0}\int_0^a\frac{1}{\left(\mathrm{a}\frac{H}{H_0}\right)^3}d\mathrm{a}$$ exact solution

$$\delta(a)\approx\frac{5a}{2}\Omega_m(a)\left[\Omega_m^{4/7}(a)-\Omega_\Lambda(a)+\left(1+\frac{\Omega_m(a)}{2}\right)\left(1+\frac{\Omega_\Lambda(a)}{70}\right)\right]^{-1}$$ approx. solution

- evolution of density contrast $\delta(t)$ for *dark matter*

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - 4\pi G\overline{\rho}\delta = 0$$

<u>solutions:</u>

$$\delta(a) = \frac{5}{2}\Omega_{m,0}\frac{H}{H_0}\int_0^a \frac{1}{\left(\mathrm{a}\frac{H}{H_0}\right)^3}d\mathrm{a}$$ exact solution

$$\delta(a) \approx \frac{5a}{2}\Omega_m(a)\left[\Omega_m^{4/7}(a) - \Omega_\Lambda(a) + \left(1+\frac{\Omega_m(a)}{2}\right)\left(1+\frac{\Omega_\Lambda(a)}{70}\right)\right]^{-1}$$ approx. solution

- there are various ways to characterize/quantify the growth of perturbations, e.g.
  - growth **factor** $g = \delta/\mathrm{a}$
  - logarithmic growth **rate** $f = \mathrm{d}\ln\delta/\mathrm{d}\ln a$

- evolution of density contrast $\delta(t)$ for *dark matter*

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - 4\pi G\bar{\rho}\delta = 0$$

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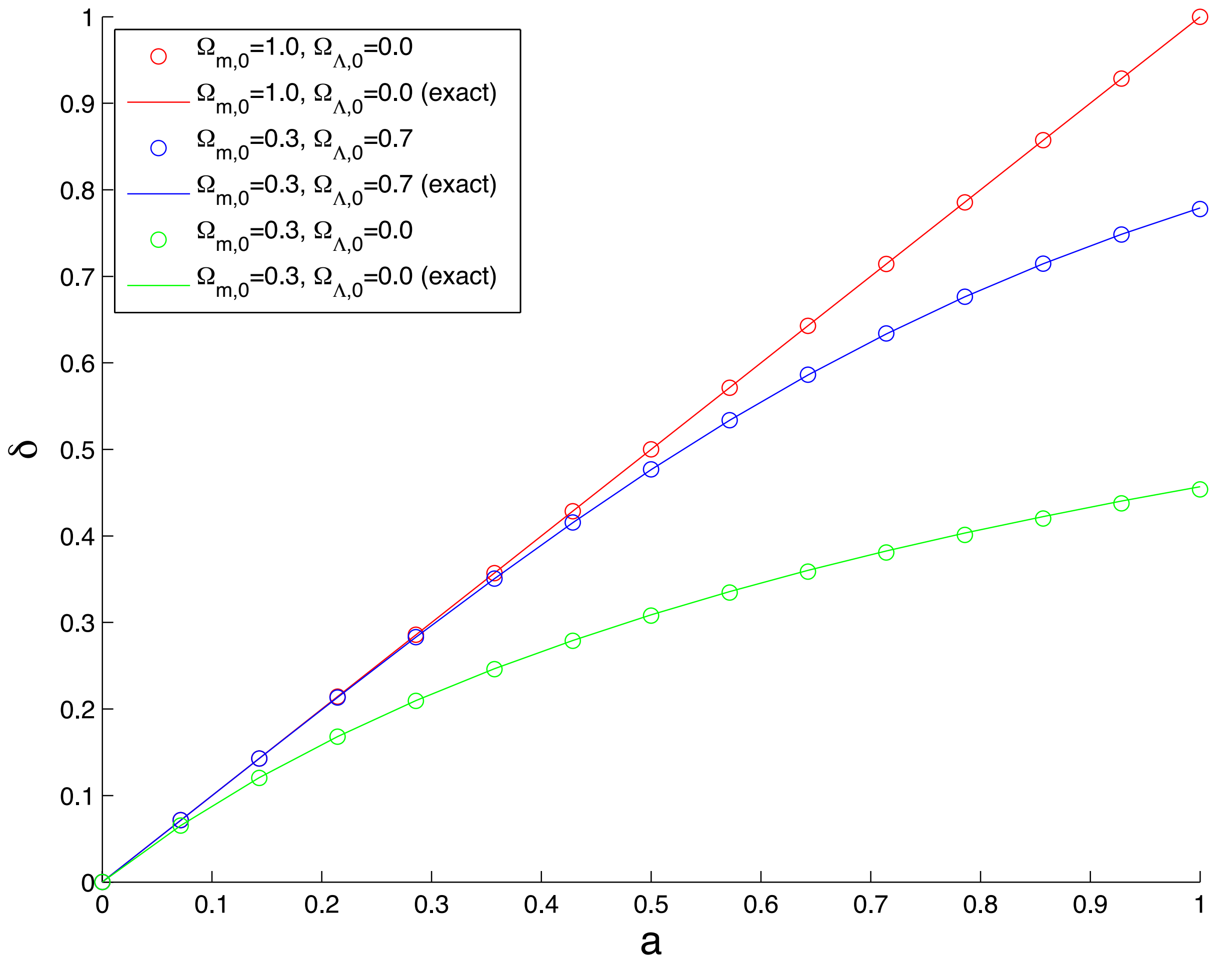

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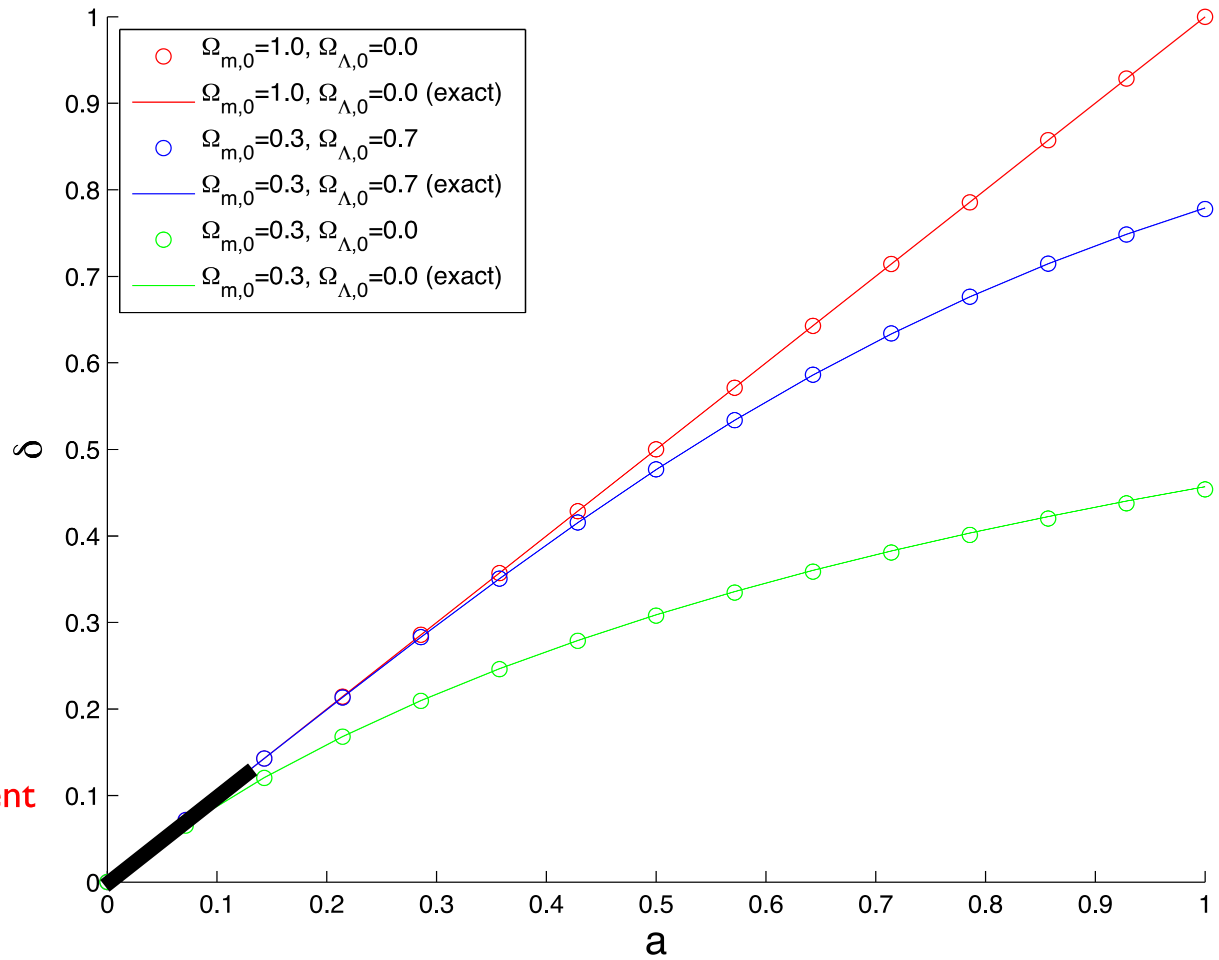

early evolution independent of cosmological model

- evolution of density contrast $\delta(t)$ for *dark matter*

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} - 4\pi G\bar{\rho}\delta = 0$$

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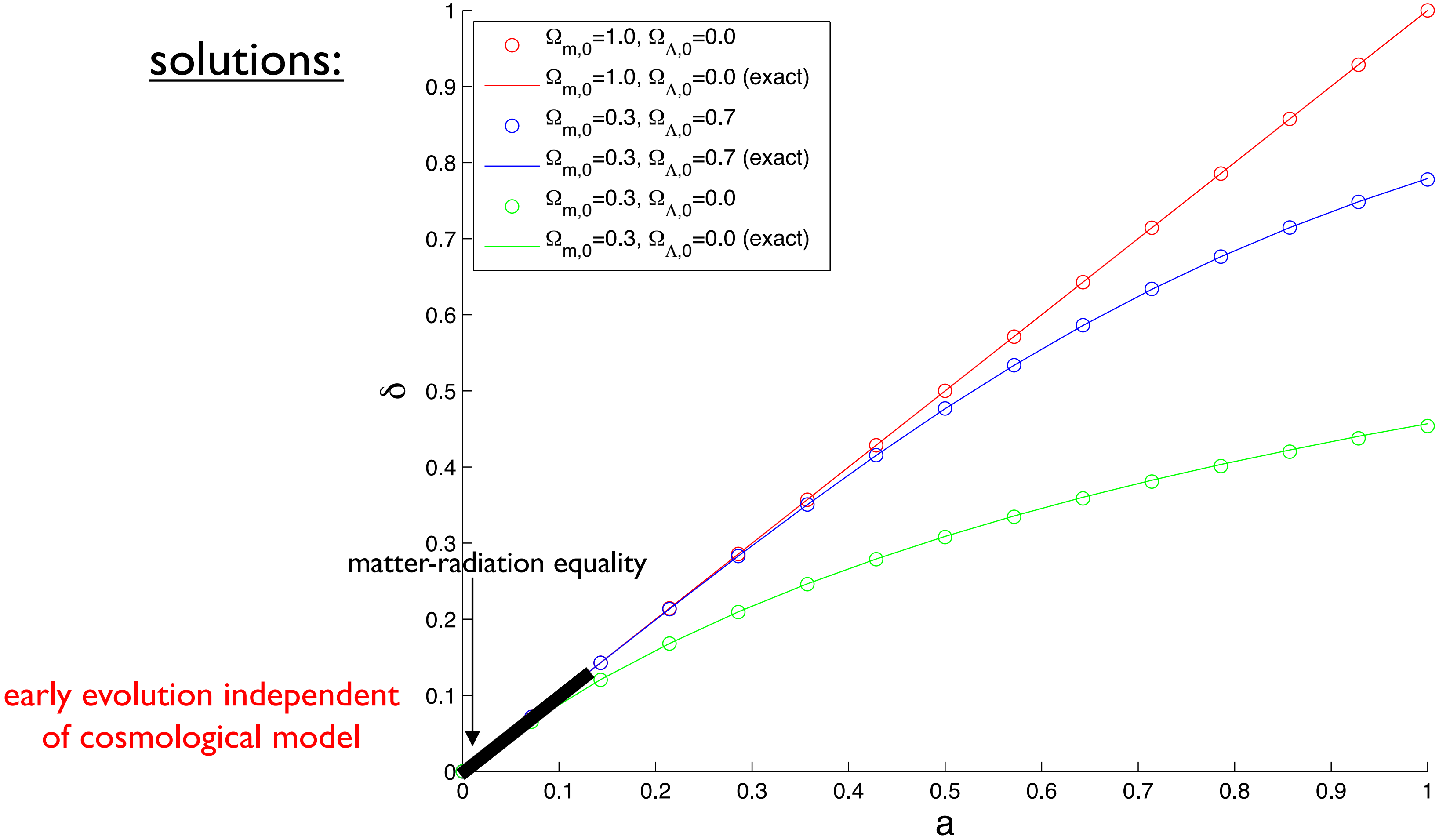

- evolution of density contrast $\delta(t)$ for *dark matter*

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} - 4\pi G \bar{\rho}\delta = 0$$

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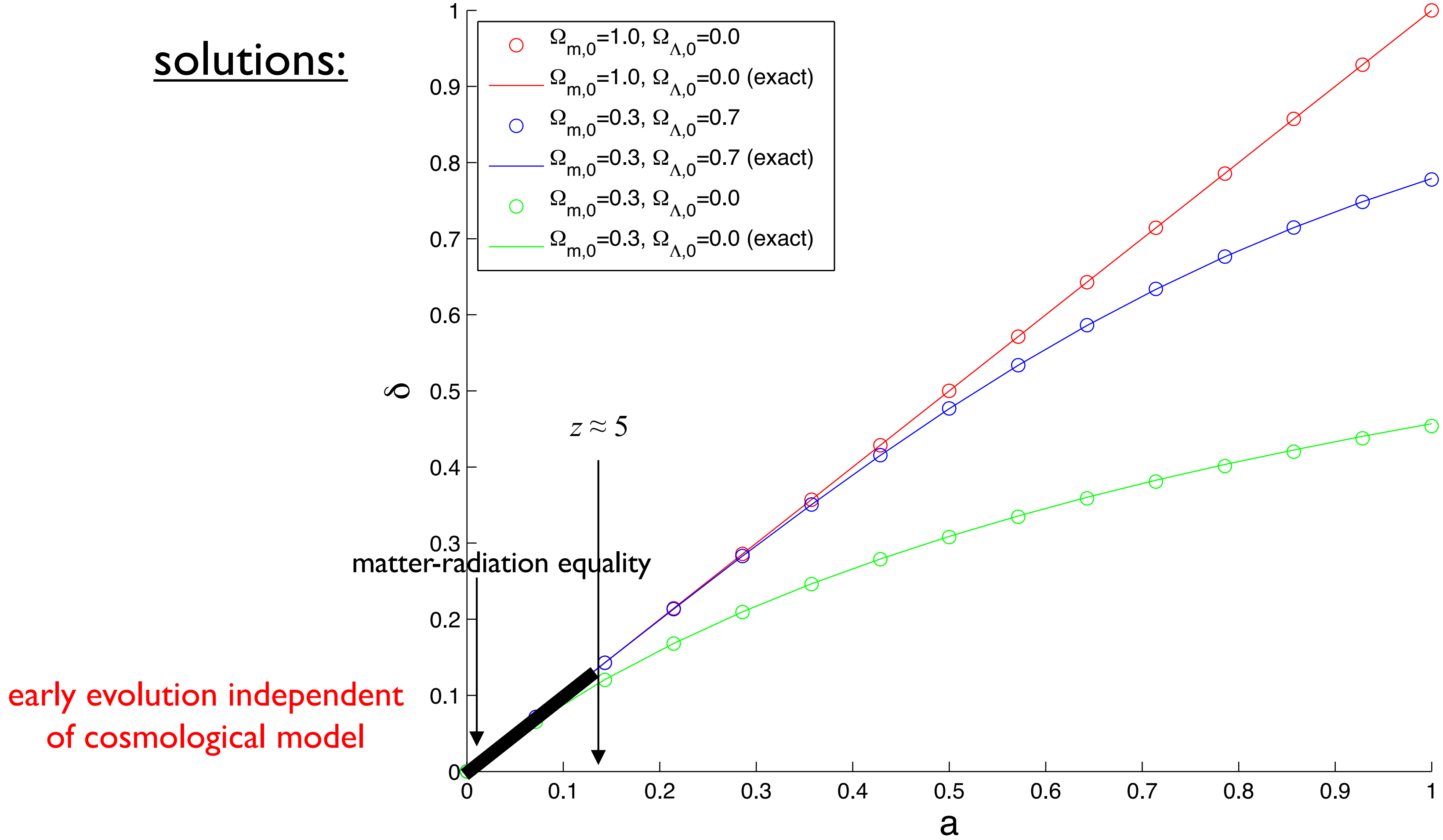

- evolution of density contrast $\delta(t)$ for *dark matter* during matter domination

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} - 4\pi G\bar{\rho}\delta = 0$$

- evolution of density contrast $\delta(t)$ for *dark matter* during matter domination

$$\frac{\partial^2\delta}{\partial t^2} + 2\boxed{\frac{\dot{a}}{a}}\frac{\partial\delta}{\partial t} - 4\pi G\boxed{\bar{\rho}}\delta = 0$$

matter domination:

$\Omega_m=1$ solution for $a(t)$:

$$\boxed{\begin{aligned} \frac{\dot{a}}{a} &= \frac{2}{3t} \\ 4\pi G\bar{\rho} &= \frac{2}{3t^2} \end{aligned}}$$

- evolution of density contrast $\delta(t)$ for *dark matter* during matter domination

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} - 4\pi G\bar{\rho}\delta = 0$$

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$$4\pi G\bar{\rho} = \frac{2}{3t^2}$$

$$\frac{\partial^2 \delta}{\partial t^2} + \frac{4}{3t}\frac{\partial \delta}{\partial t} - \frac{2}{3t^2}\delta = 0$$

- evolution of density contrast $\delta(t)$ for *dark matter* during matter domination

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<u>Ansatz:</u>

$$\delta = Ct^n$$

$$\dot\delta = nCt^{n-1}$$

$$\ddot\delta = n(n-1)Ct^{n-2}$$

- evolution of density contrast $\delta(t)$ for *dark matter* during matter domination

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matter domination:

Ω_m=1 solution for $a(t)$:

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$$\delta = Ct^n$$
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$$\delta = C_1 t^{2/3} + C_2 t^{-1}$$ (growing mode + decaying mode)

- evolution of density contrast $\delta(t)$ for *dark matter* during matter domination

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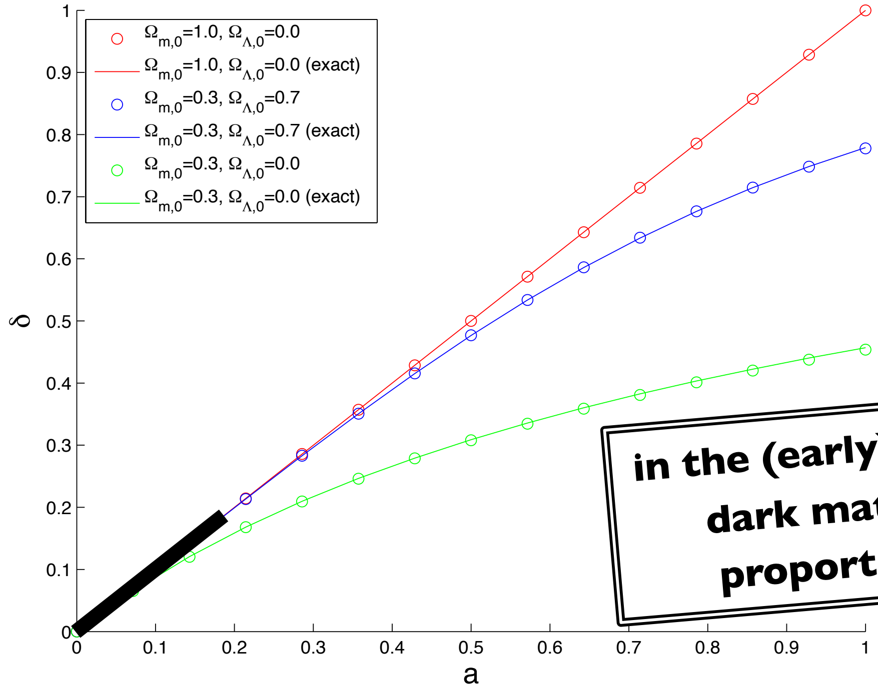
$$\delta=C_1t^{2/3}+C_2t^{-1}$$ (growing mode + decaying mode)

$$\boxed{\delta\propto a}$$

- consider growing mode only…
- remember $a\sim t^{2/3}$ (for $\Omega_m=1$, cf. FRW lecture)

- evolution of density contrast $\delta(t)$ for *dark matter* during matter domination

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - 4\pi G\bar{\rho}\delta = 0$$



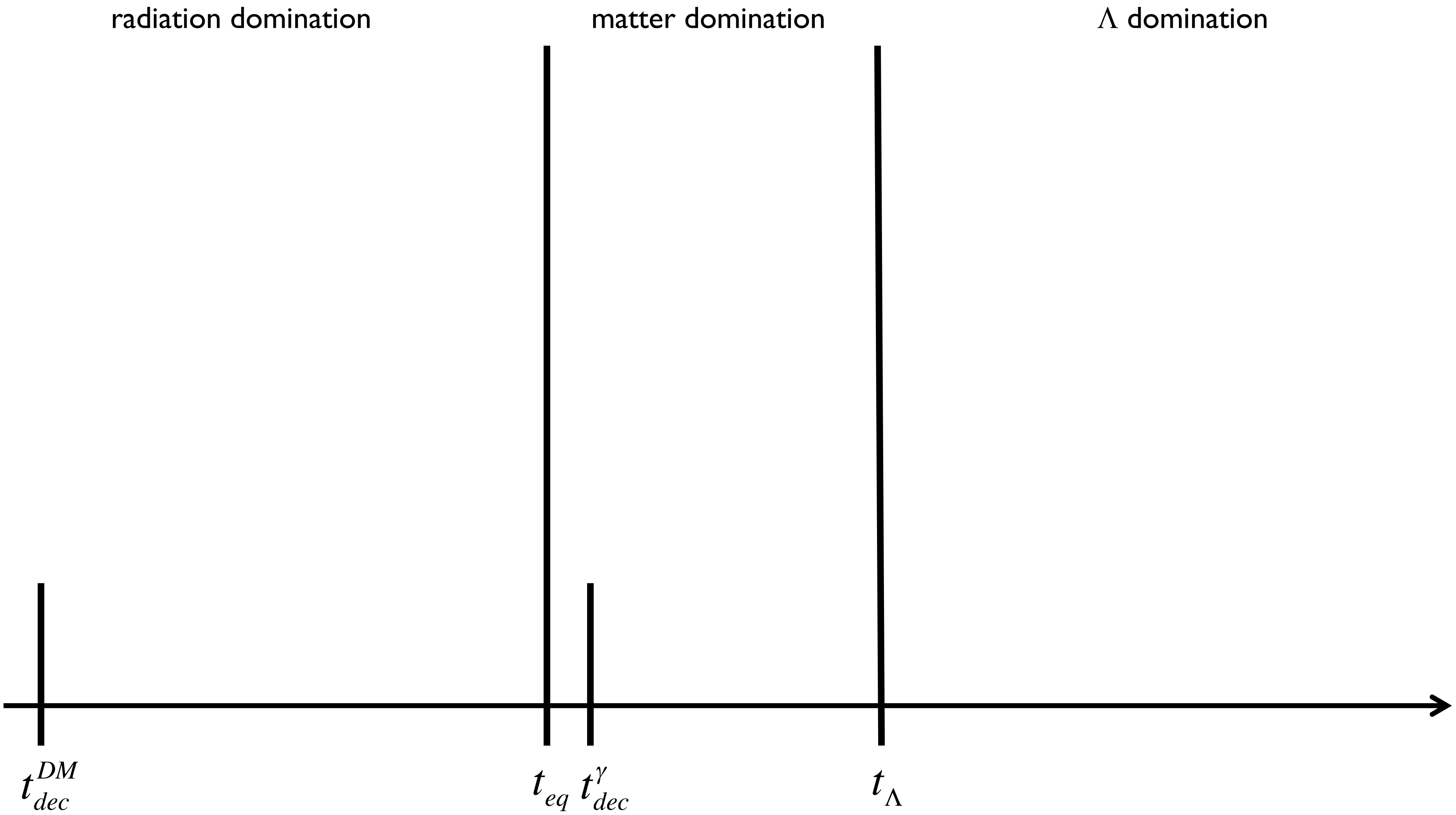
in the (early) era of matter domination dark matter perturbations grow proportional to the scale factor

$$\boxed{\delta \propto a}$$

- consider growing mode only…
- remember $a \sim t^{2/3}$ (for Ω_m=1, cf. FRW lecture)

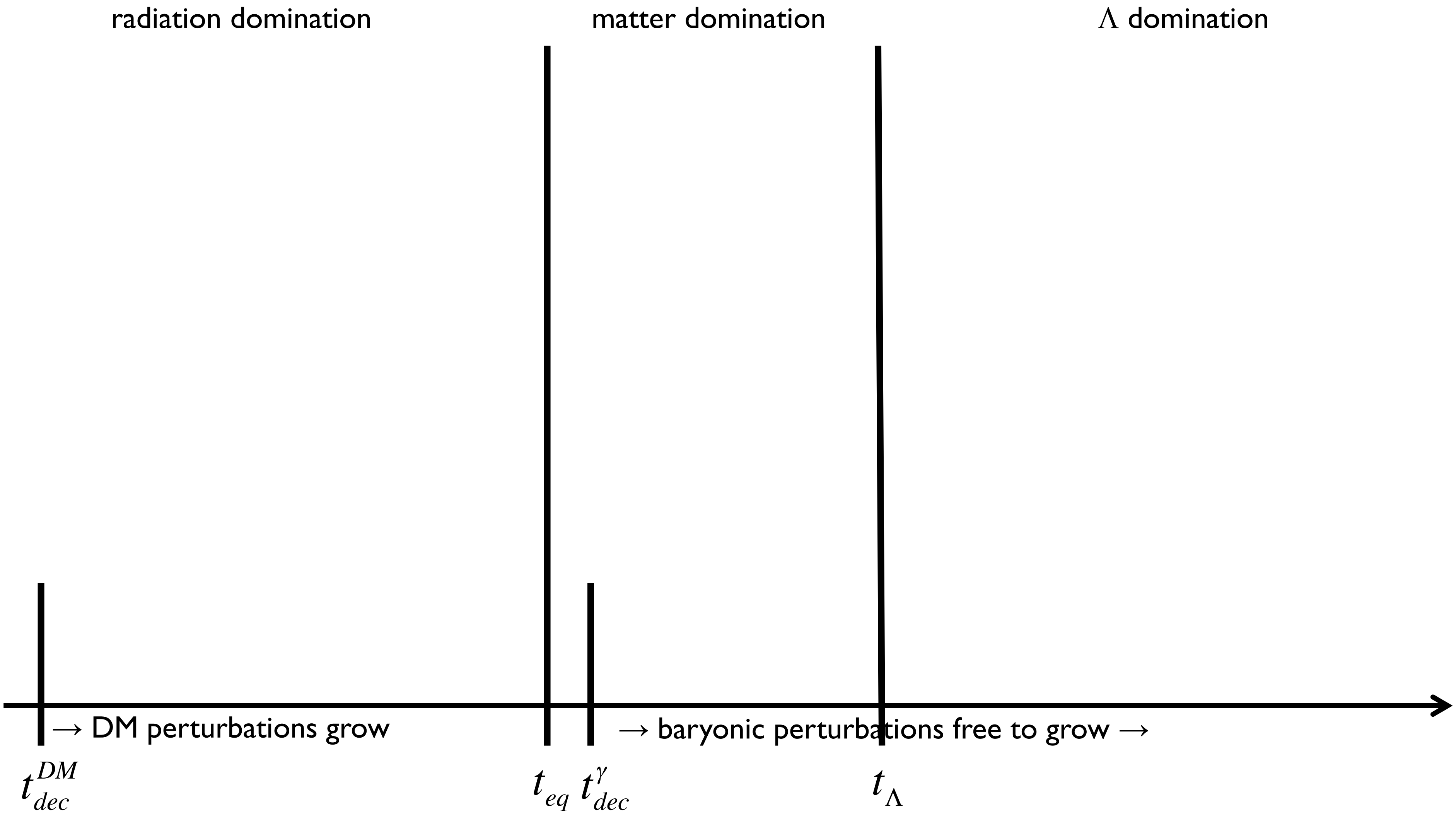
- dark matter perturbations – during all epochs

- dark matter perturbations – during all epochs



(time axis not to scale!)

- dark matter perturbations – during all epochs



(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination | matter domination | Λ domination

Thermal History lecture:

$$1+z_{eq} = \frac{\rho_{m,0}}{\rho_{r,0}} = 24000\ \Omega_{m,0}h^2$$

$$z_{eq} \approx 3500$$

FRW lecture:

$$1 = \frac{\Omega_m}{\Omega_\Lambda} = \frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}(1+z_\Lambda)^3 \Rightarrow \quad 1+z_\Lambda = \left(\frac{1-\Omega_{m,0}}{\Omega_{m,0}}\right)^{1/3}$$

$$z_\Lambda \approx 0.3$$

$\rightarrow$ DM perturbations grow

$\rightarrow$ baryonic perturbations free to grow $\rightarrow$

t_{dec}^{DM} $\quad t_{eq}$ $\quad t_{dec}^{\gamma}$ $\quad t_\Lambda$

(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination

matter domination

Λ domination

$$\delta_m(a) \propto a$$

→ DM perturbations grow

→ baryonic perturbations free to grow →

t_{dec}^{DM}

t_{eq} t_{dec}^{γ}

t_{Λ}

$z_{eq} \approx 3500$

$z_{\Lambda} \approx 0.3$

(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination | matter domination | Λ domination

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_m}{\partial t} - 4\pi G\bar{\rho}\left(\frac{\bar{\rho}_m}{\bar{\rho}}\delta_m + \frac{\bar{\rho}_r}{\bar{\rho}}\delta_r\right) = 0$$

$$\delta_m(a) \propto a$$

→ DM perturbations grow

→ baryonic perturbations free to grow →

t_{dec}^{DM} t_{eq} t_{dec}^{γ} t_{Λ}

$z_{eq} \approx 3500$ $z_{\Lambda} \approx 0.3$

(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination

$$\frac{\partial^2\delta_m}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta_m}{\partial t} - 4\pi G\bar{\rho}\underbrace{\left(\frac{\bar{\rho}_m}{\bar{\rho}}\delta_m + \frac{\bar{\rho}_r}{\bar{\rho}}\delta_r\right)}_{} = 0$$

we are interested in (dark!) matter perturbations…

…but potential perturbations
are sourced by all components!

$$\delta = \frac{\bar{\rho}_m}{\bar{\rho}}\delta_m + \frac{\bar{\rho}_r}{\bar{\rho}}\delta_r$$

matter domination

$$\delta_m(a) \propto a$$

Λ domination

→ DM perturbations grow

→ baryonic perturbations free to grow →

t_{dec}^{DM} t_{eq} t_{dec}^{γ} t_Λ

$z_{eq} \approx 3500$ $z_\Lambda \approx 0.3$

(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination | matter domination | Λ domination

$$\frac{\partial^2\delta_m}{\partial t^2}+2\frac{\dot a}{a}\frac{\partial\delta_m}{\partial t}-4\pi G\bar\rho\underbrace{\left(\frac{\bar\rho_m}{\bar\rho}\delta_m+\frac{\bar\rho_r}{\bar\rho}\delta_r\right)}=0$$

we are interested in (dark!) matter perturbations…

…but potential perturbations are sourced by all components!

$$\delta=\frac{\bar\rho_m}{\bar\rho}\delta_m+\frac{\bar\rho_r}{\bar\rho}\delta_r$$

remember:

$$\Delta\Phi=4\pi Ga^2\bar\rho_{tot}\left(\frac{\bar\rho}{\bar\rho_{tot}}\delta+\frac{\bar\rho_X}{\bar\rho_{tot}}\delta_X+\frac{\bar\rho_Y}{\bar\rho_{tot}}\delta_Y+\cdots\right)$$

$$\delta_m(a)\propto a$$

→ DM perturbations grow

→ baryonic perturbations free to grow →

t_{dec}^{DM} t_{eq} t_{dec}^{γ} t_Λ

$z_{eq}\approx 3500$ $z_\Lambda\approx 0.3$

(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_m}{\partial t} - 4\pi G \bar{\rho}\left(\frac{\bar{\rho}_m}{\bar{\rho}}\delta_m + \frac{\bar{\rho}_r}{\bar{\rho}}\delta_r\right) = 0$$

$\delta_r \approx 0$ (they oscillate*)

matter domination

$$\delta_m(a) \propto a$$

Λ domination

→ DM perturbations grow

→ baryonic perturbations free to grow →

t_{dec}^{DM} t_{eq} t_{dec}^{γ} t_{Λ}

$z_{eq} \approx 3500$ $z_{\Lambda} \approx 0.3$

*proof requires GR calculations

(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination | matter domination | Λ domination

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2\frac{\dot a}{a}\frac{\partial \delta_m}{\partial t} - 4\pi G\bar\rho\left(\frac{\bar\rho_m}{\bar\rho}\delta_m + \frac{\bar\rho_r}{\bar\rho}\delta_r\right) = 0$$

$$\delta_m(a) \propto a$$

$\delta_r \approx 0$ (they oscillate) → why don't we have $\delta_r = 4/3\ \delta_m$?

→ DM perturbations grow

→ baryonic perturbations free to grow →

t_{dec}^{DM} t_{eq} t_{dec}^{γ} t_{Λ}

$z_{eq} \approx 3500$ $z_{\Lambda} \approx 0.3$

(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_m}{\partial t} - 4\pi G \bar{\rho}_m \delta_m = 0$$

$\delta_r \approx 0$ (they oscillate)

matter domination

$$\delta_m(a) \propto a$$

Λ domination

$\rightarrow$ DM perturbations grow

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t_{dec}^{DM} t_{eq} t_{dec}^{γ} t_{Λ}

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(time axis not to scale!)

- dark matter perturbations – during all epochs

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$\delta_r \approx 0$ (they oscillate)

$$\underbrace{4\pi G \bar{\rho}_r = \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2}_{\Omega_r = \frac{8\pi G \bar{\rho}_r}{3H^2} = 1}$$

matter domination

$$\delta_m(a) \propto a$$

Λ domination

t_{dec}^{DM} → DM perturbations grow

t_{eq} t_{dec}^{γ} → baryonic perturbations free to grow → t_{Λ}

$z_{eq} \approx 3500$ $z_{\Lambda} \approx 0.3$

(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_m}{\partial t} - 4\pi G \bar{\rho}_m \delta_m = 0$$

$\delta_r \approx 0$ (they oscillate)

$$4\pi G \bar{\rho}_m \ll 4\pi G \bar{\rho}_r = \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2$$

matter domination

$$\delta_m(a) \propto a$$

Λ domination

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t_{dec}^{DM} t_{eq} t_{dec}^{γ} t_{Λ}

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- dark matter perturbations – during all epochs

radiation domination

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2\frac{\dot a}{a}\frac{\partial \delta_m}{\partial t} - 4\pi G\bar\rho_m\delta_m = 0$$

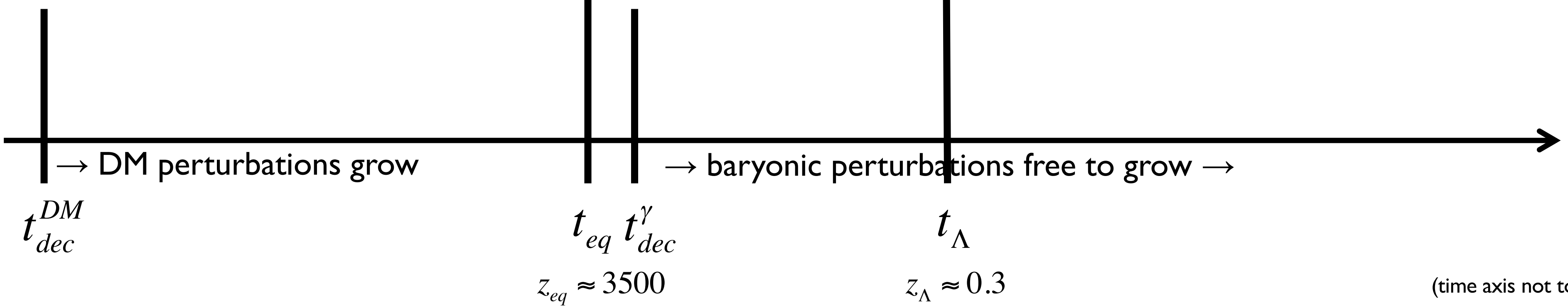
$\delta_r \approx 0$ (they oscillate)

$$4\pi G\boxed{\bar\rho_m \ll} 4\pi G\bar\rho_r = \frac{3}{2}\boxed{\left(\frac{\dot a}{a}\right)}^2$$

matter domination

$$\delta_m(a) \propto a$$

Λ domination



(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination | matter domination | Λ domination

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2\underbrace{\boxed{\frac{\dot a}{a}}}_{\text{dominates}}\frac{\partial \delta_m}{\partial t} - \underbrace{\cancel{4\pi G \bar\rho_m \delta_m}}_{\text{can be ignored}} = 0$$

$\delta_r \approx 0$ (they oscillate)

$$4\pi G\boxed{\bar\rho_m \ll} 4\pi G\bar\rho_r = \frac{3}{2}\boxed{\left(\frac{\dot a}{a}\right)}^2$$

$$\delta_m(a) \propto a$$

→ DM perturbations grow

→ baryonic perturbations free to grow →

t_{dec}^{DM} t_{eq} t_{dec}^{γ} t_Λ

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radiation domination | matter domination | Λ domination

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$$\frac{\partial^2 \delta_m}{\partial t^2} + \frac{1}{t}\frac{\partial \delta_m}{\partial t} \approx 0$$

$$\delta_m(a) \propto a$$

$\rightarrow$ DM perturbations grow

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t_{dec}^{DM} t_{eq} t_{dec}^{γ} t_Λ

$z_{eq} \approx 3500$ $z_\Lambda \approx 0.3$

(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination | matter domination | Λ domination

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_m}{\partial t} - 4\pi G\bar{\rho}_m \delta_m = 0$$

$\delta_r \approx 0$ (they oscillate)

$$4\pi G\bar{\rho}_m \ll 4\pi G\bar{\rho}_r = \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2$$

$$\frac{\partial^2 \delta_m}{\partial t^2} + \frac{1}{t}\frac{\partial \delta_m}{\partial t} \approx 0$$

$$\frac{\partial y}{\partial t} + \frac{1}{t}y = 0\,, \quad y = \frac{\partial \delta_m}{\partial t}$$

$$\delta_m(a) \propto \ln(t)$$

$$\delta_m(a) \propto a$$

→ DM perturbations grow

→ baryonic perturbations free to grow →

t_{dec}^{DM} t_{eq} t_{dec}^{γ} t_{Λ}

$z_{eq} \approx 3500$ $z_{\Lambda} \approx 0.3$

(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_m}{\partial t} - 4\pi G \bar{\rho}_m \delta_m = 0$$

$\delta_r \approx 0$ (they oscillate)

$$4\pi G \bar{\rho}_m \ll 4\pi G \bar{\rho}_r = \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2$$

$$\frac{\partial^2 \delta_m}{\partial t^2} + \frac{1}{t}\frac{\partial \delta_m}{\partial t} \approx 0$$

$$\frac{\partial y}{\partial t} + \frac{1}{t} y = 0\,, \quad y = \frac{\partial \delta_m}{\partial t}$$

$$\delta_m(a) \propto \ln(t) \propto \ln(a)$$

$a \propto t^{1/2}$ (cf. FRW lecture)

matter domination

$$\delta_m(a) \propto a$$

Λ domination

→ DM perturbations grow

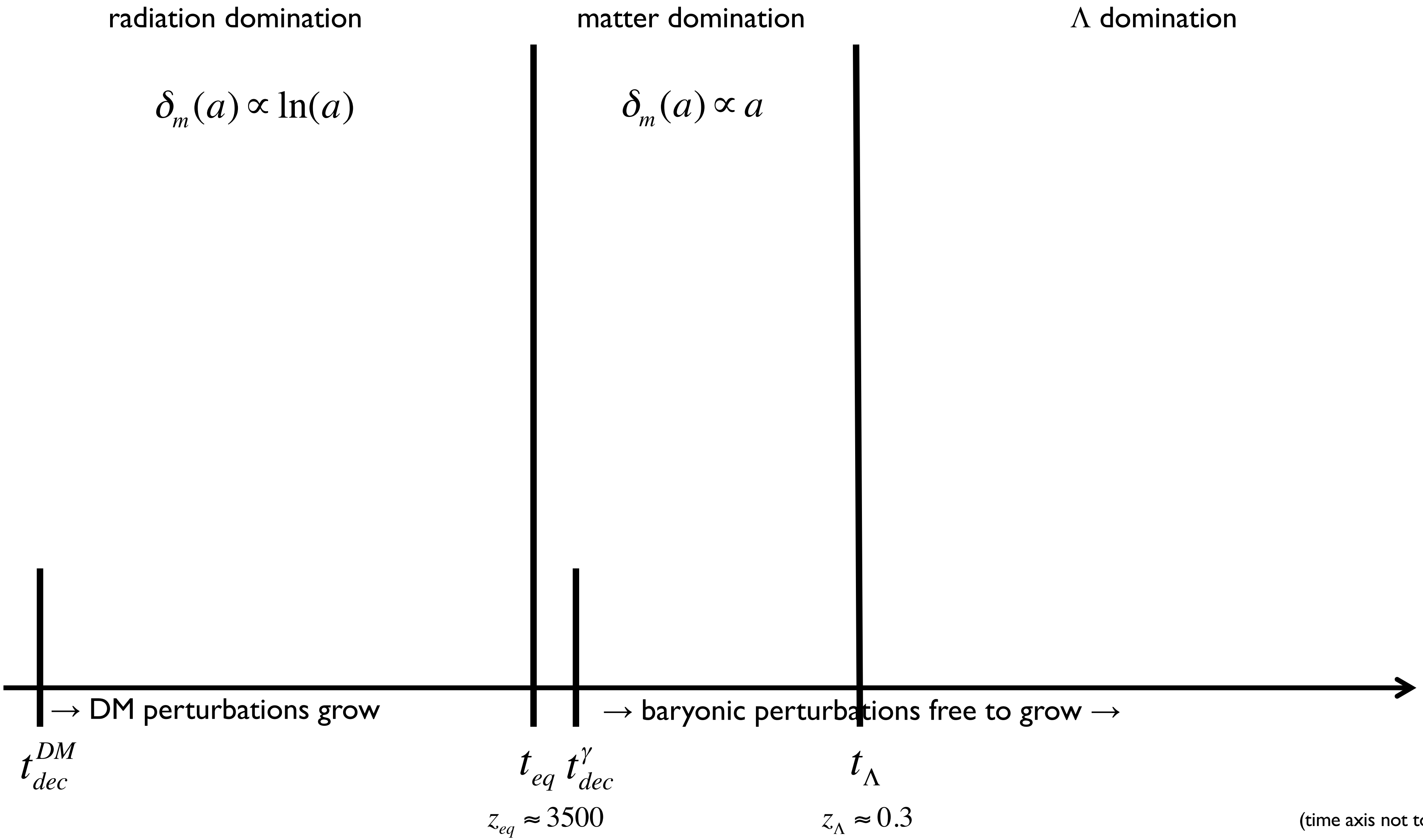
→ baryonic perturbations free to grow →

t_{dec}^{DM} t_{eq} t_{dec}^{γ} t_{Λ}

$z_{eq} \approx 3500$ $z_{\Lambda} \approx 0.3$

(time axis not to scale!)

- dark matter perturbations – during all epochs



(time axis not to scale!)

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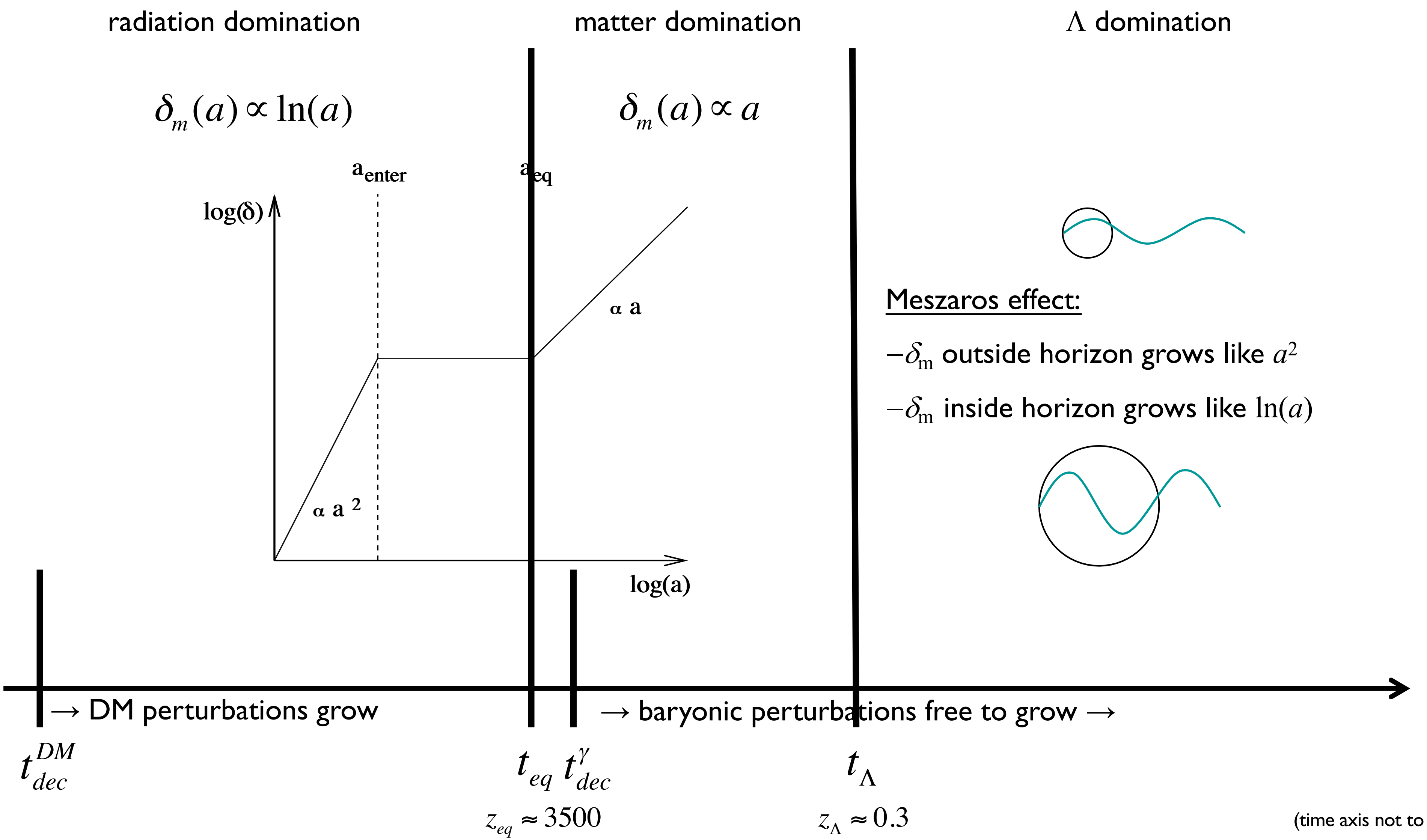
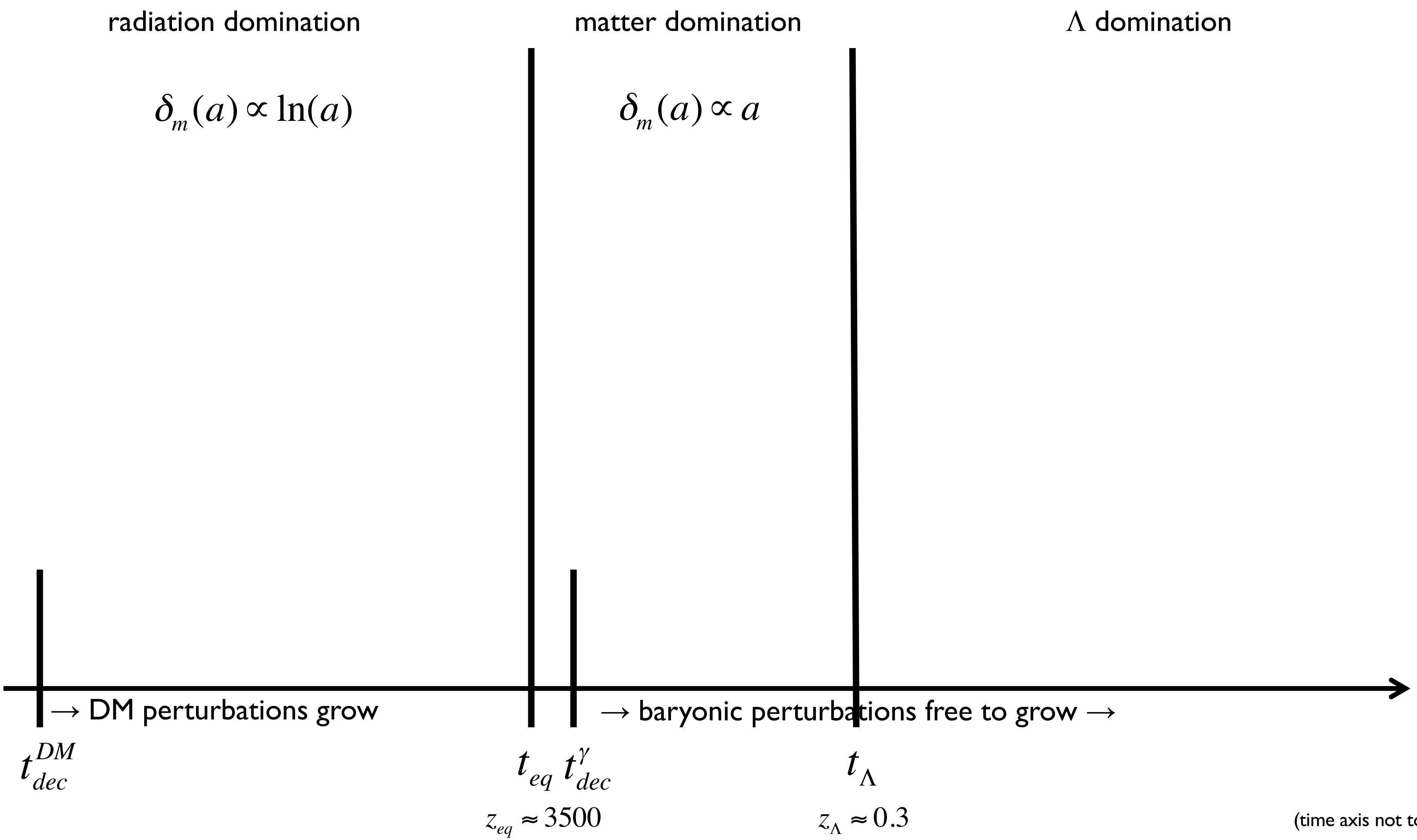


- dark matter perturbations – during all epochs



(time axis not to scale!)

- dark matter perturbations – during all epochs

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$$\delta_m(a) \propto \ln(a)$$

matter domination

$$\delta_m(a) \propto a$$

Λ domination

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_m}{\partial t} - 4\pi G \bar{\rho}_m \delta_m = 0$$

$\delta_\Lambda \approx 0$ (afaswk)

→ DM perturbations grow

→ baryonic perturbations free to grow →

t_{dec}^{DM}

t_{eq} t_{dec}^{γ}

$z_{eq} \approx 3500$

t_Λ

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(time axis not to scale!)

- dark matter perturbations – during all epochs

radiation domination

$$\delta_m(a) \propto \ln(a)$$

matter domination

$$\delta_m(a) \propto a$$

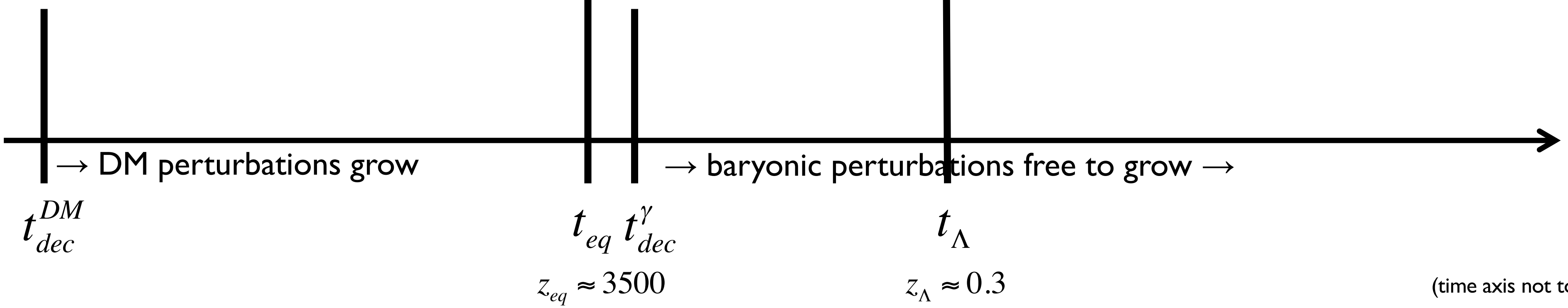
Λ domination

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2\boxed{\frac{\dot{a}}{a}}\frac{\partial \delta_m}{\partial t} - \cancel{4\pi G\bar{\rho}_m\delta_m} = 0$$

dominates   can be ignored

$\delta_\Lambda \approx 0$ (afask)

$$4\pi G\bar{\rho}_m \ll 4\pi G\bar{\rho}_\Lambda = \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2$$



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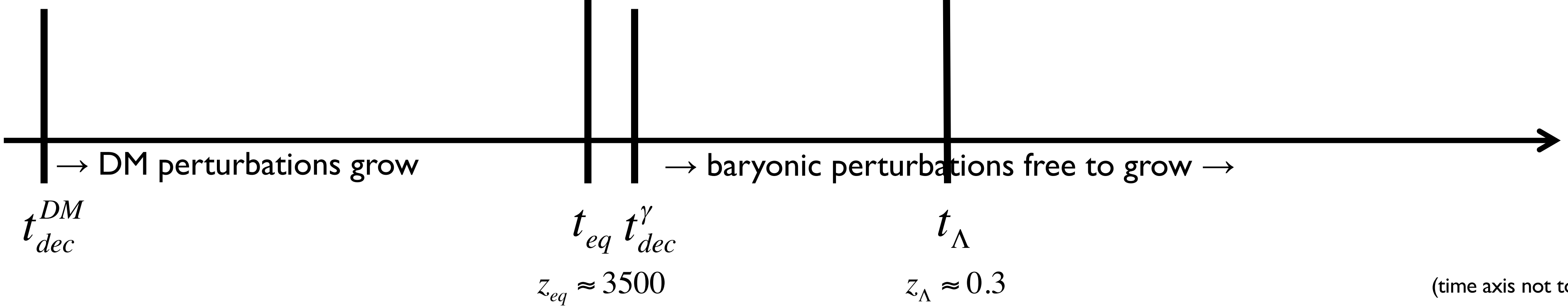
$$4\pi G\bar{\rho}_m \ll 4\pi G\bar{\rho}_\Lambda = \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2$$

$a \propto e^{H_0 t}$ (cf. FRW lecture)

$$\frac{\partial^2 \delta_m}{\partial t^2} + 2H_0\frac{\partial \delta_m}{\partial t} \approx 0$$



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$$\frac{\partial y}{\partial t} + 2H_0 y = 0\,, \quad y = \frac{\partial \delta_m}{\partial t}$$

$$\delta_m(a) \propto a^{-2}$$

$\rightarrow$ DM perturbations grow

$\rightarrow$ baryonic perturbations free to grow $\rightarrow$

t_{dec}^{DM} t_{eq} t_{dec}^{γ} t_Λ

$z_{eq} \approx 3500$ $z_\Lambda \approx 0.3$

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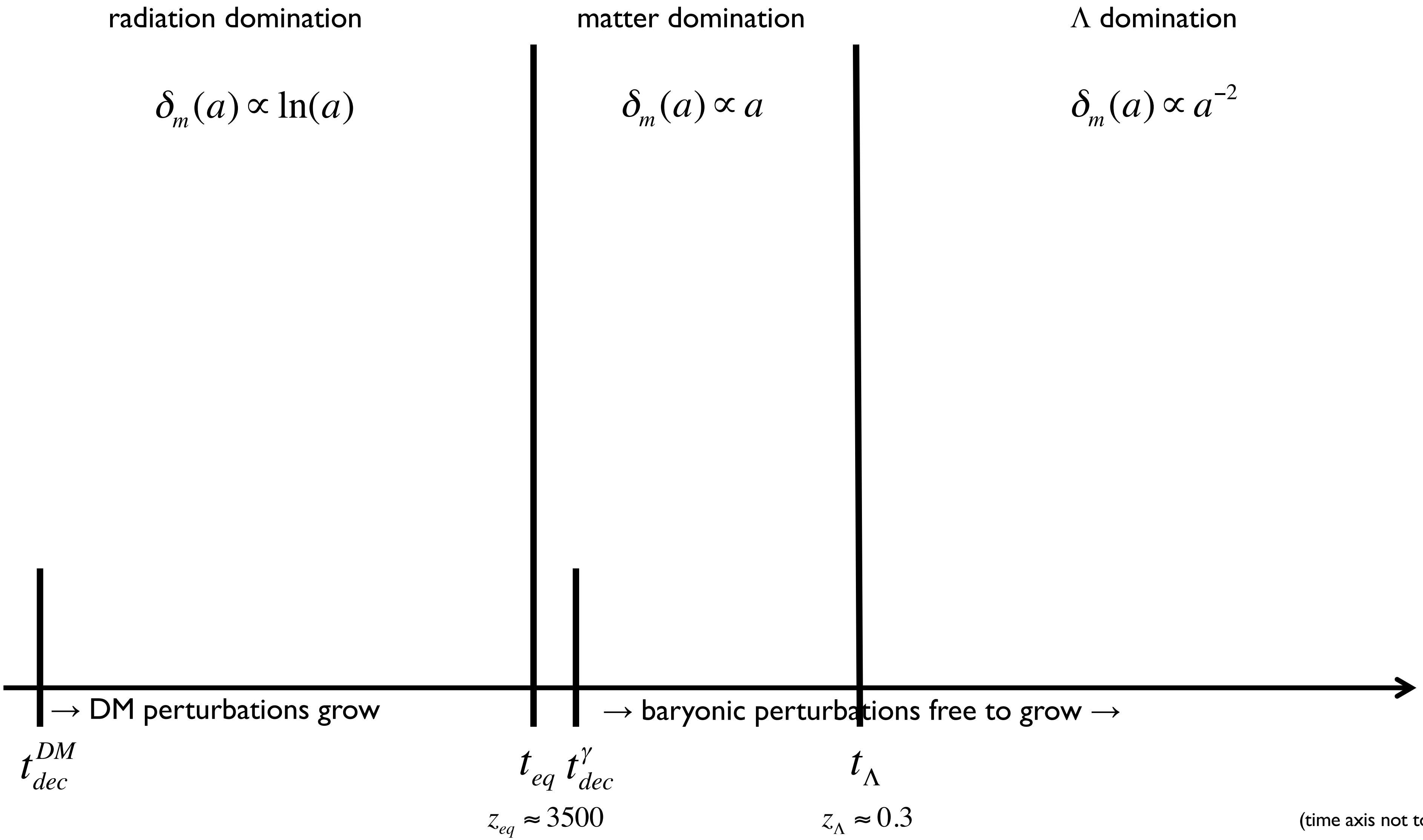
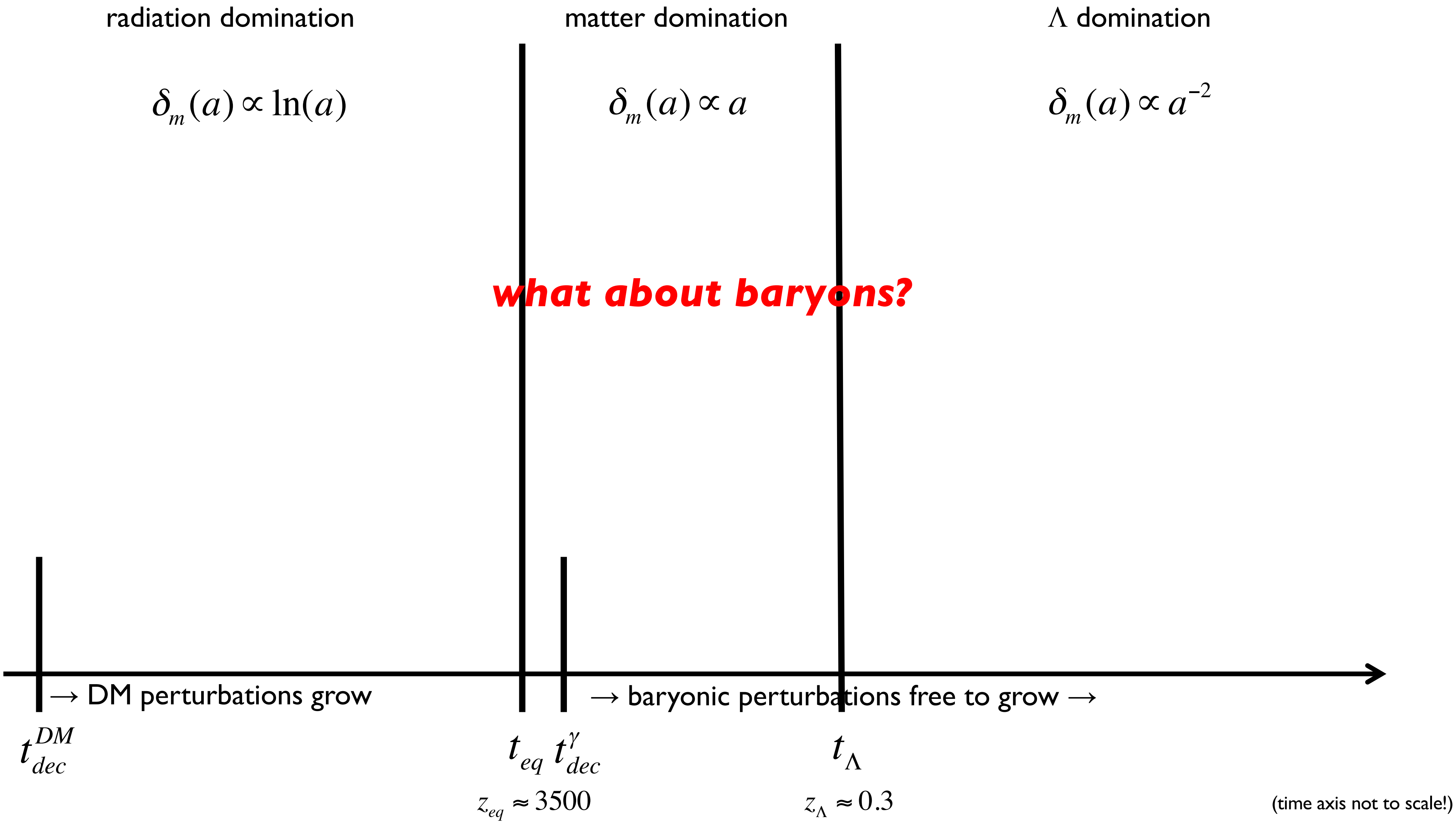
- dark matter perturbations – during all epochs

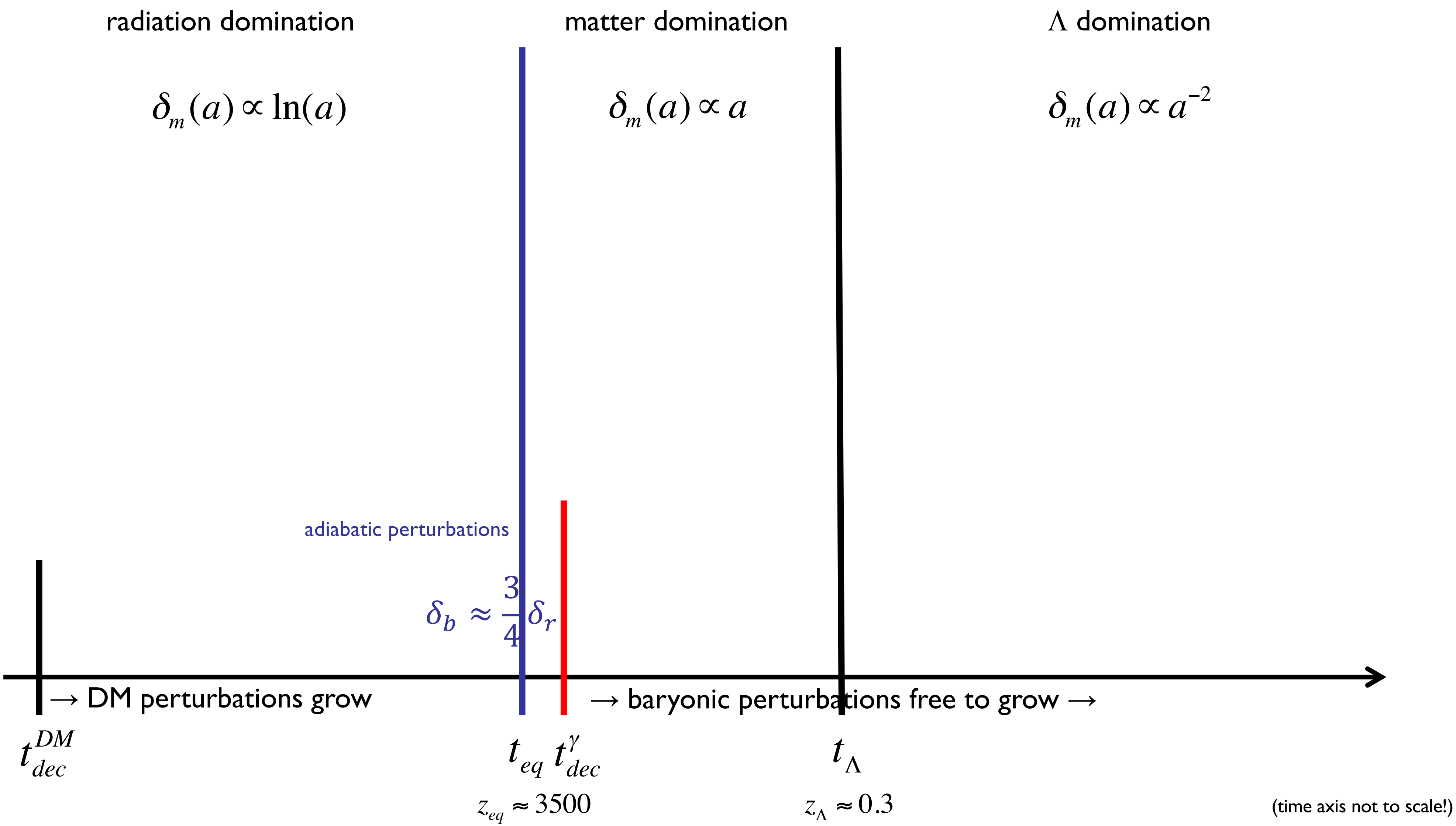


(time axis not to scale!)

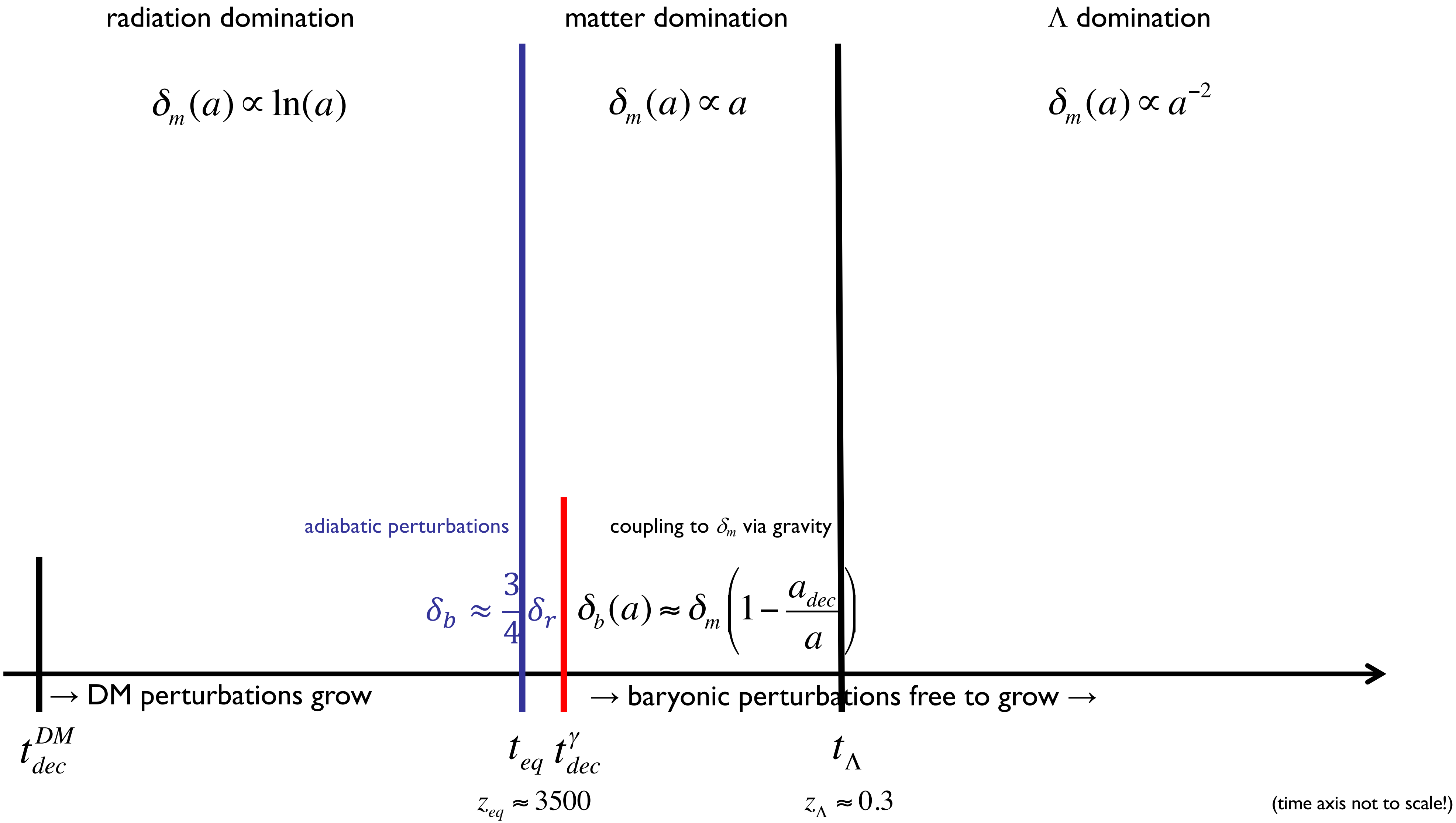
- dark matter perturbations – during all epochs



- dark matter perturbations – during all epochs



- dark matter perturbations – during all epochs



- dark matter perturbations – during all epochs

radiation domination	matter domination	Λ domination
$\delta_m(a) \propto \ln(a)$	$\delta_m(a) \propto a$	$\delta_m(a) \propto a^{-2}$

$$\delta_b(a) \approx \delta_m\left(1 - \frac{a_{dec}}{a}\right) \xrightarrow{a>>a_{dec}} \delta_m$$

$\rightarrow$ DM perturbations grow

$\rightarrow$ baryonic perturbations free to grow $\rightarrow$ catching up with δ_m

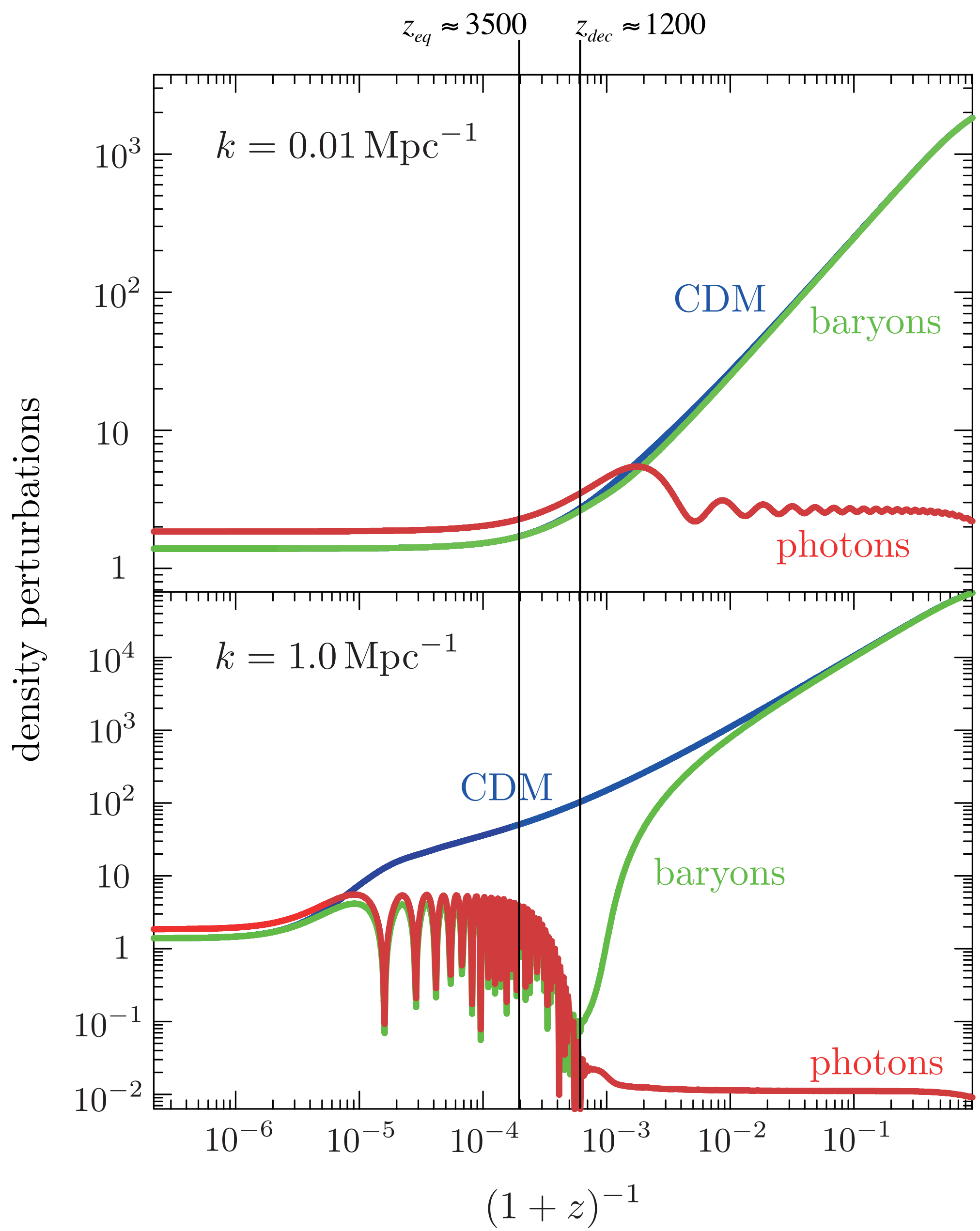
t_{dec}^{DM} $\quad t_{eq}$ $\quad t_{dec}^{\gamma}$ $\quad t_{\Lambda}$

$z_{eq} \approx 3500$ $\quad z_{\Lambda} \approx 0.3$

(time axis not to scale!)

- matter perturbations:

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_k}{\partial t} + \left(\frac{c_s^2}{a^2}k^2 - 4\pi G\bar{\rho}\right)\delta_k = 0$$



- Jeans Mass analysis – epochs & components:

$$\lambda_J(a) = c_s(a)\sqrt{\frac{\pi}{G\bar{\rho}(a)}} \qquad M_{J,w}(a) = \frac{4\pi}{3}\left(\frac{\lambda_J(a)}{2}\right)^3 \bar{\rho}_w(a)$$

radiation domination	**matter domination**	
$\bar{\rho} \approx \rho_r \propto a^{-4}$	$\bar{\rho} \approx \rho_m \propto a^{-3}$	
$M_J^{CDM} = const.$ $c_s \approx \sigma_v \propto a^{-1}$	$M_J^{CDM} \propto a^{-3/2}$ $c_s \approx \sigma_v \propto a^{-1}$	
$M_J^b \propto a^3$ $c_s = \frac{c}{\sqrt{3}}$	$M_J^b \approx const.$ $c_s \propto \frac{c}{\sqrt{3}}\left(\frac{4\rho_r}{3\rho_b}\right)^{1/2} \propto a^{-1/2}$	$M_J^b \approx a^{-3/2}$ $c_s \propto \left(\frac{3kT}{2m_p}\right)^{1/2} \propto a^{-1}$ $\left(T \propto a^{-2}\right)$

$$c_s = \frac{c}{\sqrt{3}}\left(1 + \frac{\delta\rho_b}{\delta\rho_r}\right)^{-1/2}$$

$$\left(\frac{\delta\rho_b}{\rho_b} = \frac{3}{4}\frac{\delta\rho_r}{\rho_r}\right)$$

t_{dec}^{DM} → DM perturbations grow; t_{eq}; t_{dec}^{γ} → baryonic perturbations free to grow

(time axis not to scale!)

- governing equations
- growth of matter perturbations
- **statistics of perturbations**
- non-linear structure formation

- evolution of density contrast $\delta_k(t)$

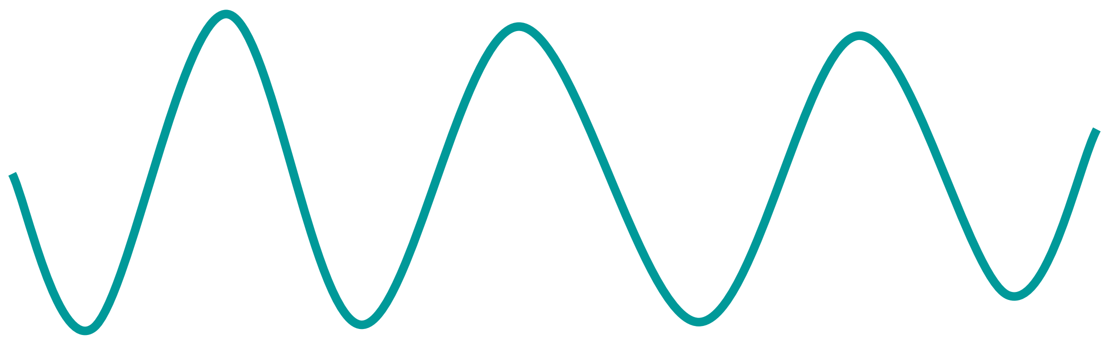
$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - 4\pi G\overline{\rho}\delta = 0 \quad \Rightarrow \qquad \delta(\vec{x},t) = \delta_k(t)\, e^{i\vec{k}\cdot\vec{x}}$$

- decomposition of $\delta(x,t)$ into waves

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} - 4\pi G\bar{\rho}\delta = 0 \quad \Rightarrow \quad \delta(\vec{x},t) = \sum_{\vec{k}} \delta_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}}$$

long wavelength

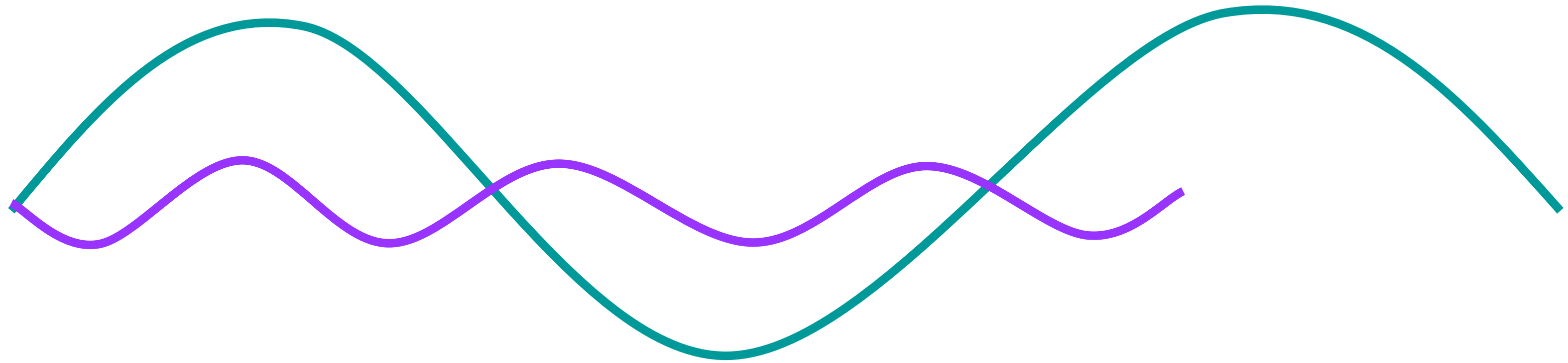
short wavelength,
large amplitude



short wavelength,
small amplitude

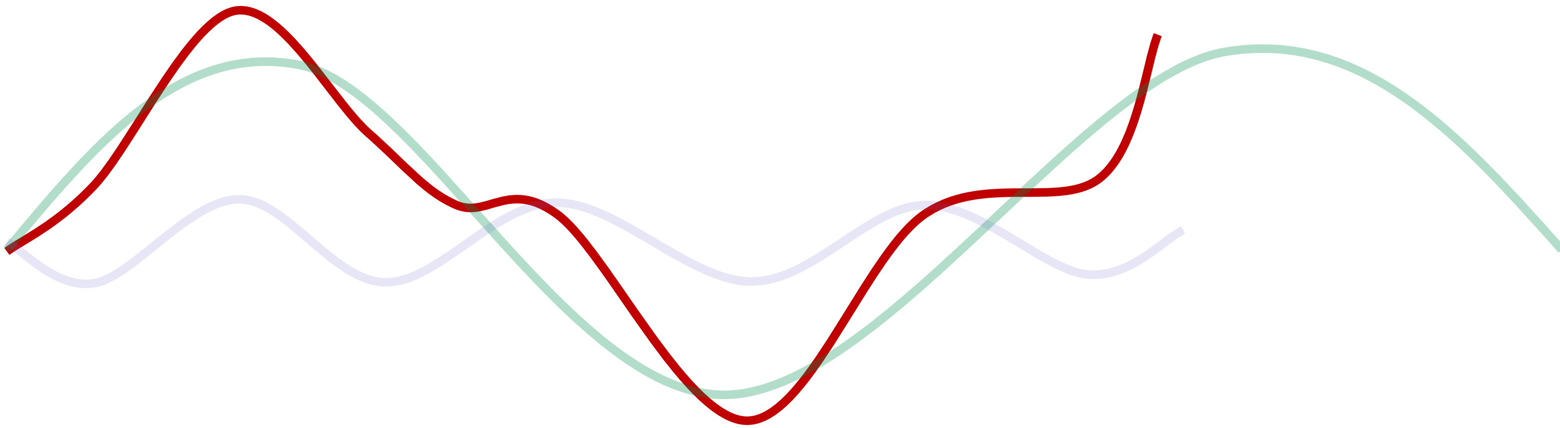
- decomposition of $\delta(x,t)$ into waves

$$\frac{\partial^2\delta}{\partial t^2}+2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t}-4\pi G\bar{\rho}\delta=0 \quad\Rightarrow\quad \delta(\vec{x},t)=\sum_{\vec{k}}\delta_{\vec{k}}(t)\,e^{i\vec{k}\cdot\vec{x}}$$



- decomposition of $\delta(x,t)$ into waves

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} - 4\pi G\bar{\rho}\delta = 0 \quad \Rightarrow \quad \delta\left(\vec{x},t\right) = \sum_{\vec{k}} \delta_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}}$$



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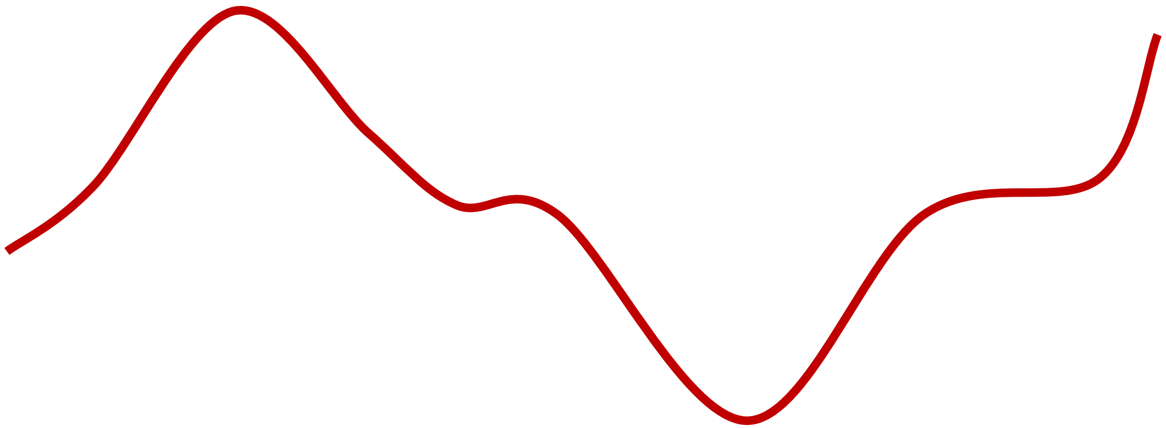

- decomposition of $\delta(x,t)$ into waves: $$\delta\left(\vec{x},t\right) = \sum_{\vec{k}} \delta_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}}$$

    - perturbation equation is linear => $\delta_{\vec{k}}\left(t\right)$ grow independently

    - first moment

      $\left\langle \delta\left(\vec{x},t\right)\right\rangle = 0$ => $\left\langle \delta_{\vec{k}}\left(t\right)\right\rangle = 0\,,\quad \delta_{\vec{k}}\left(t\right) = \delta_{-\vec{k}}\left(t\right)$

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  - higher order moments:

$$\xi_2 = \left\langle \delta\left(\vec{x}_1,t\right)\delta\left(\vec{x}_2,t\right)\right\rangle$$
$$\xi_3 = \left\langle \delta\left(\vec{x}_1,t\right)\delta\left(\vec{x}_2,t\right)\delta\left(\vec{x}_3,t\right)\right\rangle$$
$$\xi_4 = \left\langle \delta\left(\vec{x}_1,t\right)\delta\left(\vec{x}_2,t\right)\delta\left(\vec{x}_3,t\right)\delta\left(\vec{x}_4,t\right)\right\rangle$$
$$\ldots$$

- decomposition of $\delta(x,t)$ into waves: $$\delta\left(\vec{x},t\right) = \sum_{\vec{k}} \delta_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}}$$

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    $$\xi_4 = \left\langle \delta\left(\vec{x}_1,t\right)\delta\left(\vec{x}_2,t\right)\delta\left(\vec{x}_3,t\right)\delta\left(\vec{x}_4,t\right)\right\rangle$$
    ...

    homogeneity & isotropy =>

    $$\xi_2 = \xi_2\left(\left|\vec{x}_1 - \vec{x}_2\right|\right)$$
    ...

- decomposition of $\delta(x,t)$ into waves: $$\delta(\vec{x},t) = \sum_{\vec{k}} \delta_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{x}}$$

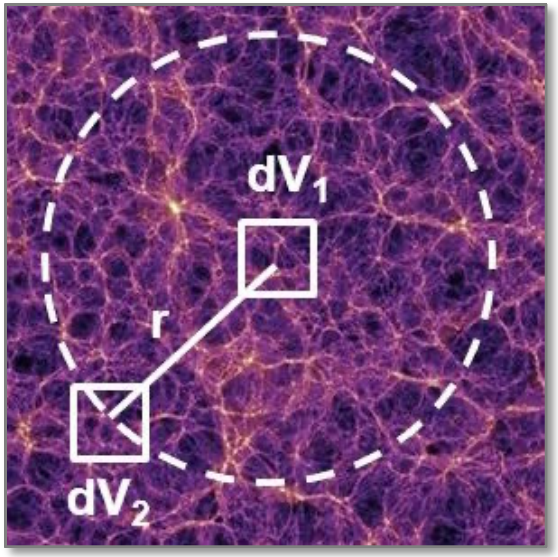
  - perturbation equation is linear => $\delta_{\vec{k}}(t)$ grow independently

  - first moment

    $\langle \delta(\vec{x},t) \rangle = 0$ => $\langle \delta_{\vec{k}}(t) \rangle = 0\,,\quad \delta_{\vec{k}}(t) = \delta_{-\vec{k}}(t)$

  - two-point correlation function (2nd moment)

$$\xi_2(\vec{x}) = \frac{n_{pair}(\vec{x}+d\vec{x})}{n_{random}(\vec{x}+d\vec{x})} - 1$$



- decomposition of $\delta(x,t)$ into waves: $$\delta\left(\vec{x},t\right)=\sum_{\vec{k}}\delta_{\vec{k}}(t)\,e^{i\vec{k}\cdot\vec{x}}$$

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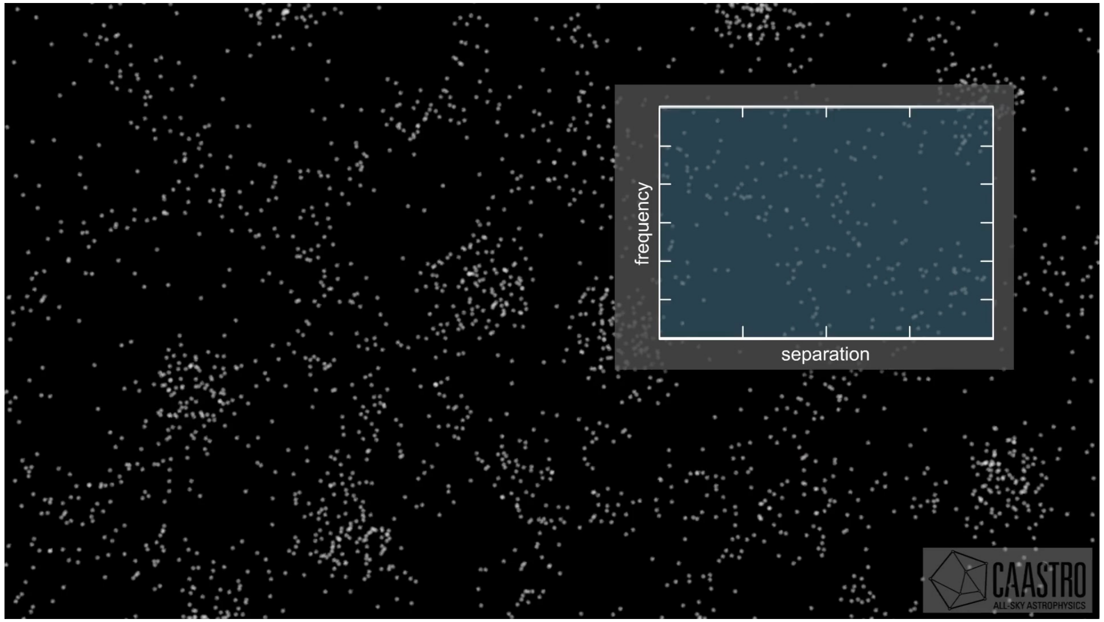

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  - two-point correlation function

$$\xi_2\left(\vec{x}\right) = \frac{n_{pair}(\vec{x}+d\vec{x})}{n_{random}(\vec{x}+d\vec{x})} - 1$$

  - power spectrum

$$\xi_2\left(\vec{x}\right) = \frac{1}{\left(2\pi\right)^3}\int P(k)\, e^{-i\vec{k}\cdot\vec{x}} d^3k = \frac{1}{2\pi^3}\int P(k)\,\frac{\sin(\vec{k}\cdot\vec{x})}{\vec{k}\cdot\vec{x}} k^2 dk$$

$$P(k) = \left\langle \left|\delta_{\vec{k}}\right|^2 \right\rangle_{\left|\vec{k}\right|=k}$$

- decomposition of $\delta(x,t)$ into waves: $$\delta(\vec{x},t) = \sum_{\vec{k}} \boxed{\delta_{\vec{k}}}(t)\, e^{i\vec{k}\cdot\vec{x}}$$

  - perturbation equation is linear => $\delta_{\vec{k}}(t)$ grow independently

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    $\langle \delta(\vec{x},t)\rangle = 0$ => $\langle \delta_{\vec{k}}(t)\rangle = 0\,,\quad \delta_{\vec{k}}(t) = \delta_{-\vec{k}}(t)$

  - two-point correlation function

$$\xi_2(\vec{x}) = \frac{n_{pair}(\vec{x}+d\vec{x})}{n_{random}(\vec{x}+d\vec{x})} - 1$$

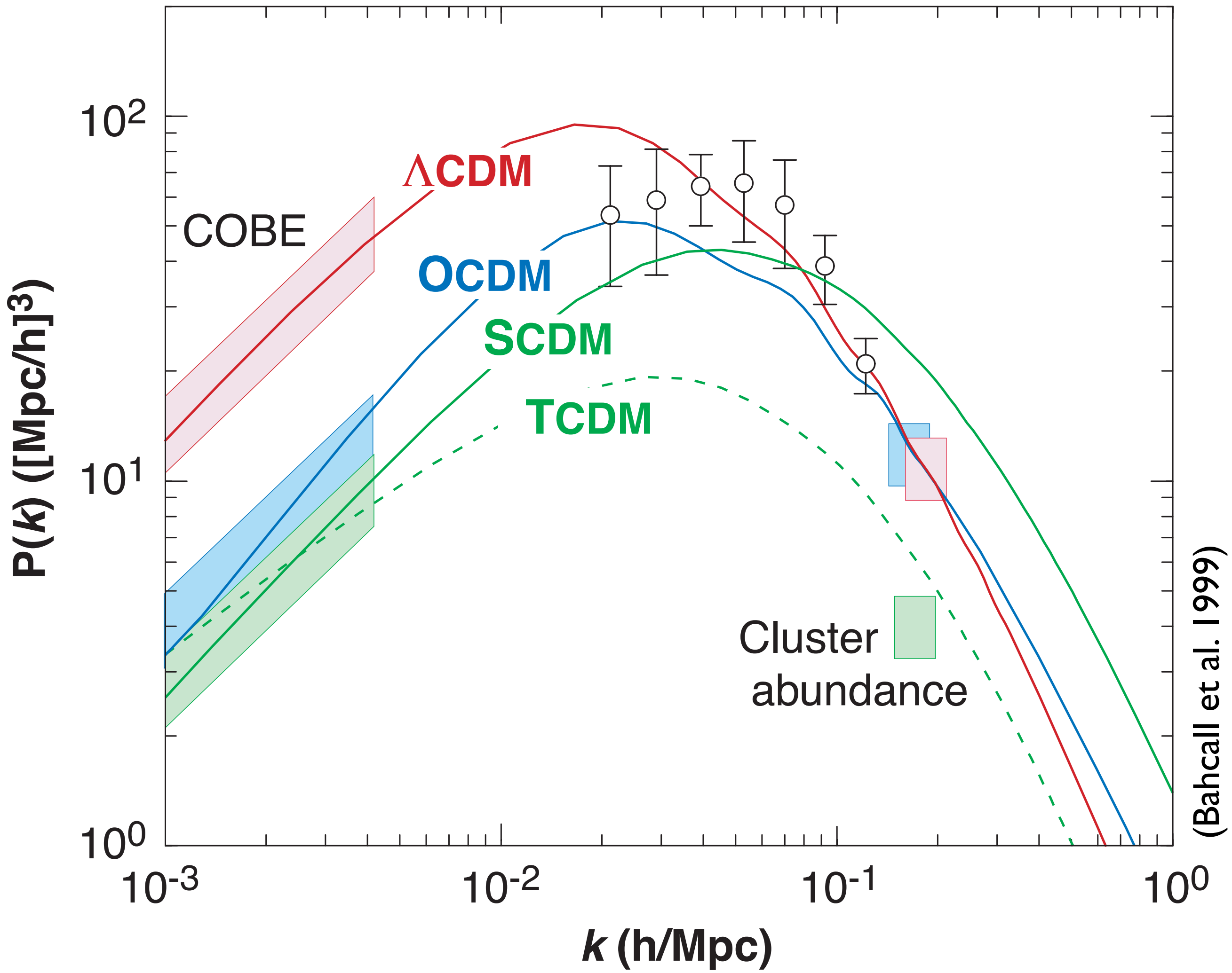
what is the initial shape?

  - power spectrum

$$\xi_2(\vec{x}) = \frac{1}{(2\pi)^3}\int P(k)\, e^{-i\vec{k}\cdot\vec{x}} d^3k = \frac{1}{2\pi^3}\int P(k)\,\frac{\sin(\vec{k}\cdot\vec{x})}{\vec{k}\cdot\vec{x}} k^2 dk$$

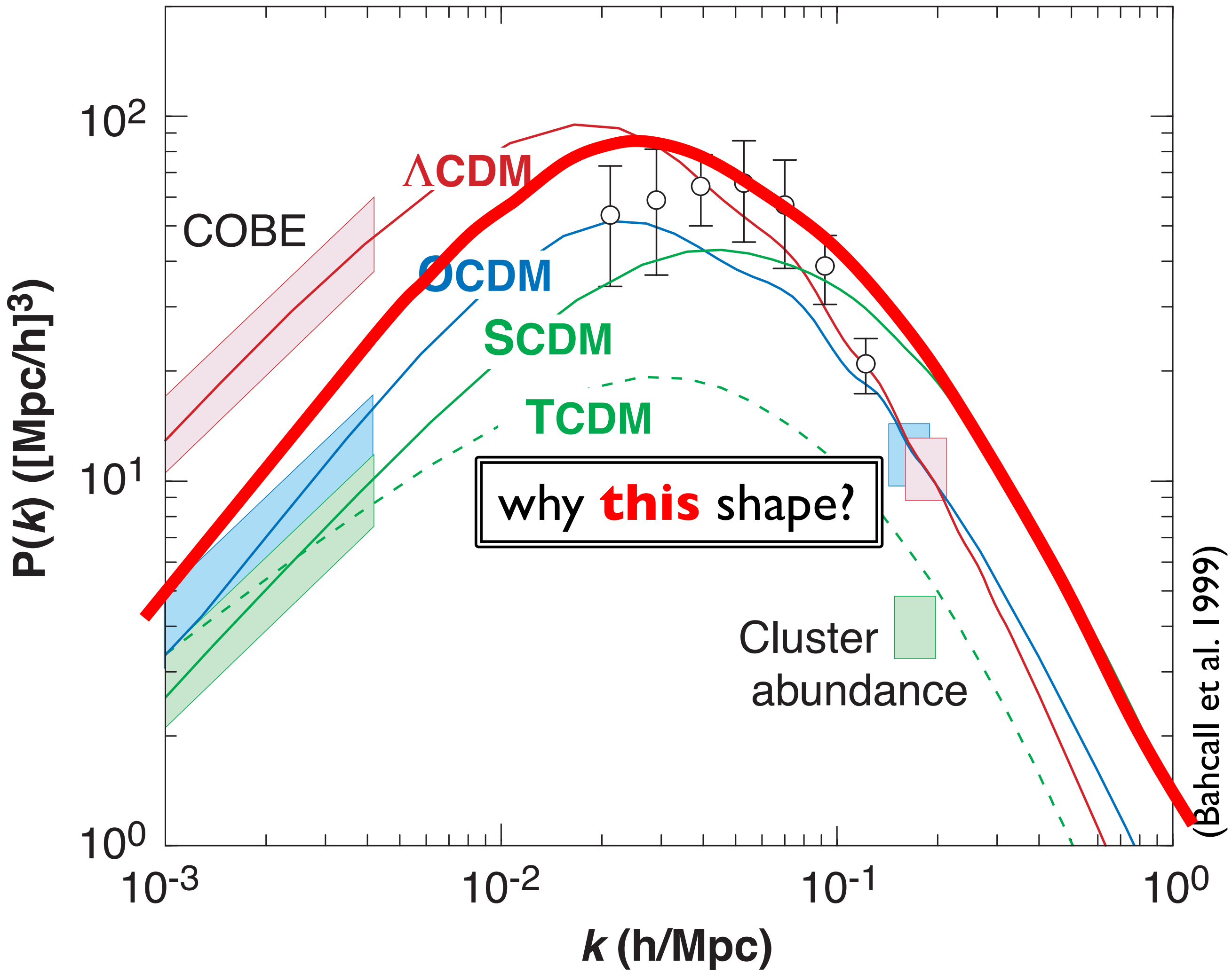
$$P(k) = \left\langle \boxed{\left|\delta_{\vec{k}}\right|}^2 \right\rangle_{|\vec{k}|=k}$$

- power spectrum $P(k)$ of perturbations

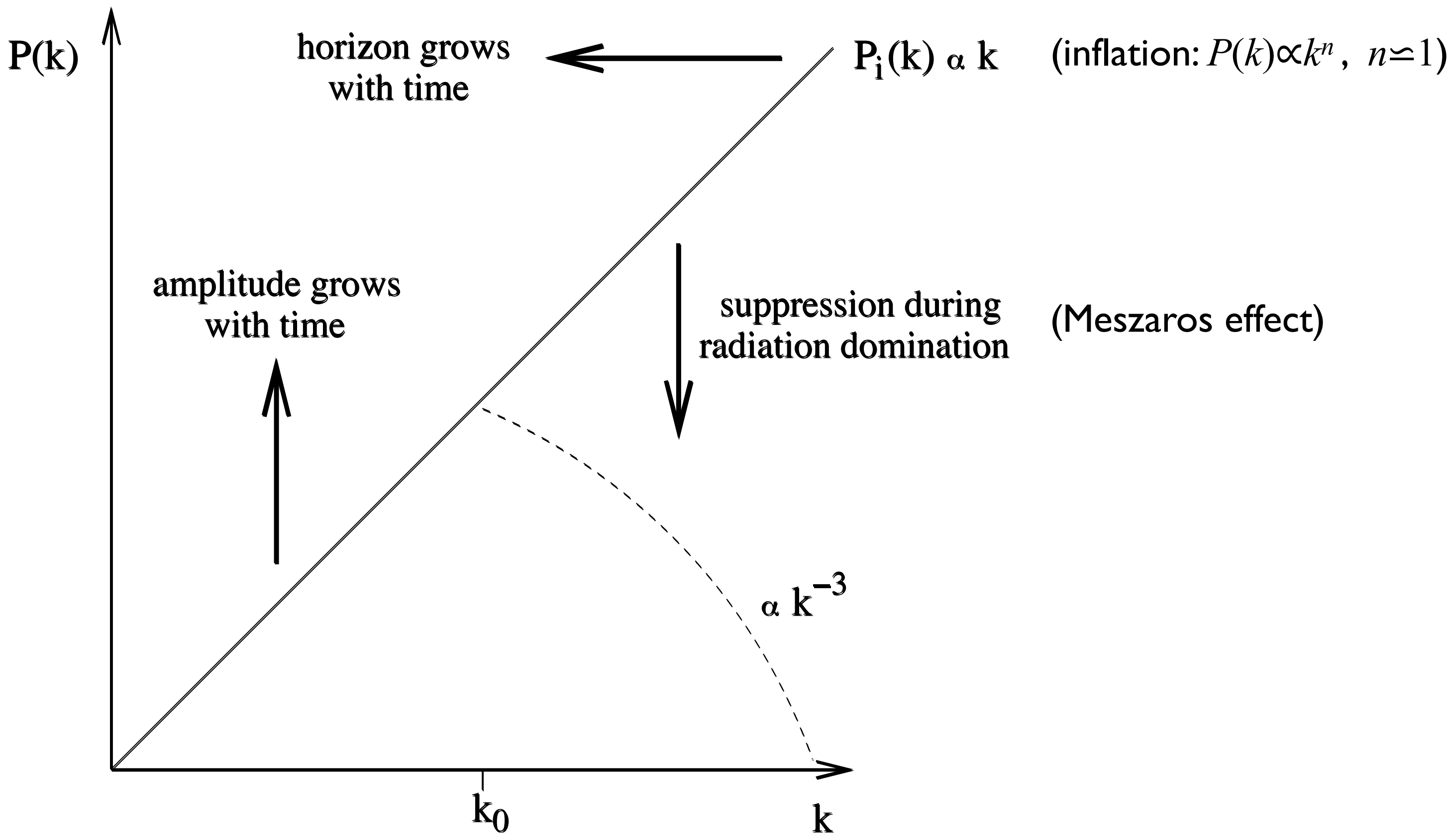


(Bahcall et al. 1999)

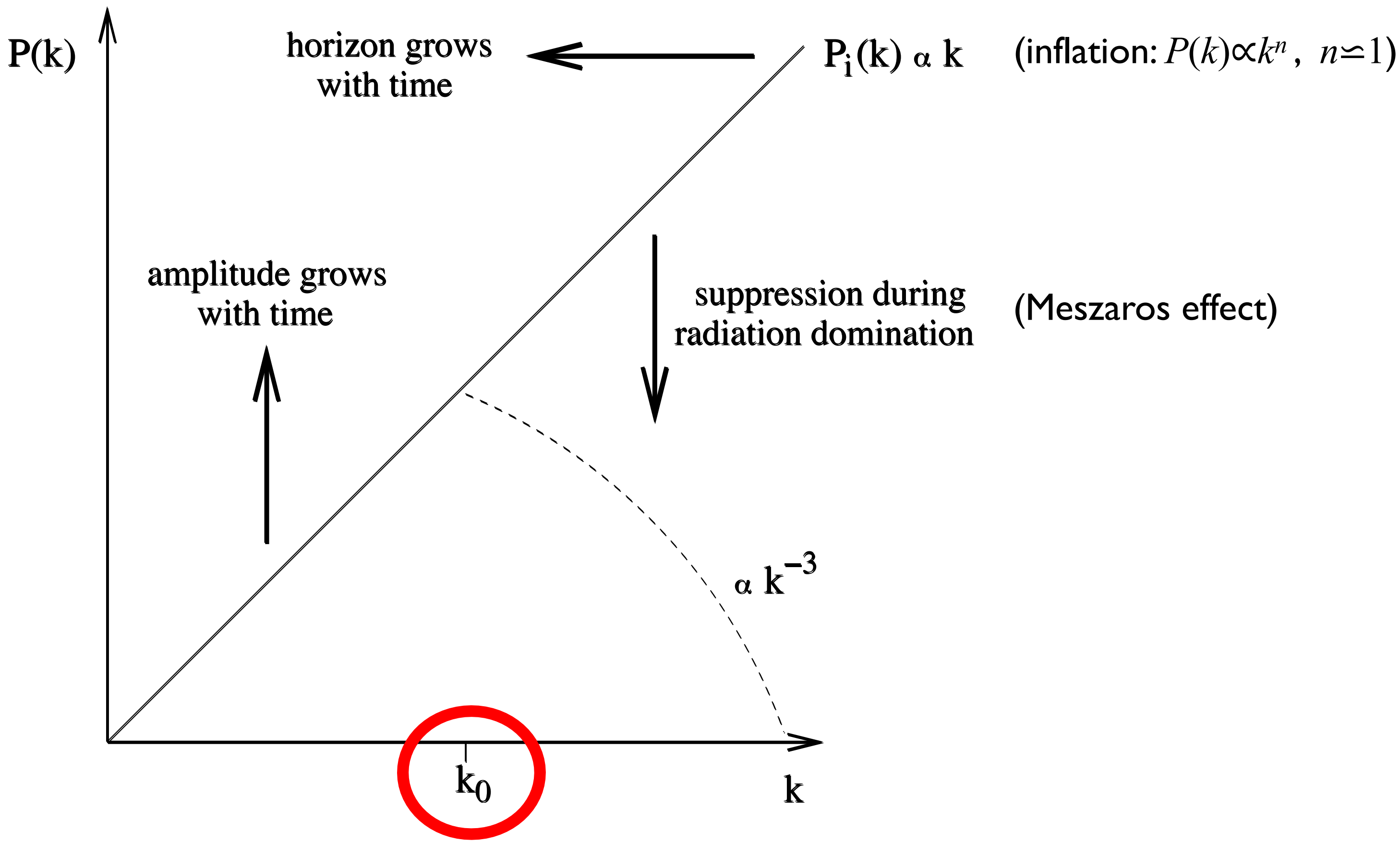
- power spectrum $P(k)$ of perturbations



- power spectrum $P(k)$ of perturbations – general shape



- power spectrum $P(k)$ of perturbations – general shape



what is this characteristic scale?

- power spectrum $P(k)$ of perturbations – growth during matter domination

$$\delta\left(\vec{x},a\right) = \sum_{\vec{k}} \delta_{\vec{k}}(a)\, e^{i\vec{k}\cdot\vec{x}}$$

- power spectrum $P(k)$ of perturbations – growth during matter domination

$$\delta\left(\vec{x},a\right)=\sum_{\vec{k}}\delta_{\vec{k}}(a)\,e^{i\vec{k}\cdot\vec{x}}$$

$$\delta_{\vec{k}}(a)=\frac{D(a)}{D(a_0)}\delta_{\vec{k}}(a_0)\quad;\quad 0=\frac{\partial^2 D}{\partial t^2}+2\frac{\dot{a}}{a}\frac{\partial D}{\partial t}-4\pi G\bar{\rho}D$$

- power spectrum $P(k)$ of perturbations – growth during matter domination

$$\delta\left(\vec{x},a\right)=\sum_{\vec{k}}\delta_{\vec{k}}(a)\,e^{i\vec{k}\cdot\vec{x}}$$

$$\delta_{\vec{k}}(a)=\frac{D(a)}{D(a_0)}\delta_{\vec{k}}(a_0) \quad ; \quad 0=\frac{\partial^2 D}{\partial t^2}+2\frac{\dot{a}}{a}\frac{\partial D}{\partial t}-4\pi G\bar{\rho}D$$

$$P(k)=\left\langle\left|\delta_{\vec{k}}\right|^2\right\rangle_{\left|\vec{k}\right|=k} \quad \Rightarrow \quad P(k)=\left(\frac{D(a)}{D(a_0)}\right)^2 P_0(k)$$

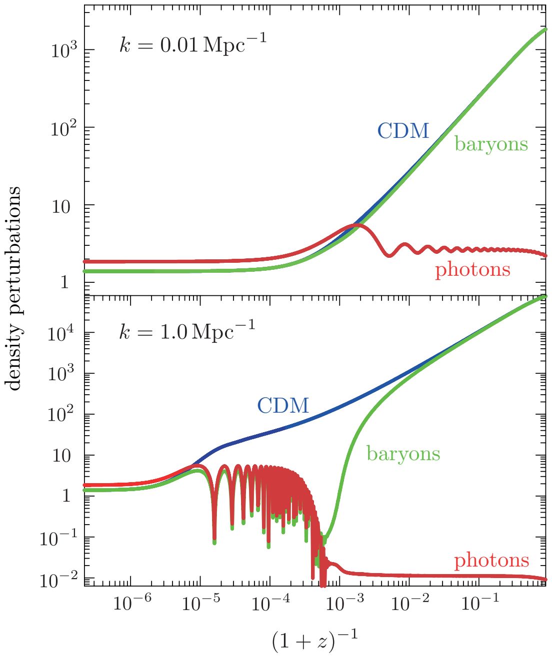
- power spectrum $P(k)$ of perturbations – growth during matter domination

$$\delta(\vec{x},a) = \sum_{\vec{k}} \delta_{\vec{k}}(a)\, e^{i\vec{k}\cdot\vec{x}}$$

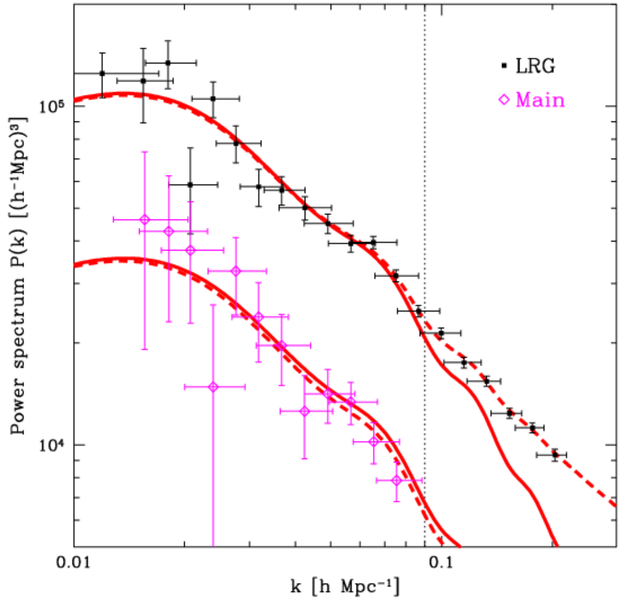
$$\delta_{\vec{k}}(a) = \frac{D(a)}{D(a_0)} \delta_{\vec{k}}(a_0) \quad ; \quad 0 = \frac{\partial^2 D}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial D}{\partial t} - 4\pi G\bar{\rho}D$$

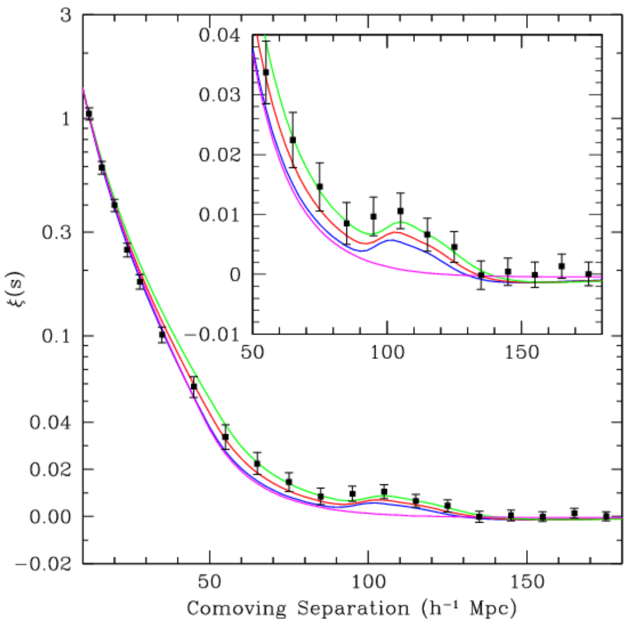
$$P(k) = \left\langle \left|\delta_{\vec{k}}\right|^2 \right\rangle_{|\vec{k}|=k} \quad \Rightarrow \quad \boxed{P(k) = \left(\frac{D(a)}{D(a_0)}\right)^2 P_0(k)}$$

- power spectrum $P(k)$ of perturbations



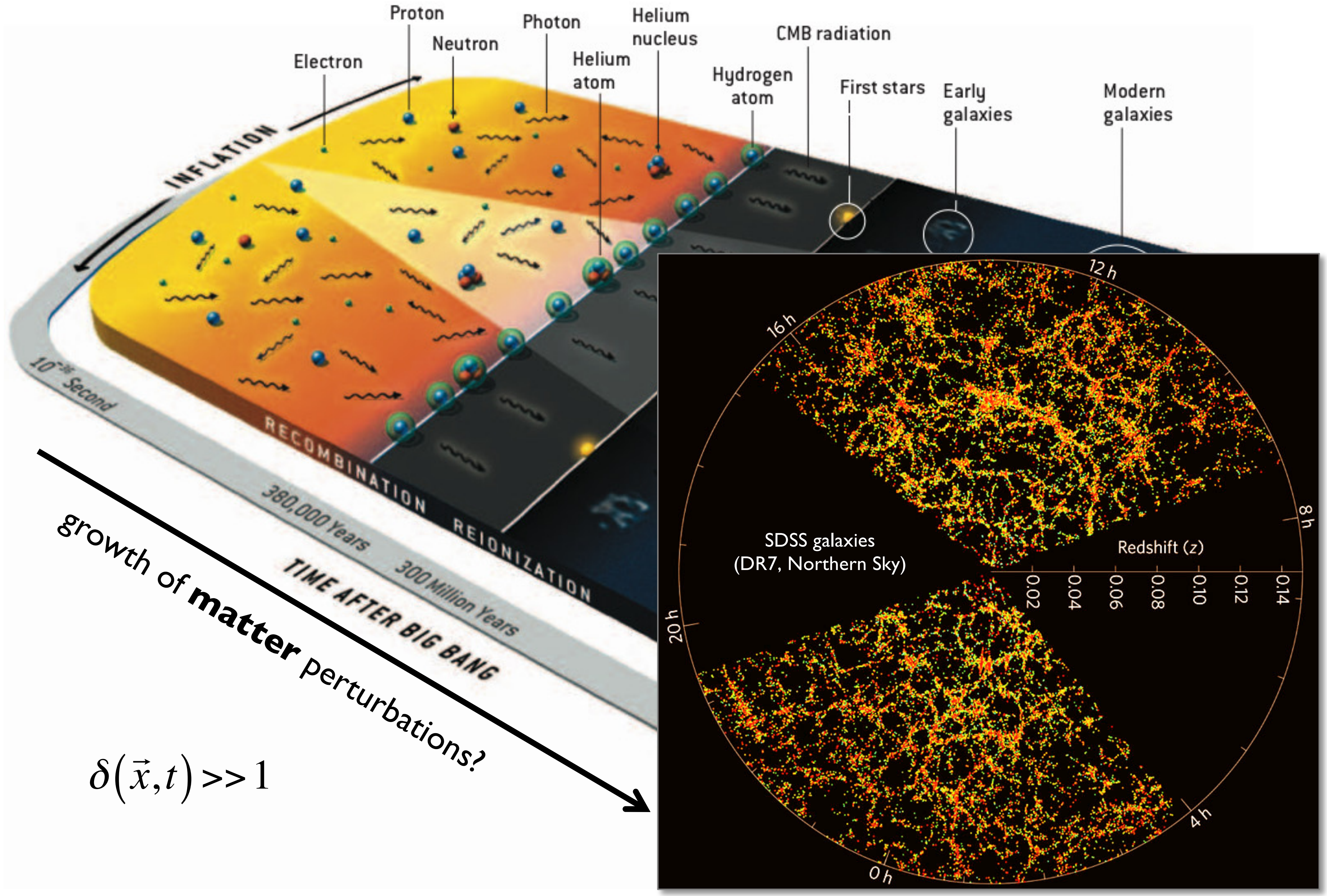
oscillations leave
distinct feature in $P(k)$ and $\xi(x)$





(Basset & Hlozek, arXiv:0910.5224)

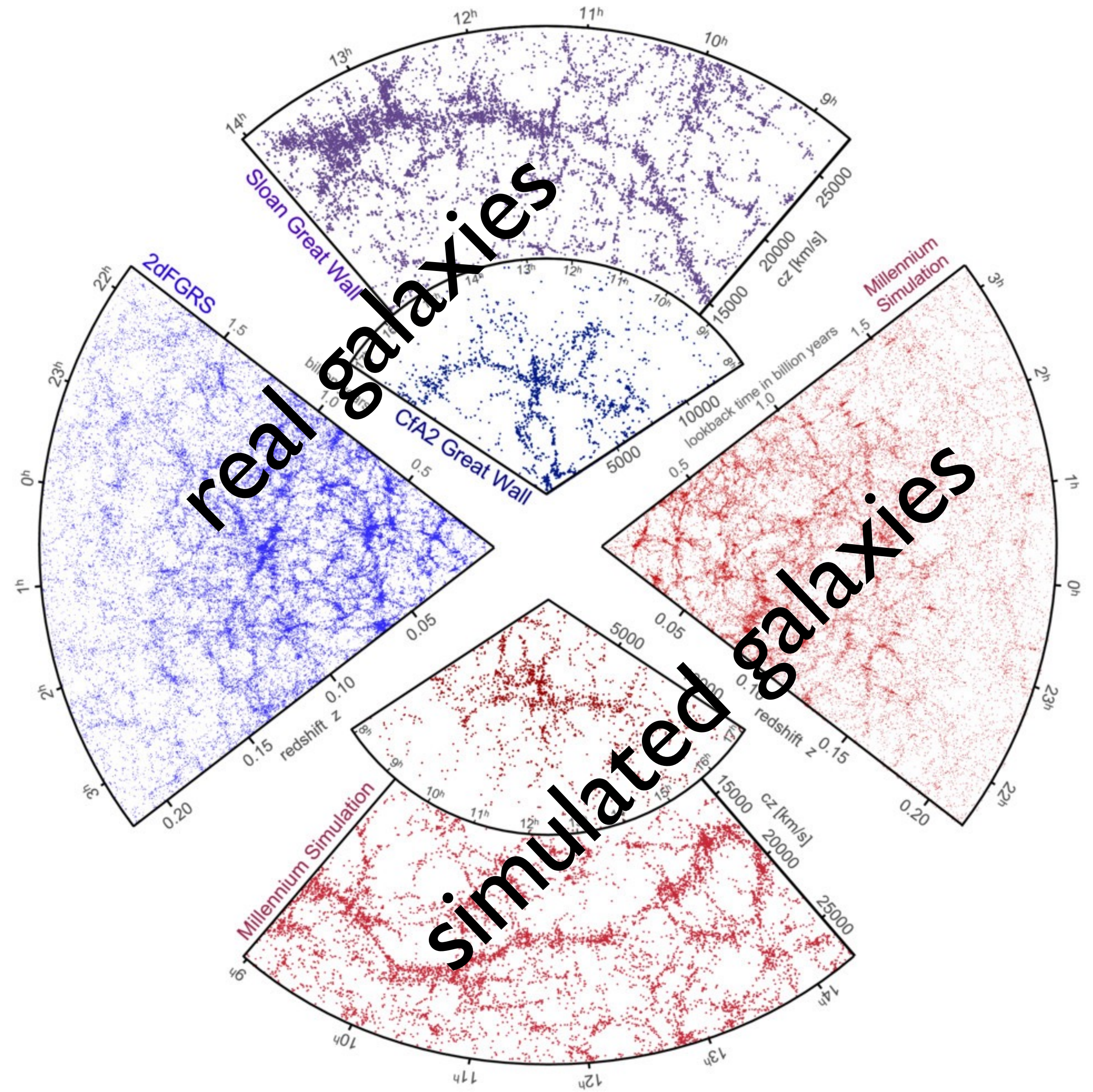
- governing equations
- growth of matter perturbations
- statistics of perturbations
- **non-linear structure formation**



growth of **matter** perturbations?

$$\delta\left(\vec{x},t\right) >> 1$$

- territory of computational cosmology…



- territory of computational cosmology…

- …but powerful, analytical (quasi-linear) approaches exist, too:

    - Zel'dovich approximation (1^{st} order Lagrangian perturbation theory)
    - Spherical Top-Hat Collapse
    - Press-Schechter halo mass function
    - …

- territory of computational cosmology…

- …but powerful, analytical (quasi-linear) approaches exist, too:

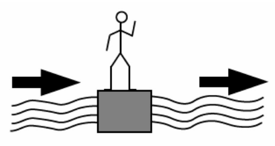
  - **Zel'dovich approximation** (1st order Lagrangian perturbation theory)
  - Spherical Top-Hat Collapse
  - Press-Schechter halo mass function
  - …

- Zel'dovich approximation

$$\vec{x}(t) = \vec{q} + D(t)\vec{S}(\vec{q})$$

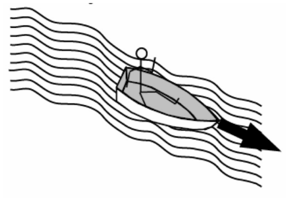
• Lagrangian viewpoint: (we follow mass elements…)

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} - 4\pi G\bar{\rho}\delta - \frac{c_s^2}{a^2}\Delta\delta = 0$$

• Eulerian viewpoint:  (we measure the flow…)

- Zel'dovich approximation*

$$\vec{x}(t) = \vec{q} + D(t)\vec{S}(\vec{q})$$

• Lagrangian viewpoint:  (we follow mass elements…)

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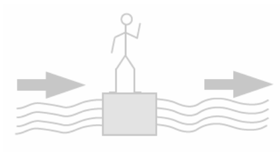
• Eulerian viewpoint:  (we measure the flow…)

*1st order Lagrangian perturbation theory (as opposed to Eulerian treatment from previous slides…)

- Zel'dovich approximation

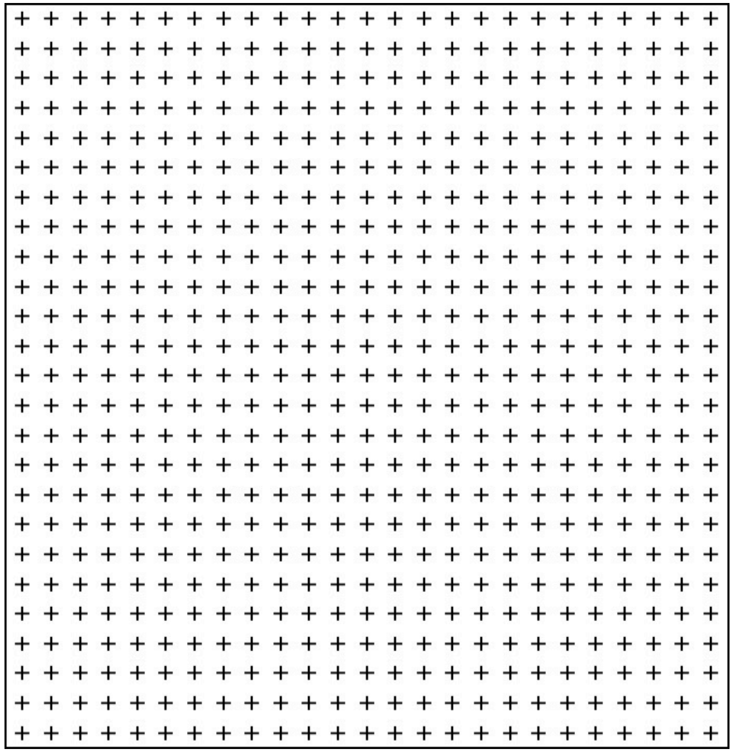
$$\vec{x}(t) = \vec{q} + D(t)\vec{S}(\vec{q})$$

? ? ?

- Zel'dovich approximation

$$\vec{x}(t) = \vec{q} + D(t)\vec{S}(\vec{q})$$

initial (unperturbed!) position



- Zel'dovich approximation

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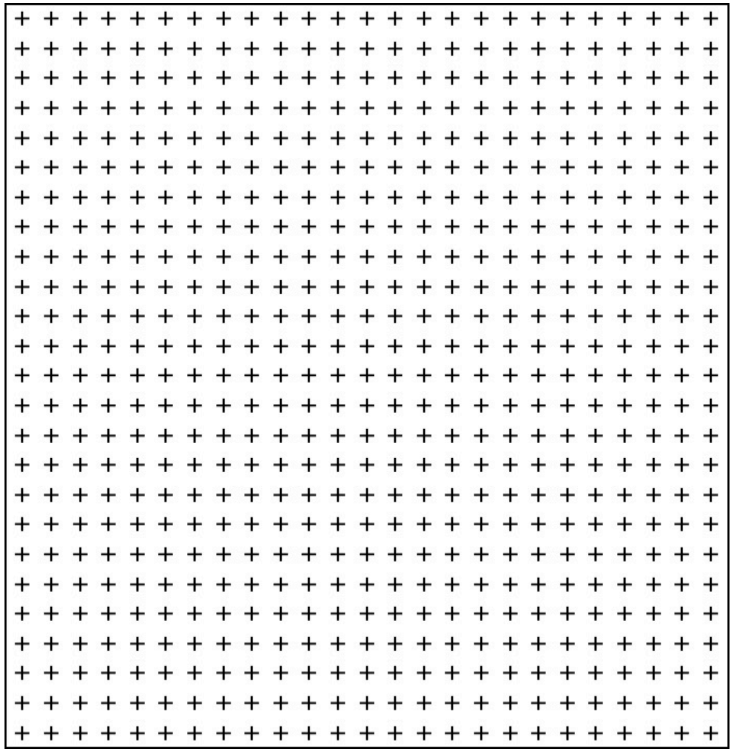
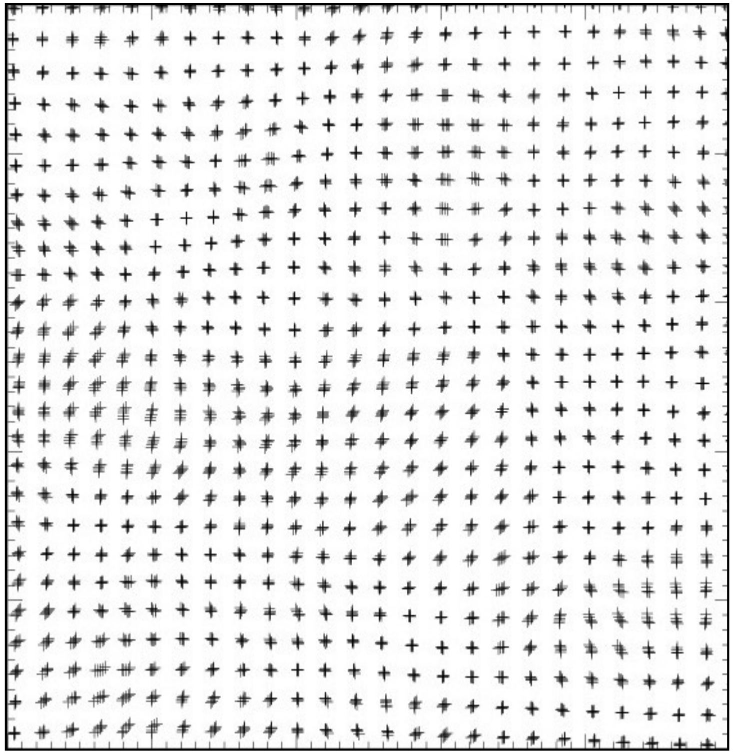
initial (unperturbed!) position



perturbations



- Zel'dovich approximation

$$\vec{x}(t) = \vec{q} + \boxed{D(t)}\vec{S}(\vec{q})$$

$$\delta(\vec{x}, t) = D(t)\delta_0(\vec{x})$$

$$0 = \frac{\partial^2 D}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial D}{\partial t} - 4\pi G\bar{\rho}D$$

$\delta_0(\vec{x})$ (initial perturbations)

- Zel'dovich approximation

$$\vec{x}(t) = \vec{q} + D(t)\boxed{\vec{S}(\vec{q})}$$

$\vec{S}(\vec{q})$? (displacement field)

- Zel'dovich approximation

$$\vec{x}(t) = \vec{q} + D(t)\vec{S}(\vec{q})$$

$\dot{\vec{x}} = \dot{D}\,\vec{S}(\vec{q})$ derivative of ZA

$\dot{\vec{x}} = \frac{1}{a}\vec{u}$ definition of peculiar velocity field

- Zel'dovich approximation

$$\vec{x}(t) = \vec{q} + D(t)\vec{S}(\vec{q})$$

$$\left.\begin{aligned} \dot{\vec{x}} &= \dot{D}\,\vec{S}(\vec{q}) \\ \dot{\vec{x}} &= \frac{1}{a}\vec{u} \end{aligned}\right\} \qquad \vec{u} = \dot{D}a\,\vec{S}(\vec{q})$$

- Zel'dovich approximation

$$\vec{x}(t) = \vec{q} + D(t)\vec{S}(\vec{q})$$

$$\left.\begin{aligned} \dot{\vec{x}} &= \dot{D}\,\vec{S}(\vec{q}) \\ \dot{\vec{x}} &= \frac{1}{a}\vec{u} \end{aligned}\right\} \qquad \begin{aligned} \vec{u} &= \dot{D}a\,\vec{S}(\vec{q}) \\ \frac{\partial \vec{u}}{\partial t} &= \dot{a}\dot{D}\vec{S}(\vec{q}) + a\ddot{D}\vec{S}(\vec{q}) \end{aligned}$$

- Zel'dovich approximation

$$\vec{x}(t) = \vec{q} + D(t)\vec{S}(\vec{q})$$

$$\left.\begin{aligned}\dot{\vec{x}} &= \dot{D}\,\vec{S}(\vec{q})\\ \dot{\vec{x}} &= \frac{1}{a}\vec{u}\end{aligned}\right\} \quad \left.\begin{aligned}\vec{u} &= \dot{D}a\,\vec{S}(\vec{q})\\ \frac{\partial\vec{u}}{\partial t} &= \dot{a}\dot{D}\vec{S}(\vec{q}) + a\ddot{D}\vec{S}(\vec{q})\end{aligned}\right\} \quad \Rightarrow \quad (2a\dot{a}\dot{D} + a^2\ddot{D})\vec{S}(\vec{q}) = -\nabla\Phi$$

$$\frac{\partial\vec{u}}{\partial t} + \frac{\dot{a}}{a}\vec{u} = -\frac{1}{a}\nabla\Phi$$

conservation of momentum
according to Eulerian perturbation theory (see above)

- Zel'dovich approximation

$$\vec{x}(t) = \vec{q} + D(t)\vec{S}(\vec{q})$$

$$(2a\dot{a}\dot{D} + a^2\ddot{D})\vec{S}(\vec{q}) = -\nabla\Phi$$

=>

$$0 = \ddot{D} + 2\frac{\dot{a}}{a}\dot{D} - 4\pi G\bar{\rho}D$$

$$\Delta\Phi = 4\pi G a^2\bar{\rho}\delta$$

$$\Phi = 4\pi G a^2\bar{\rho}\Psi$$

$$\vec{S}(\vec{q}) = -\nabla\Psi$$

$$\Delta\Psi = \delta_0(\vec{x})$$

- Zel'dovich approximation

$$\vec{x}(t) = \vec{q} + D(t)\vec{S}(\vec{q})$$

$$(2a\dot{a}\dot{D} + a^2\ddot{D})\vec{S}(\vec{q}) = -\nabla\Phi$$

=>

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$$\Delta\Phi = 4\pi G a^2\bar{\rho}\delta$$

$$\Phi = 4\pi G a^2\bar{\rho}\Psi$$

$$\vec{S}(\vec{q}) = -\nabla\Psi$$

$$\Delta\Psi = \delta_0(\vec{x})$$

Ψ = "peculiar potential",
sourced by initial perturbations

- Zel'dovich approximation

$$\vec{x}(a) = \vec{q} - D(a)\nabla\Psi$$

$$D(a) = \frac{5}{2}\Omega_{m,0}H\int_0^a \frac{1}{\left(\Omega_{m,0}a^{-3} + (1-\Omega_{m,0}-\Omega_{\Lambda,0})a^{-2} + \Omega_{\Lambda,0}\right)}da$$

$$\Delta\Psi = \delta_0(\vec{x})$$

- Zel'dovich approximation

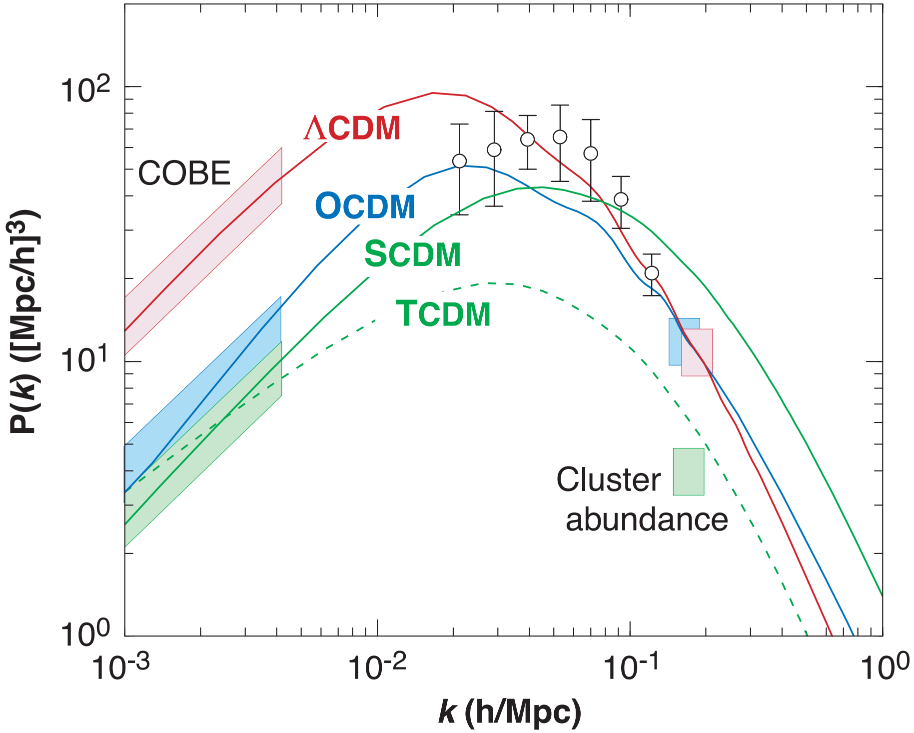
$$\vec{x}(a) = \vec{q} - D(a)\nabla\Psi$$

$$D(a) = \frac{5}{2}\Omega_{m,0} H \int_0^a \frac{1}{\left(\Omega_{m,0}a^{-3} + (1-\Omega_{m,0}-\Omega_{\Lambda,0})a^{-2} + \Omega_{\Lambda,0}\right)} da$$

$$\Delta\Psi = \delta_0(\vec{x})$$

statistical representation of $P(k)$

$$\hat{\delta}_0\,(k) = \boxed{\sqrt{P_0(k)}} R_{\vec{k}}\, e^{i\varphi_{\vec{k}}}$$



- Zel'dovich approximation

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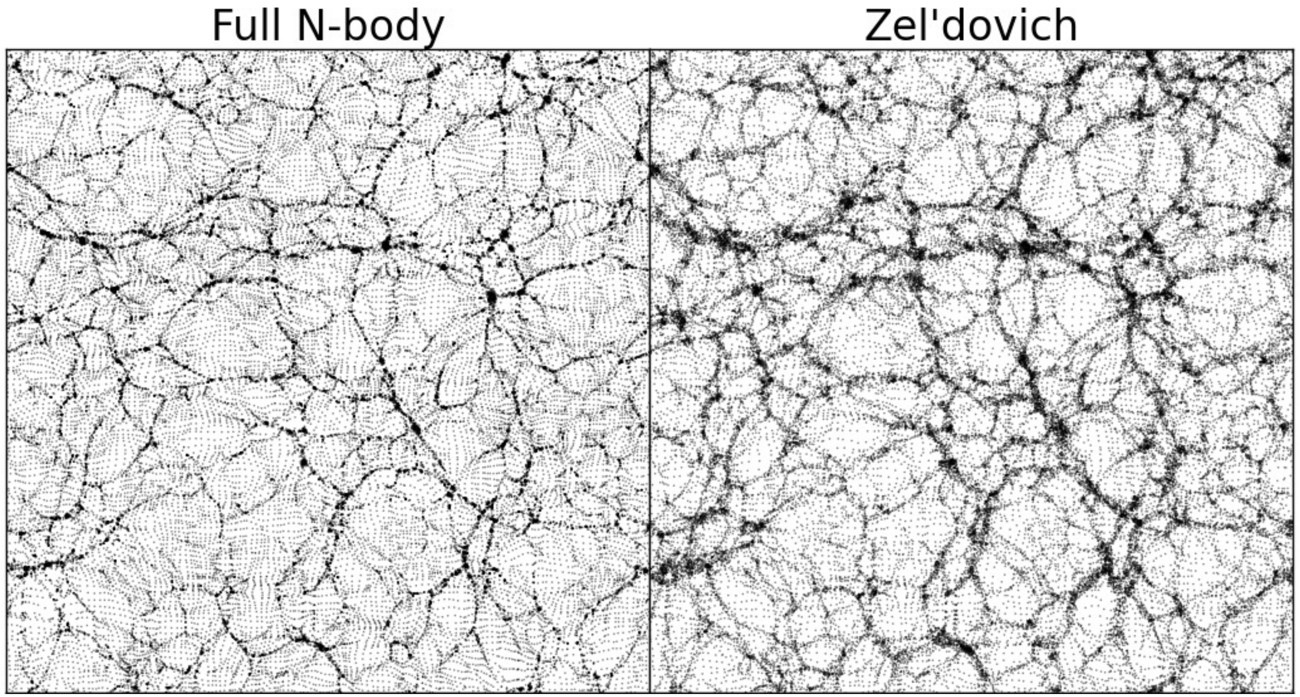
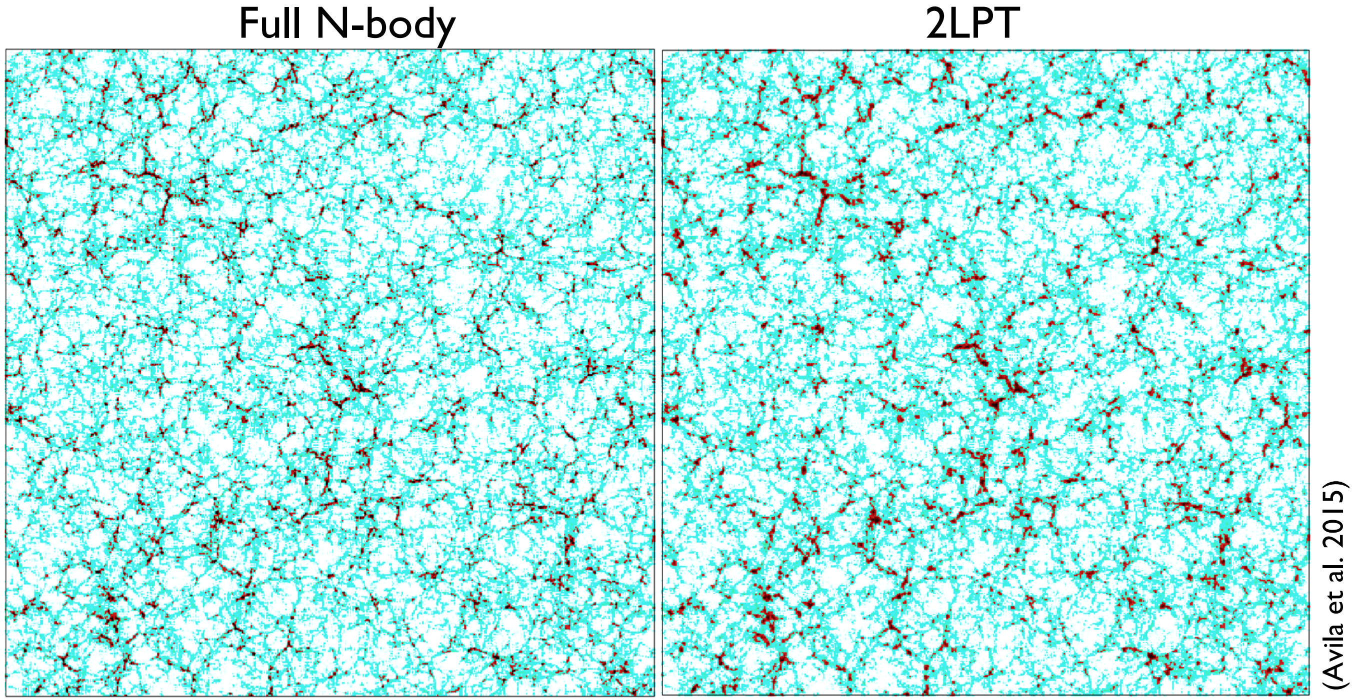


$z = 0$

(Neyrinck , arXiv:1204.1326)

- 2^{nd} order Lagrangian perturbation theory

$$\vec{x}(a) = \vec{q} - D(a)\nabla\Psi + D^{(2)}\nabla\Psi^{(2)}$$



$z = 0.5$

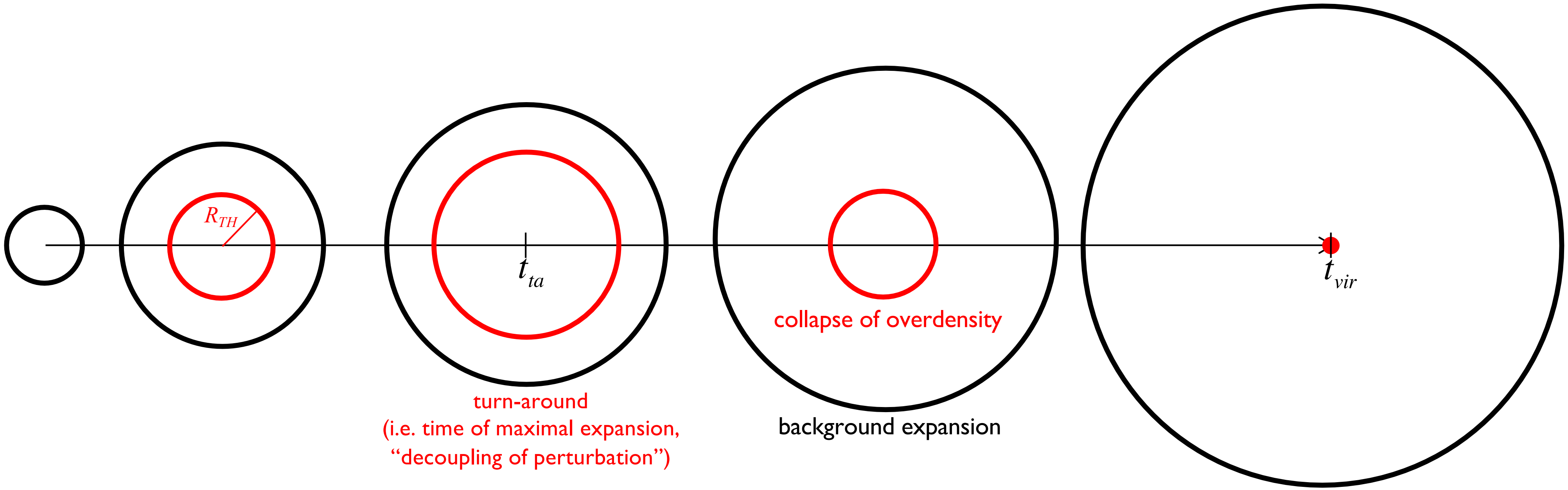
- territory of computational cosmology…

- …but powerful, analytical (quasi-linear) approaches exist, too:

    - Zel'dovich approximation (1^{st} order Lagrangian perturbation theory)
    - **Spherical Top-Hat Collapse**
    - Press-Schechter halo mass function
    - …

▪ Spherical Top-Hat Collapse

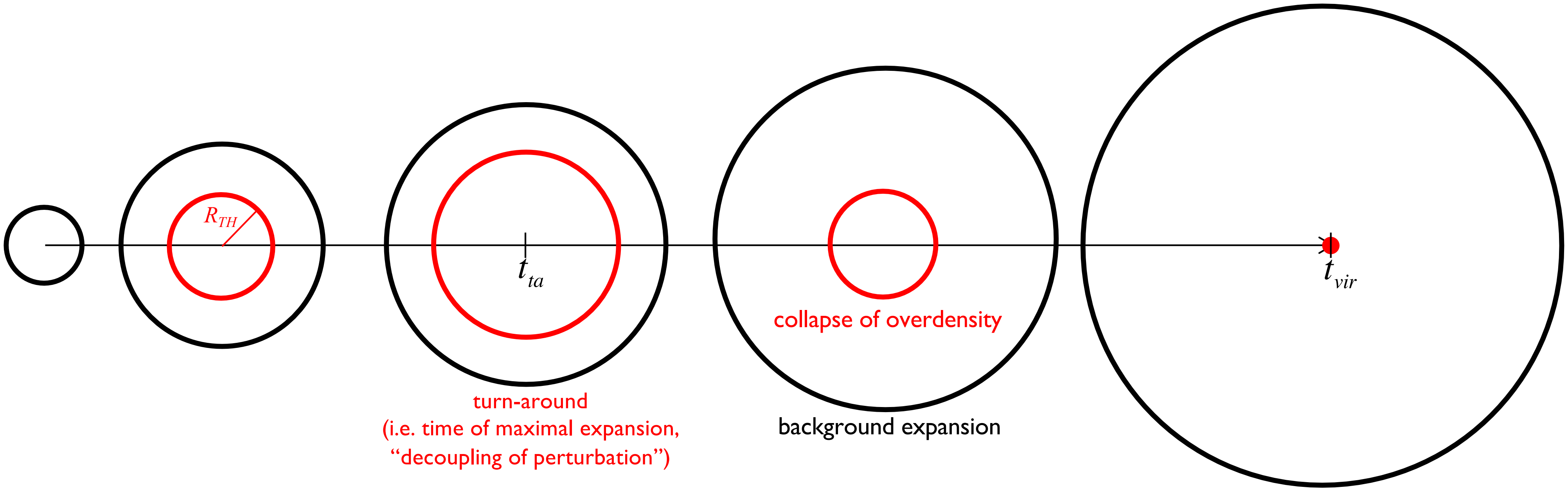
- spherical (top-hat) overdensity $R_{TH} = R_{TH}(t)\,,\, M_{TH} = const. \quad \Rightarrow \quad \delta_{TH}\,?$



▪ Spherical Top-Hat Collapse

- spherical (top-hat) overdensity $R_{TH} = R_{TH}(t)\,,\, M_{TH} = const. \quad \Rightarrow \quad \delta_{TH}\,?$

treat perturbation as 'closed universe-in-universe' with k=1

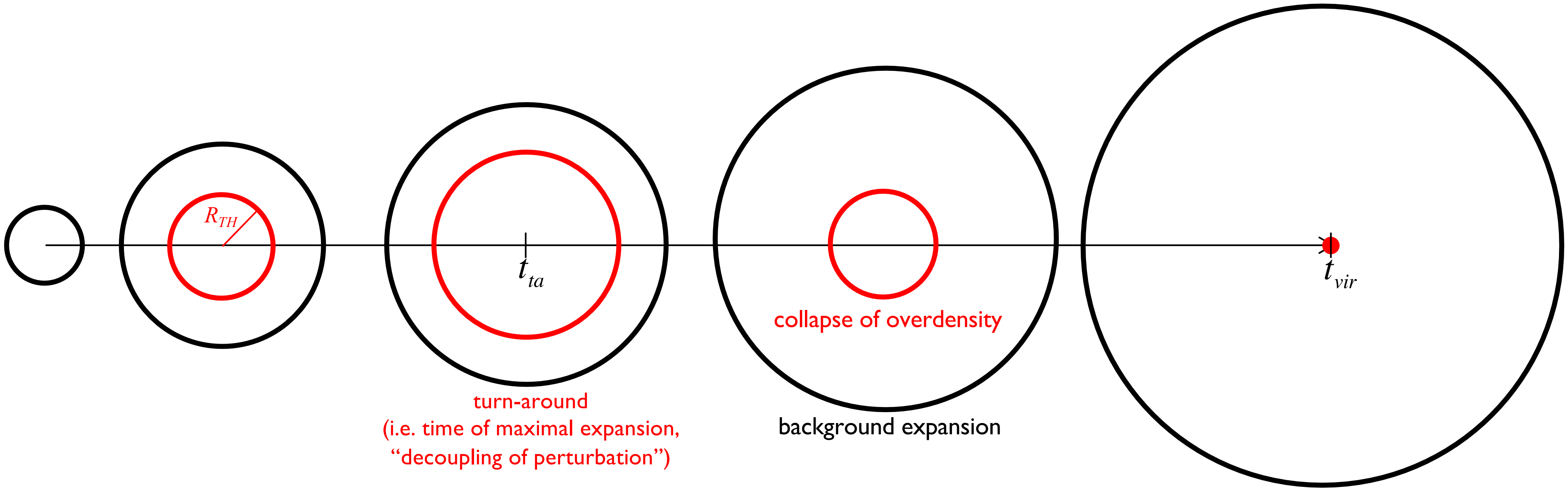


▪ Spherical Top-Hat Collapse

- spherical (top-hat) overdensity $R_{TH} = R_{TH}(t)\,,\, M_{TH} = const. \quad \Rightarrow \quad \delta_{TH}\,?$

- Friedmann equation

$$\dot{R}_{TH}^2 = \frac{8\pi G}{3}\rho_{TH}R_{TH}^2 - kc^2 = \frac{8\pi G}{3}\frac{M_{TH}}{\frac{4\pi}{3}R_{TH}^3}R_{TH}^2 - kc^2 = \frac{2GM_{TH}}{R_{TH}} - kc^2$$



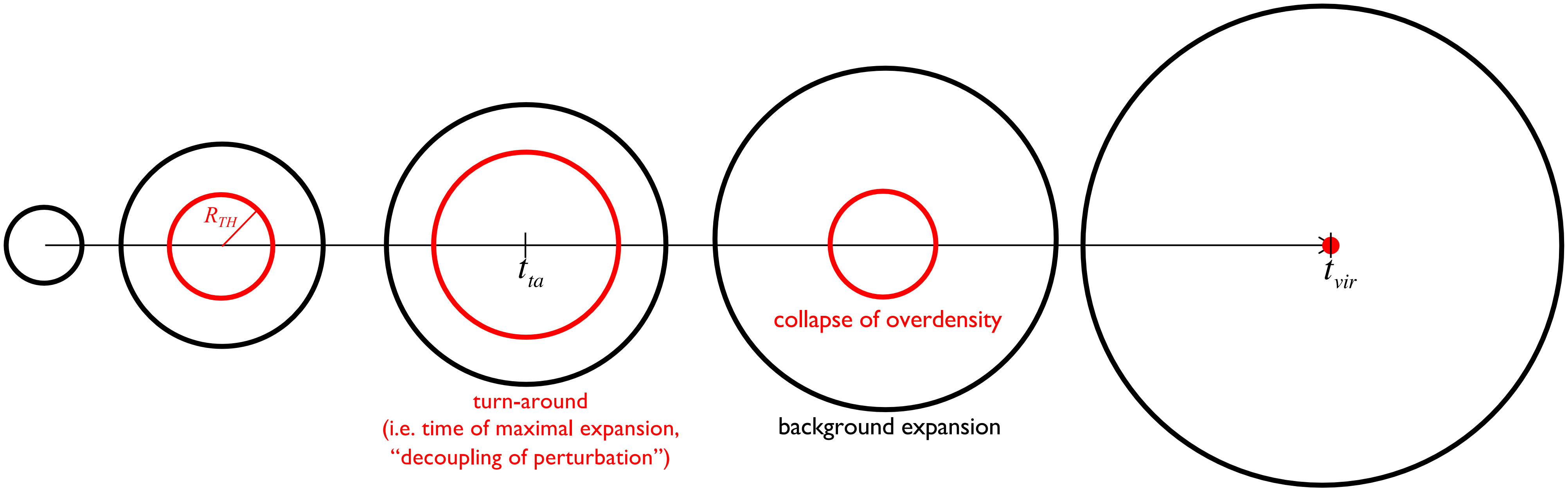
▪ Spherical Top-Hat Collapse

- spherical (top-hat) overdensity $R_{TH} = R_{TH}(t)\,,\, M_{TH} = const. \quad \Rightarrow \quad \delta_{TH}\,?$

- Friedmann equation

$$\frac{1}{2}\dot{R}_{TH}^2 - \frac{GM_{TH}}{R_{TH}} = -|k|c^2 < 0$$

(k=1 for top-hat overdensity! => closed Universe)



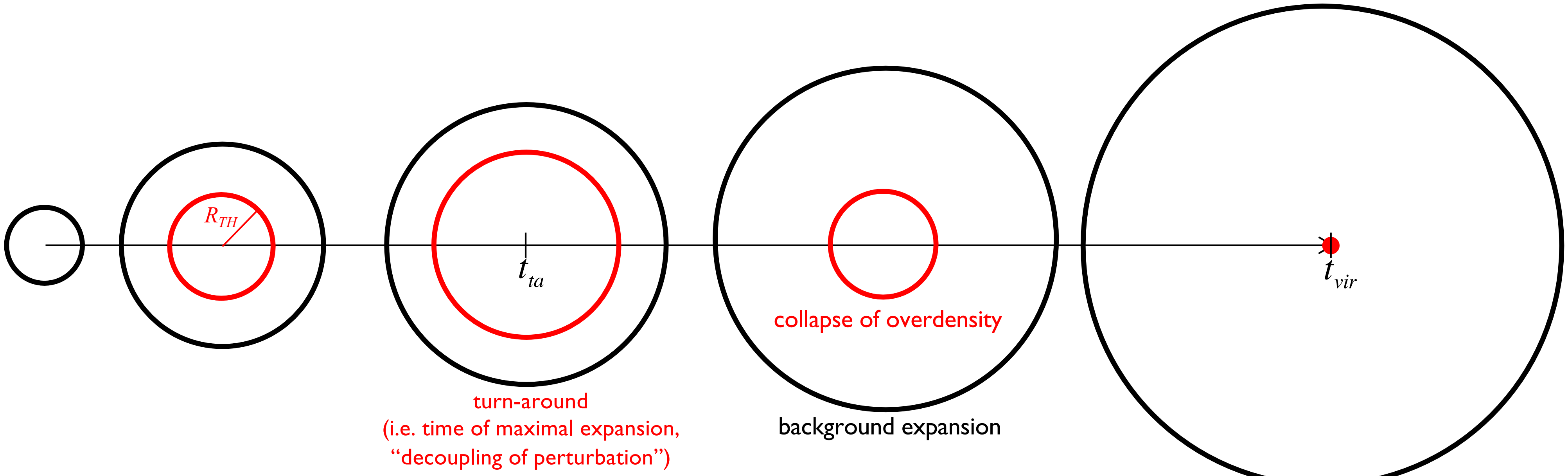
▪ Spherical Top-Hat Collapse

- spherical (top-hat) overdensity $R_{TH} = R_{TH}(t)\,, M_{TH} = const. \quad \Rightarrow \quad \delta_{TH}\,?$

- Friedmann equation

$$\frac{1}{2}\dot{R}_{TH}^2 - \frac{GM_{TH}}{R_{TH}} = -|k|c^2 < 0 \quad \overset{\text{parametric solution (cf. FRW lecture)}}{\Longrightarrow} \quad \frac{R_{TH}}{R_{ta}} = \frac{1}{2}(1-\cos\eta) \qquad R_{ta} = \frac{2GM_{TH}}{c^2}$$

$$\frac{t}{t_{ta}} = \frac{1}{\pi}(\eta - \sin\eta) \qquad t_{ta} = \frac{\pi R_{ta}}{2c} \qquad \eta \in [0, 2\pi]$$



Spherical Top-Hat Collapse

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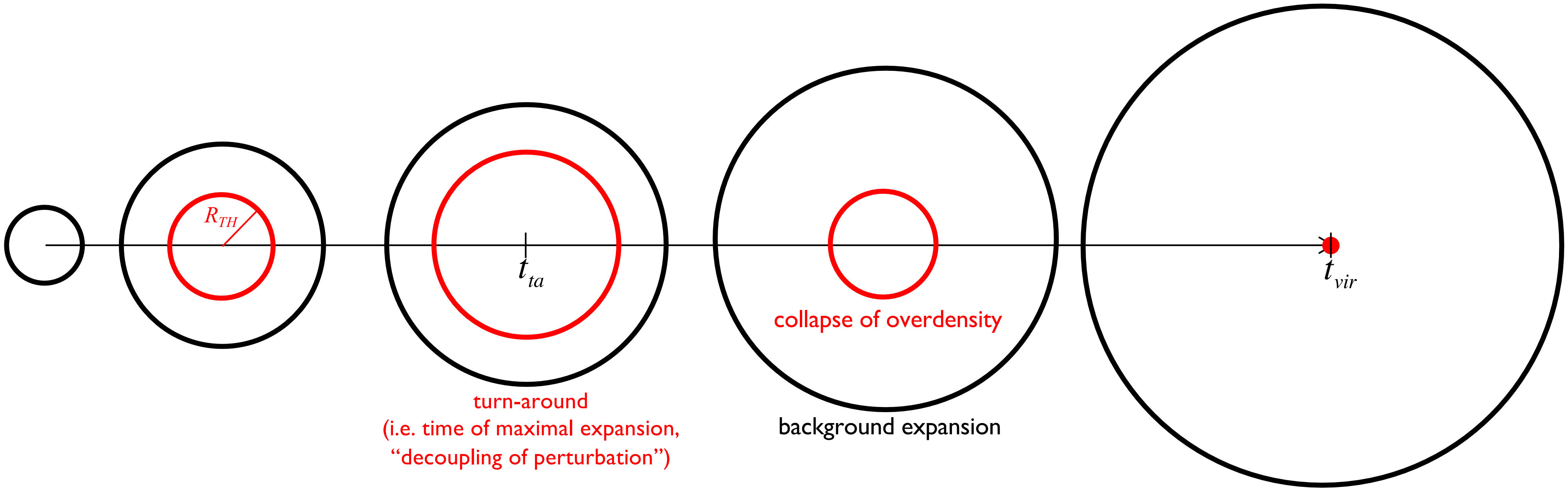

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- spherical (top-hat) overdensity $R_{TH} = R_{TH}(t)\,,\, M_{TH} = const. \quad \Rightarrow \quad \delta_{TH} = \dfrac{\rho_{TH} - \bar{\rho}}{\bar{\rho}}$

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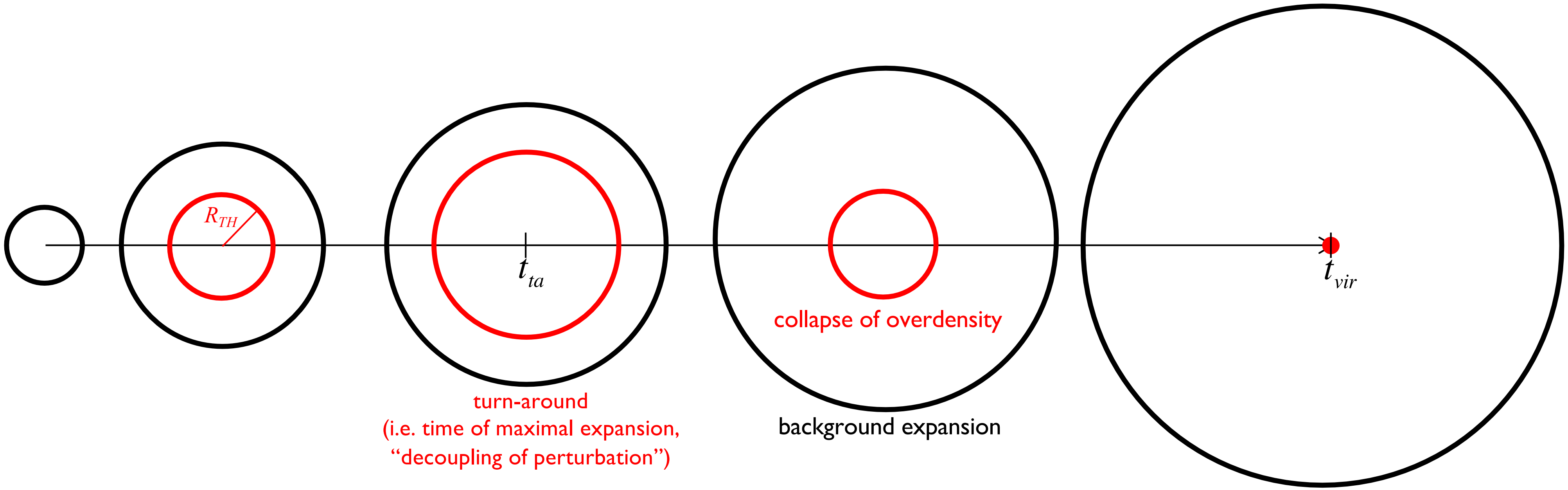


▪ Spherical Top-Hat Collapse

- spherical (top-hat) overdensity $R_{TH} = R_{TH}(t)\,,\, M_{TH} = const.$ $\Rightarrow$ $\delta_{TH} = \dfrac{\rho_{TH} - \bar{\rho}}{\bar{\rho}}$

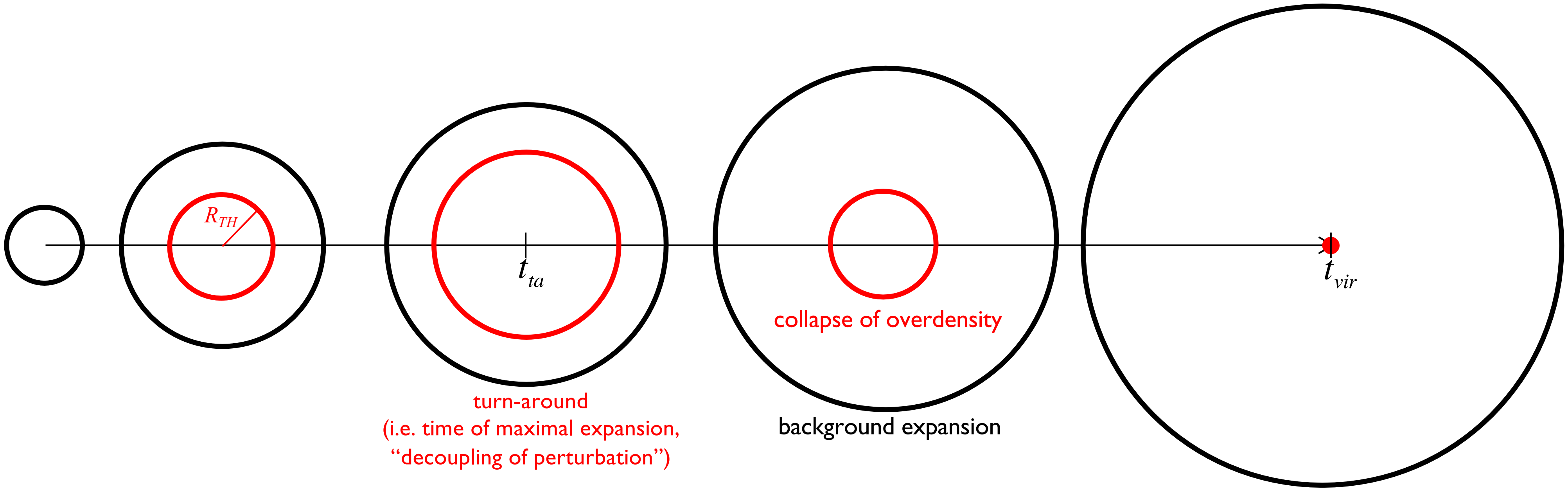
(definition)

- parametric solution

$$\frac{R_{TH}}{R_{ta}} = \frac{1}{2}(1 - \cos\eta) \qquad R_{ta} = \frac{2GM_{TH}}{c^2} \qquad \rho_{TH} = \frac{3M_{TH}}{4\pi R_{TH}^3}$$

(background = flat model with $\Omega_m = 1$)

$$\frac{t}{t_{ta}} = \frac{1}{\pi}(\eta - \sin\eta) \qquad t_{ta} = \frac{\pi R_{ta}}{2c} \qquad \bar{\rho} = \frac{1}{6\pi G t^2}$$



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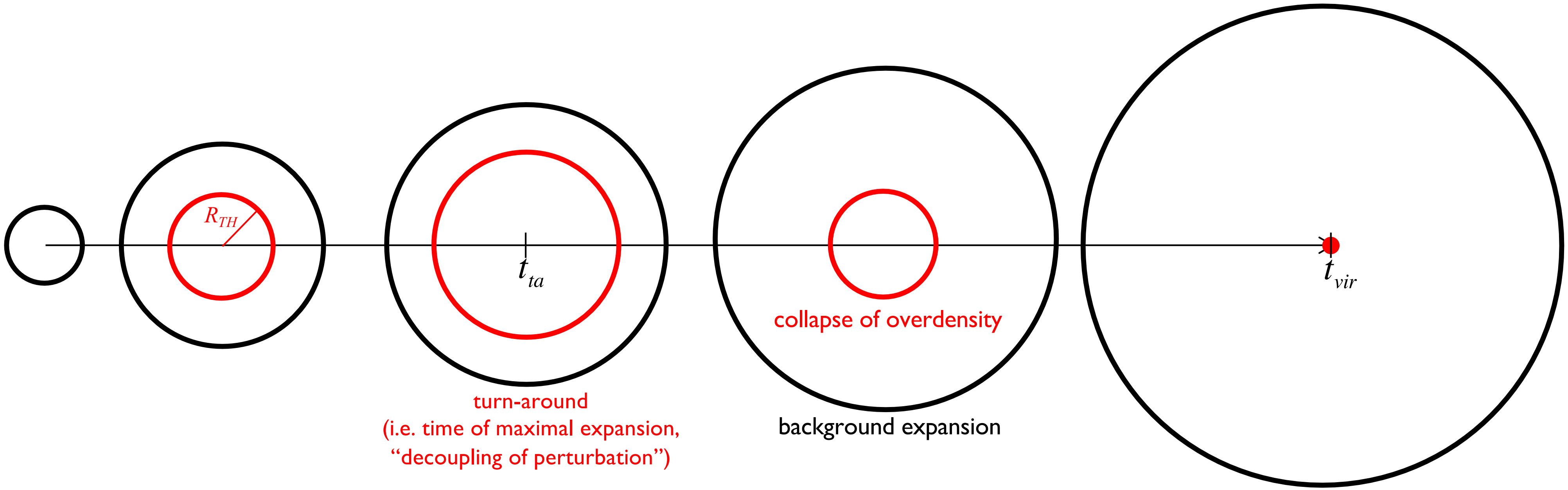

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- parametric solution

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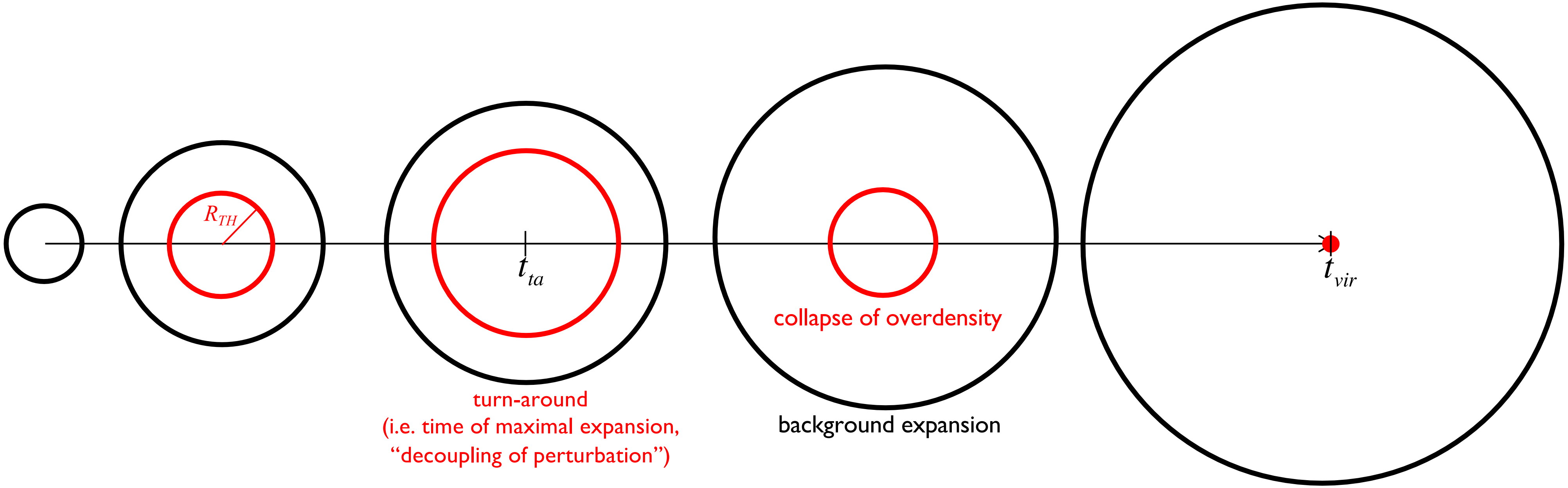
$$\Rightarrow \quad \rho_{TH} = \frac{3M_{TH}}{4\pi R_{TH}^3} = \ldots = \frac{6M_{TH}}{\pi R_{ta}^3}(1-\cos\eta)^{-3}$$

$$\bar\rho = \frac{1}{6\pi G t^2} = \ldots = \frac{\pi^2}{6\pi G t_{ta}^2}(\eta - \sin\eta)^{-2}$$



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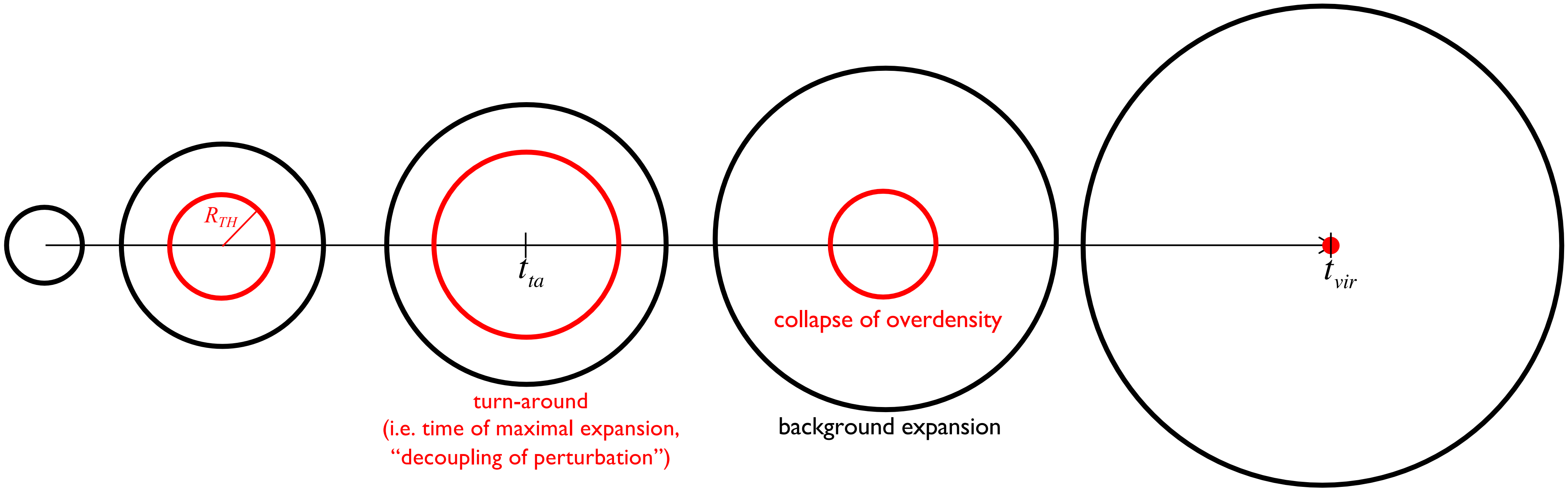

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$$\frac{t}{t_{ta}} = \frac{1}{\pi}(\eta-\sin\eta) \qquad t_{ta} = \frac{\pi R_{ta}}{2c}$$

$$\Rightarrow \quad 1+\delta_{TH} = \frac{\rho_{TH}}{\bar{\rho}} = \ldots = \frac{9}{2}\frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3}$$

<u>proof:</u>

$$\frac{\rho_{TH}}{\bar{\rho}} = \frac{6M_{TH}}{\pi R_{ta}^3}\frac{6\pi G t_{ta}^2}{\pi^2}\frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3} = \frac{t_{ta}^2}{R_{ta}^3}\frac{36GM_{TH}}{\pi^2}\frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3} = \frac{\pi^2 R_{ta}^2}{4c^2 R_{ta}^3}\frac{36GM_{TH}}{\pi^2}\frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3}$$

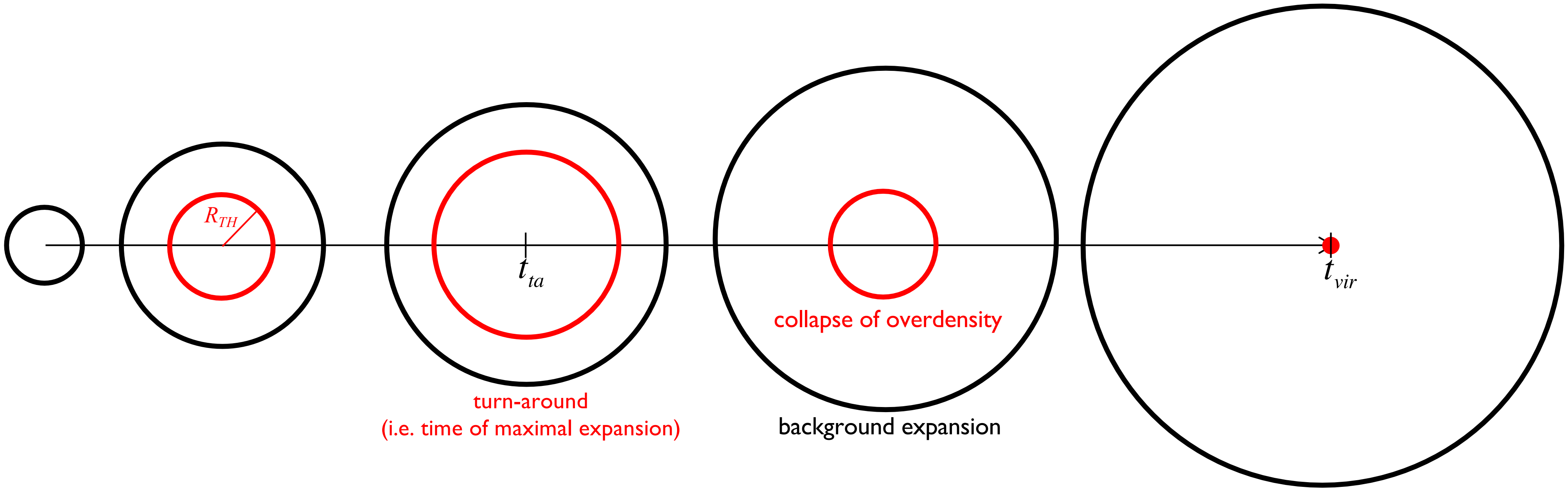
$$= \frac{1}{R_{ta}}\frac{36GM_{TH}}{4c^2}\frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3} = \frac{c^2}{2GM_{TH}}\frac{9GM_{TH}}{c^2}\frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3} = \frac{9}{2}\frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3}$$

▪ Spherical Top-Hat Collapse

- spherical (top-hat) overdensity $R_{TH} = R_{TH}(t)\,,\, M_{TH} = const.$

- solution for collapsed overdensity

$$1+\delta_{TH}(\eta) = \frac{9}{2}\frac{\left(\eta - \sin\eta\right)^2}{\left(1-\cos\eta\right)^3}$$



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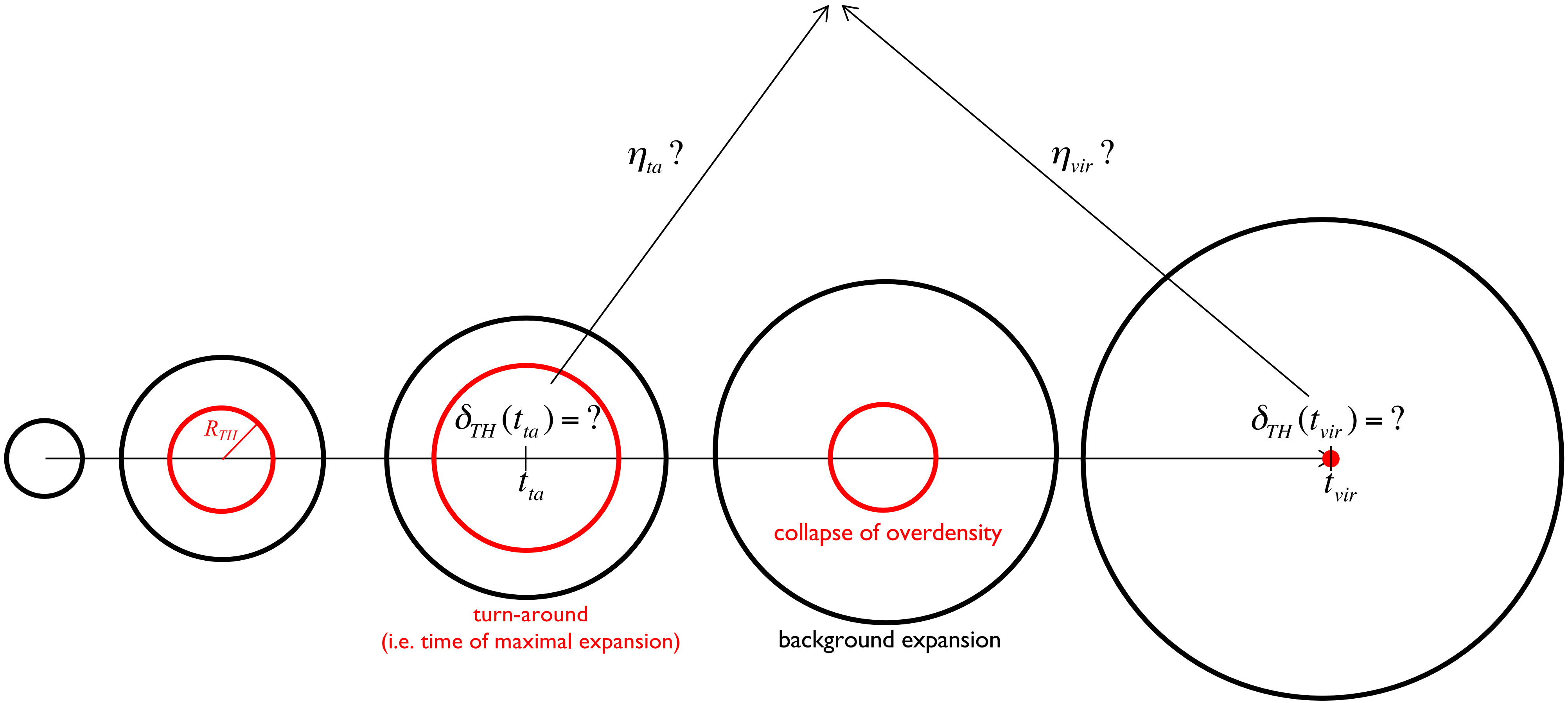


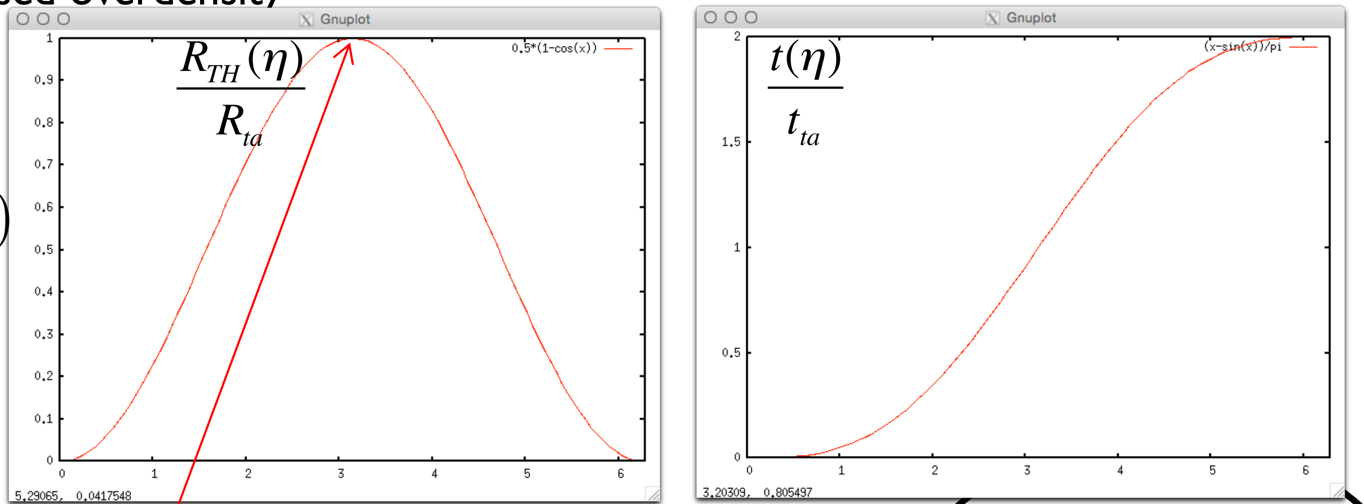
▪ Spherical Top-Hat Collapse

- spherical (top-hat) overdensity $R_{TH} = R_{TH}(t)\,,\; M_{TH} = const.$

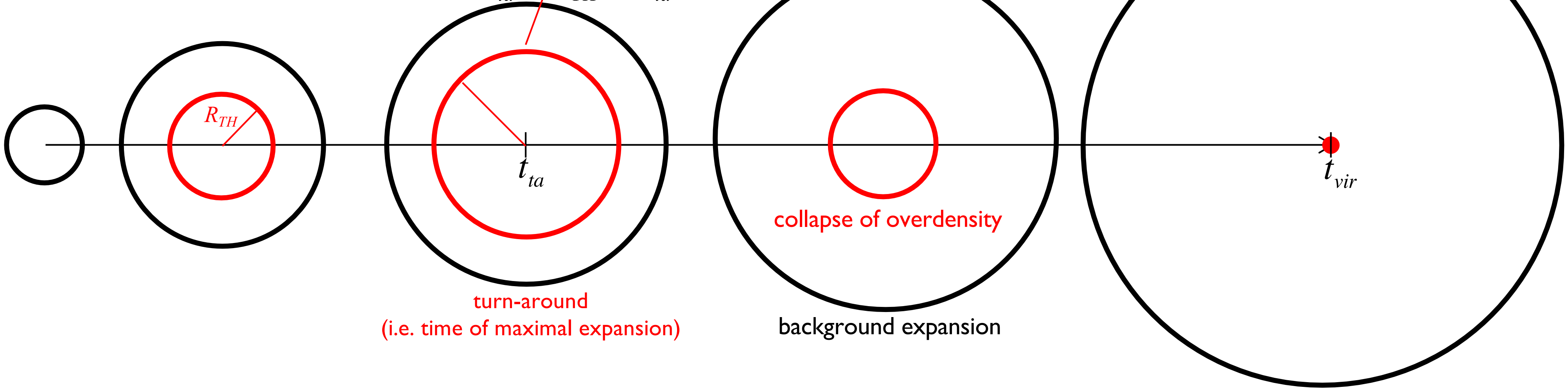
- solution for collapsed overdensity

$$\frac{R_{TH}}{R_{ta}} = \frac{1}{2}(1-\cos\eta)$$

$$\frac{t}{t_{ta}} = \frac{1}{\pi}(\eta - \sin\eta)$$



$R_{ta} = R_{TH} \equiv \eta_{ta} = \pi$



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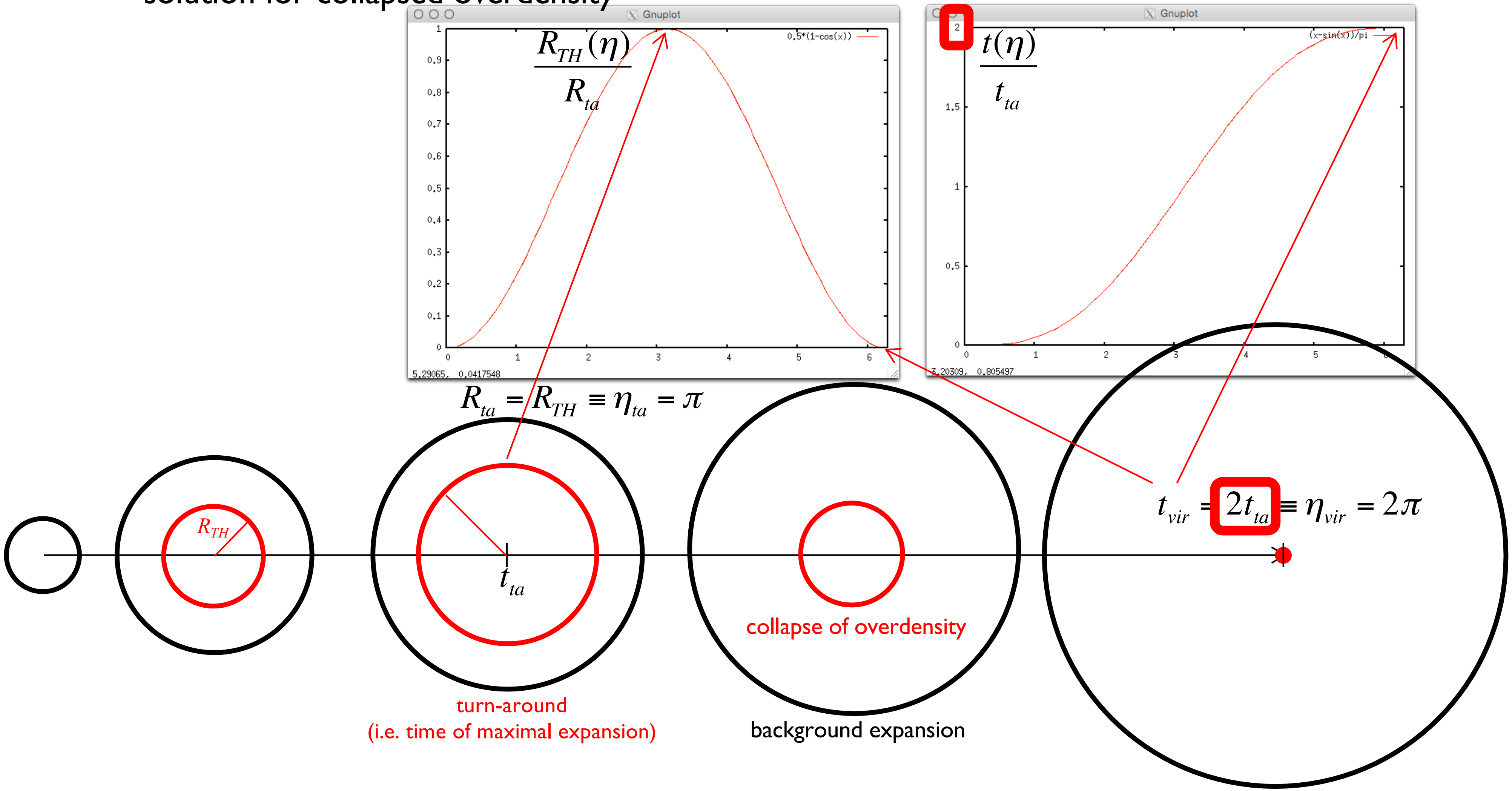
- solution for collapsed overdensity



▪ Spherical Top-Hat Collapse

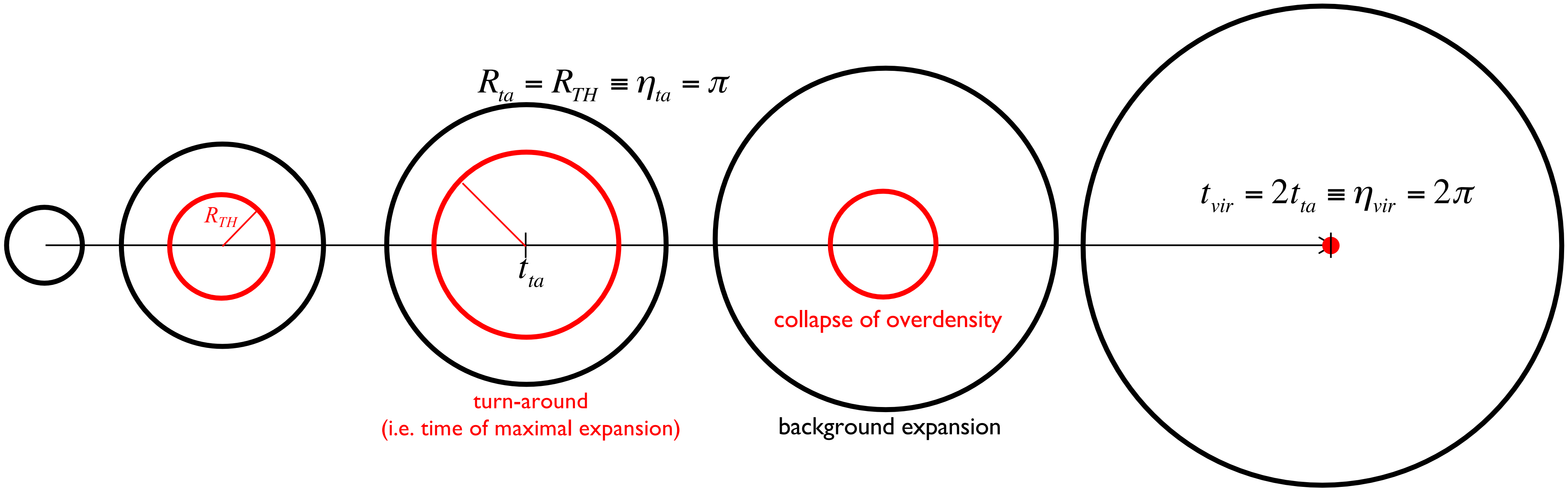
- spherical (top-hat) overdensity $R_{TH} = R_{TH}(t)\,,\, M_{TH} = const.$

- solution for collapsed overdensity

$$1+\delta_{TH}(\eta) = \frac{9}{2}\frac{(\eta-\sin\eta)^2}{(1-\cos\eta)^3}$$

$$1+\delta_{TH}(t_{ta}) = \frac{9\pi^2}{16} \approx 5.5$$

$$1+\delta_{TH}(t_{vir}) = \infty$$ (hmmmm...)



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- spherical (top-hat) overdensity $R_{TH} = R_{TH}(t)\,,\ M_{TH} = const.$

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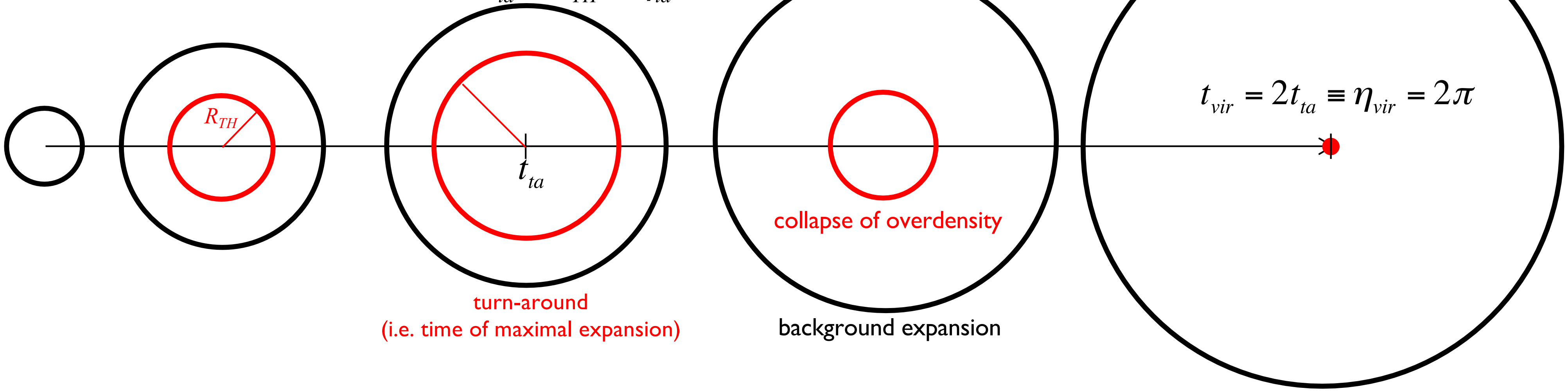
$$1+\delta_{TH}(t_{ta}) = \frac{9\pi^2}{16} \approx 5.5$$

$1+\delta_{TH}(t_{vir}) = \infty$ (hmmmm...)

virialisation!

$$R_{ta} = R_{TH} \equiv \eta_{ta} = \pi$$

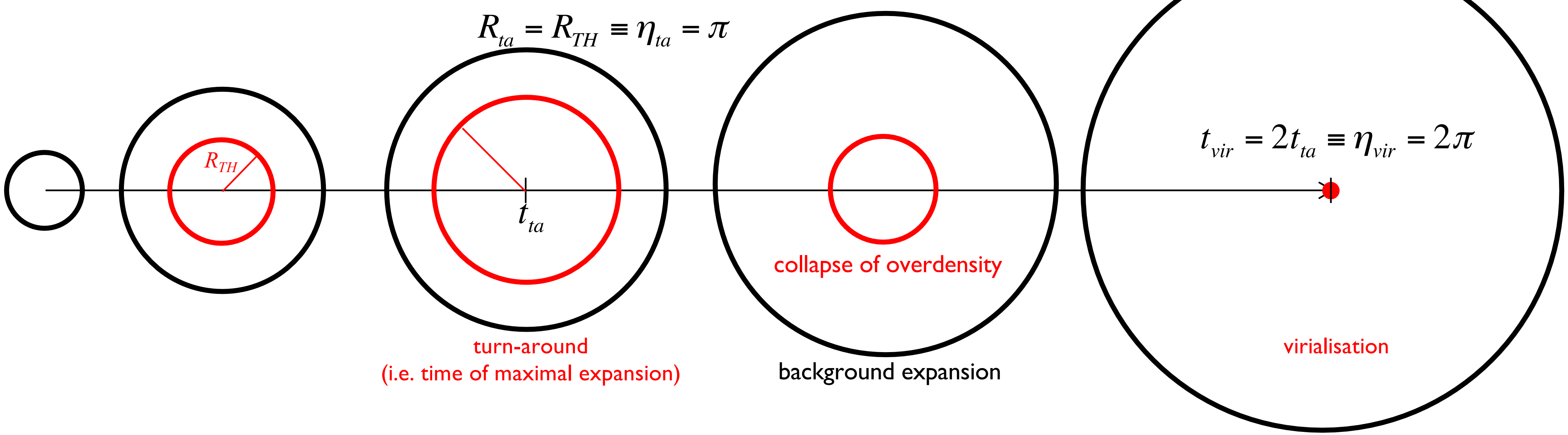


▪ Spherical Top-Hat Collapse

- spherical (top-hat) overdensity $R_{TH} = R_{TH}(t)\,,\, M_{TH} = const.$
- solution for virialized overdensity

$$E_{ta} = U_{ta}$$

$$E_{vir} = T_{vir} + U_{vir}$$



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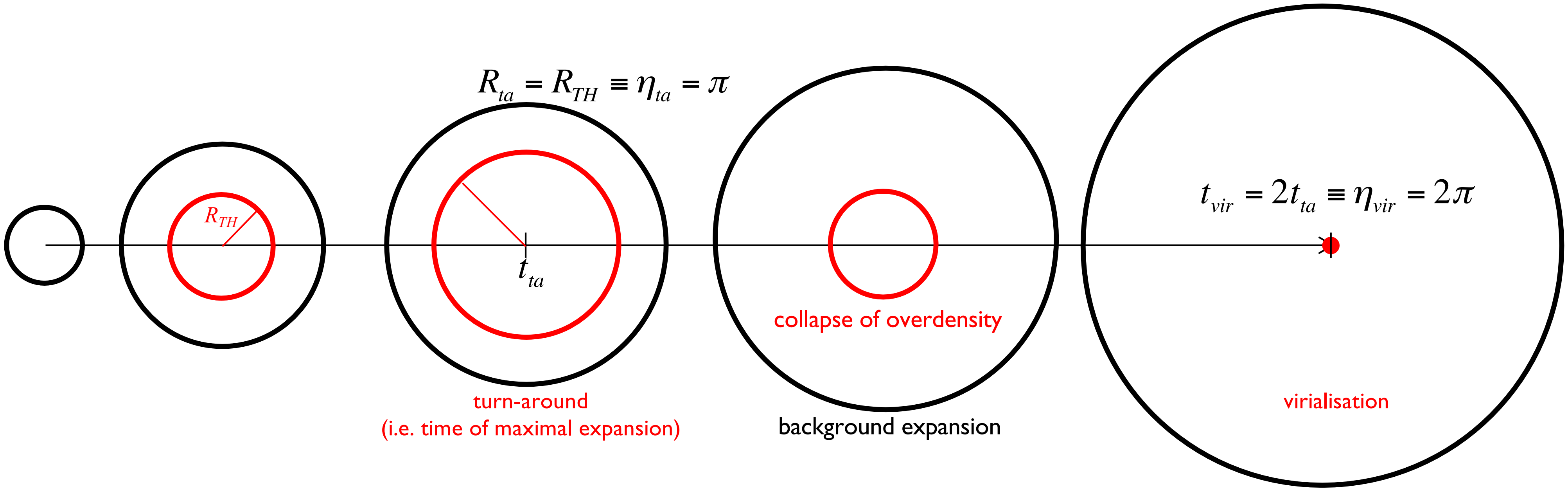

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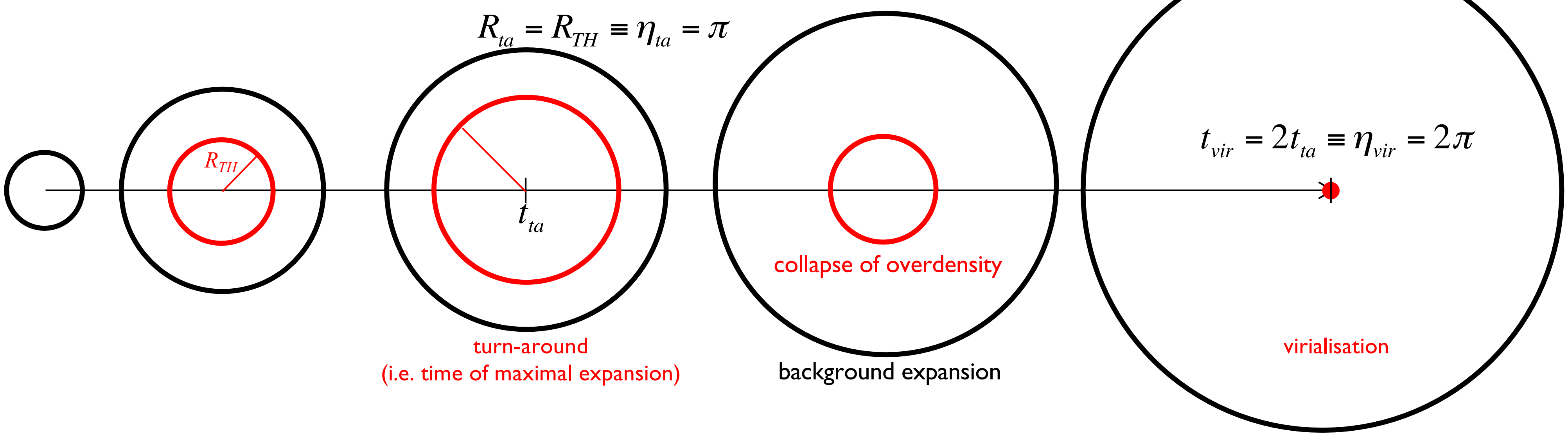
virial theorem: $0 = 2T_{vir} + U_{vir}$



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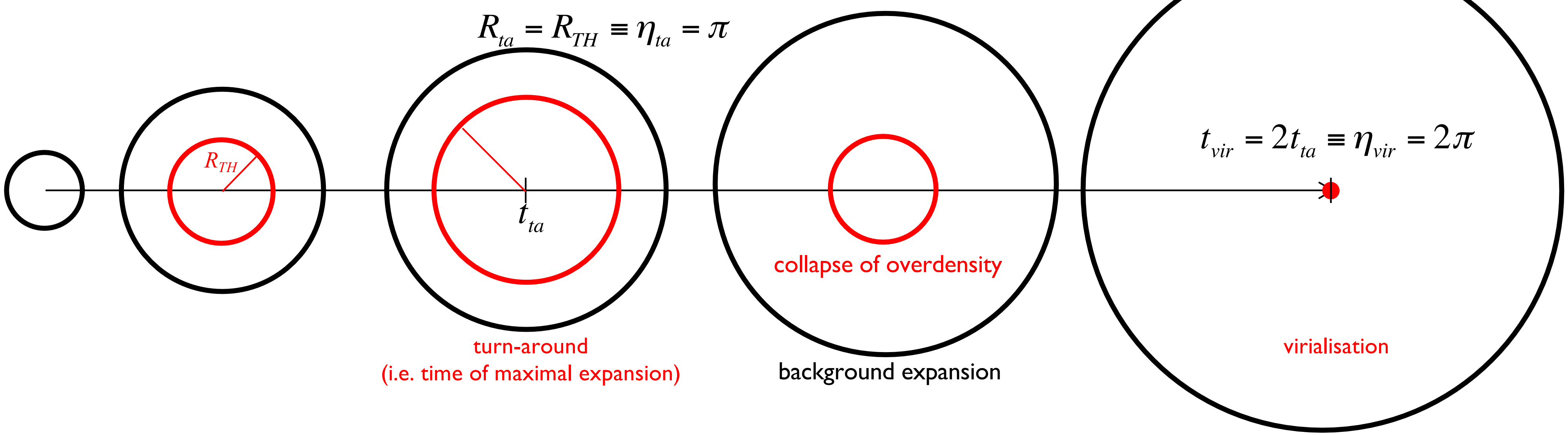

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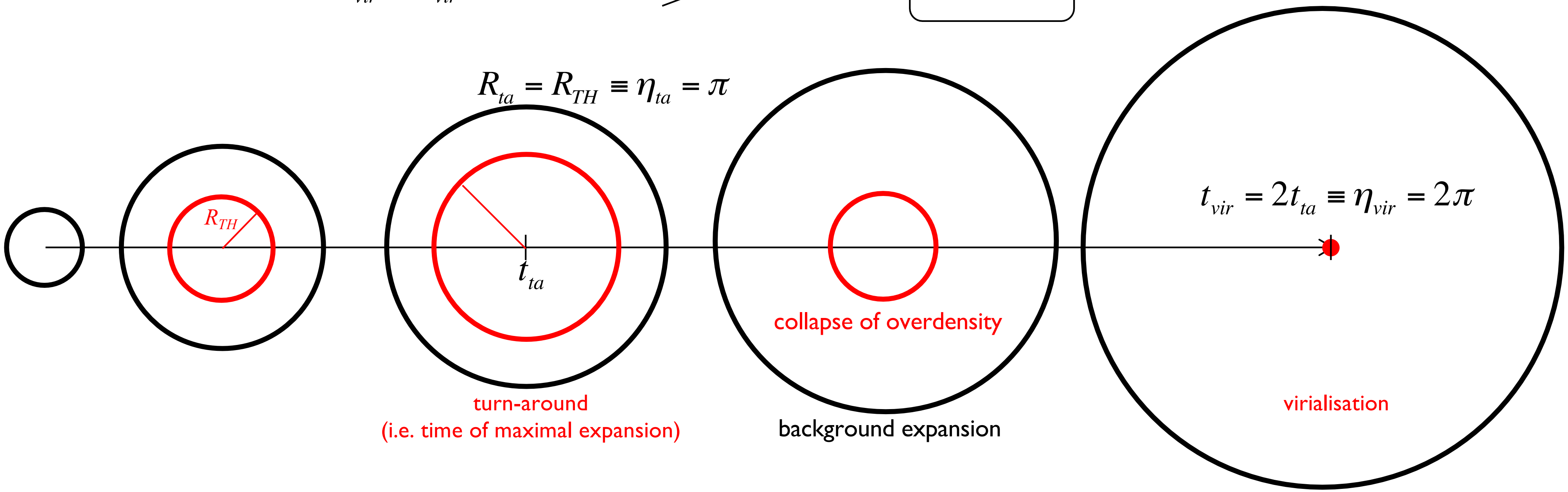
$$E_{vir} = T_{vir} + U_{vir} \qquad \boxed{U_{vir} = -\frac{3}{5}\frac{GM_{TH}^2}{R_{vir}}} \longrightarrow R_{vir} = \frac{R_{ta}}{2}$$

virial theorem: $0 = 2T_{vir} + U_{vir} \quad \Rightarrow \quad 2E_{vir} = \cancel{2T_{vir} + U_{vir}} + U_{vir} = \boxed{U_{vir} = 2U_{ta}} = 2E_{ta}$



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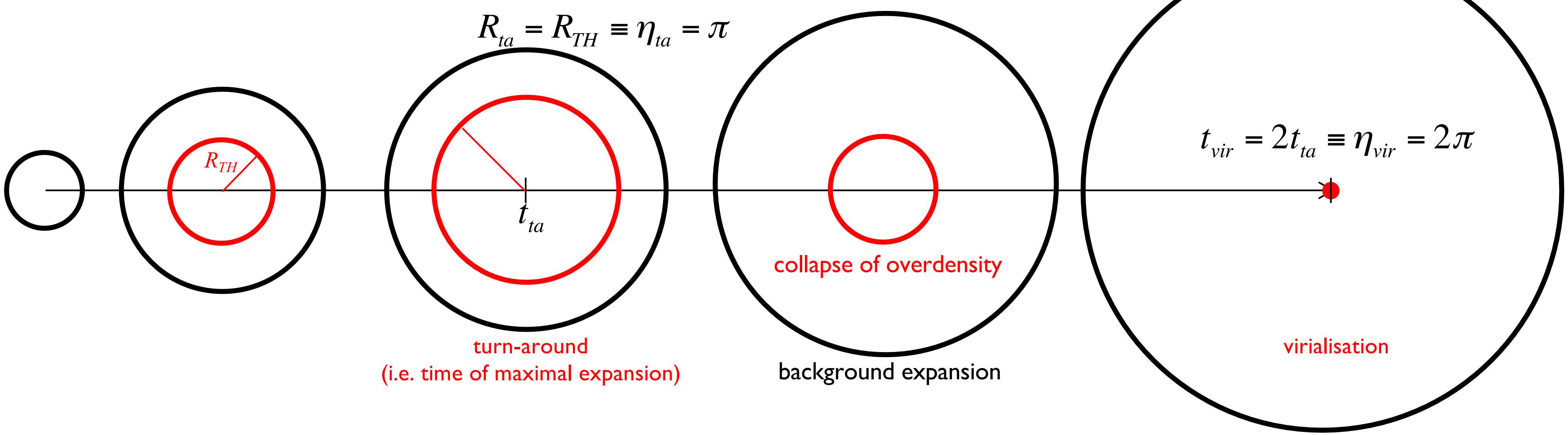
$$R_{vir} = \frac{R_{ta}}{2} \qquad \Rightarrow \qquad \rho_{TH}(t_{vir}) = 8\rho_{TH}(t_{ta})$$



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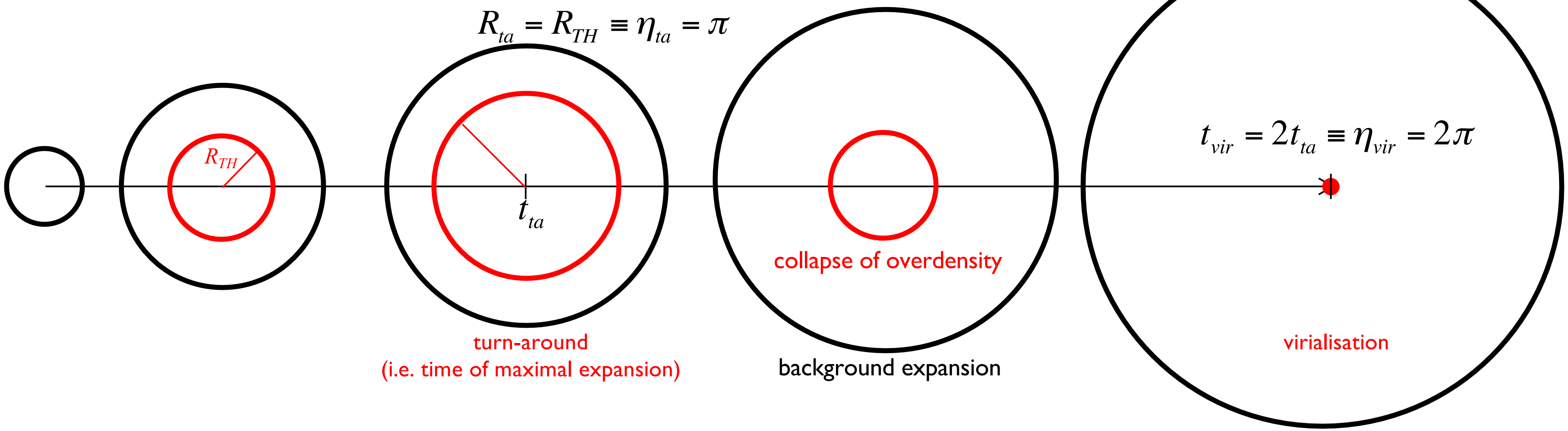
$$t_{vir} = 2t_{ta} \quad \underset{\bar{\rho} = \frac{1}{6\pi G t^2}}{=>} \quad \bar{\rho}(t_{vir}) = \frac{1}{2^2}\bar{\rho}(t_{ta})$$



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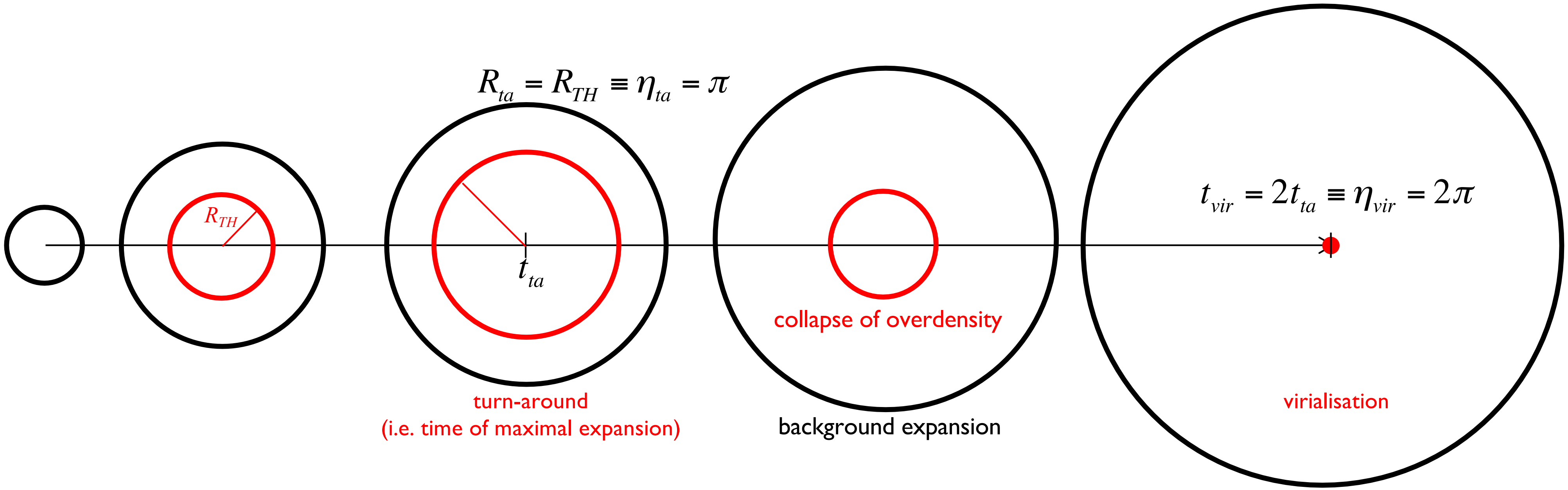
?



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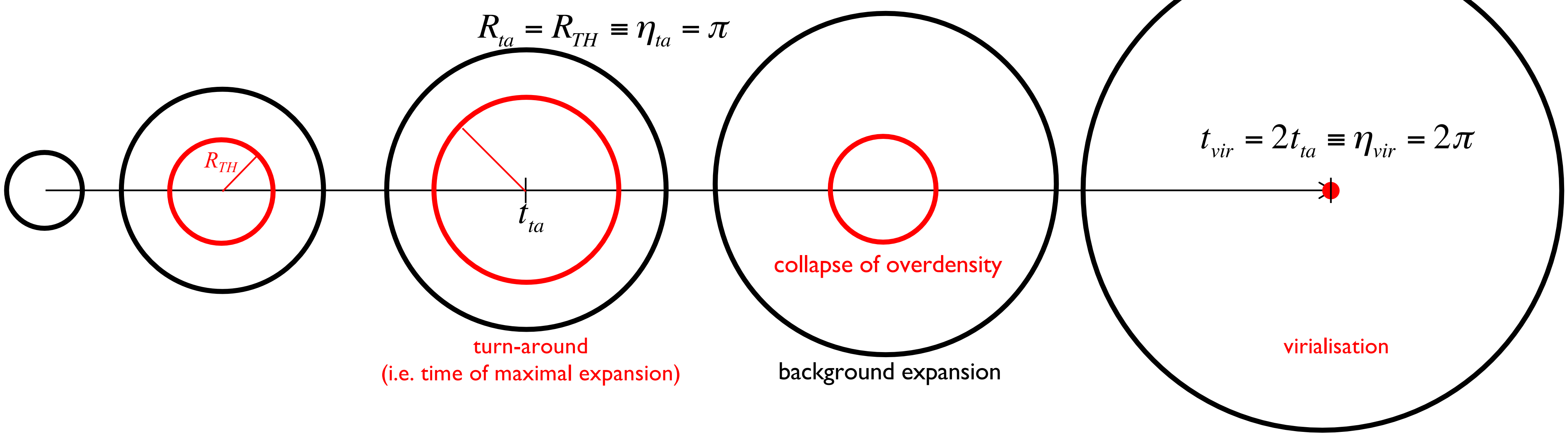
$$1+\delta_{TH}(t_{vir}) = \frac{8\rho(t_{ta})}{\bar{\rho}(t_{ta})/4} = 32\left(1+\delta(t_{ta})\right) = 32\,\frac{9\pi^2}{16} = 18\pi^2 \approx 178$$



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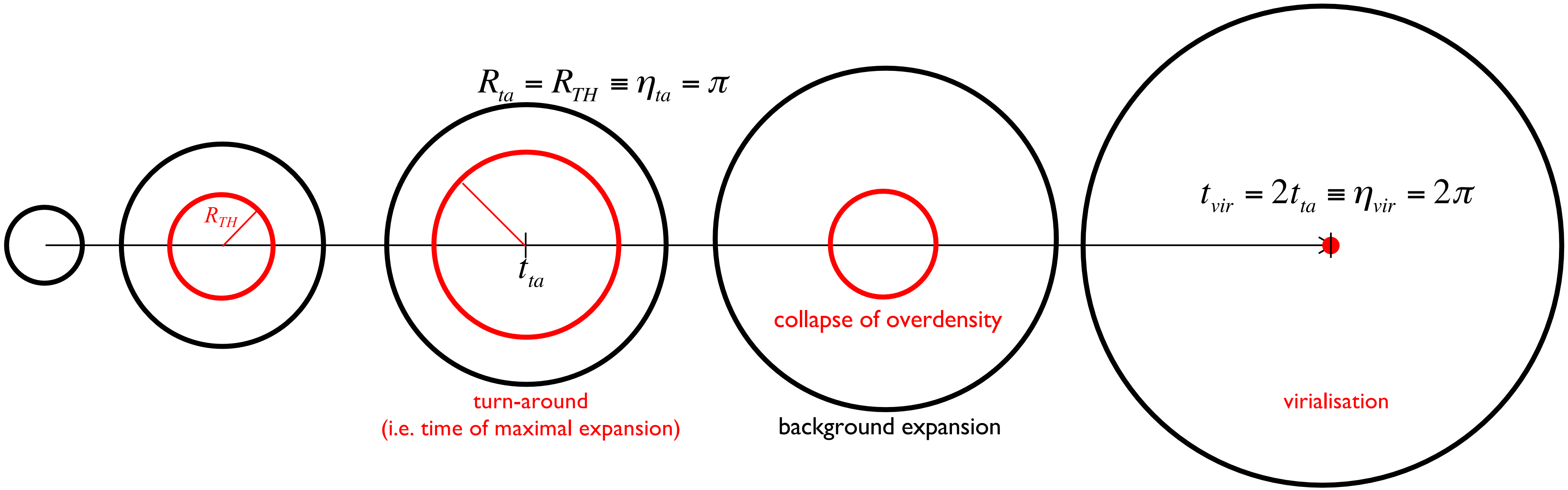

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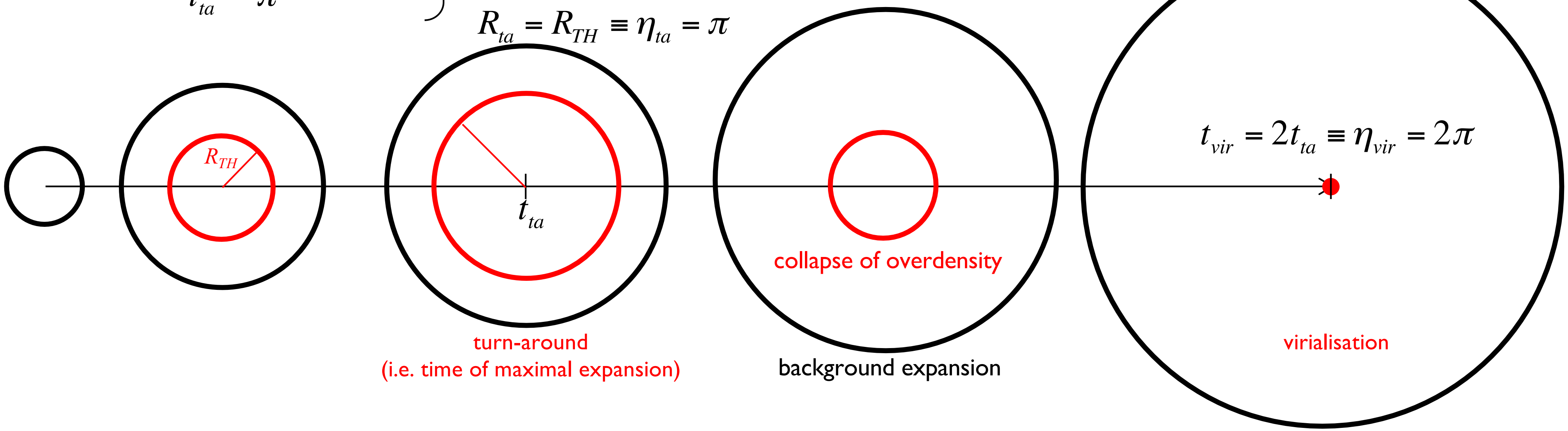
- non-singular solution for **linearized** overdensity

$$\left.\begin{aligned}\frac{R_{TH}}{R_{ta}} &= \frac{1}{2}(1-\cos\eta)\\ \frac{t}{t_{ta}} &= \frac{1}{\pi}(\eta-\sin\eta)\end{aligned}\right\} \xrightarrow[\text{and combining to } R_{TH}(t)]{\textbf{Taylor-expanding} \cos() \text{ and } \sin()} R_{TH}(t) = \frac{R_{ta}}{4}\left(\frac{6\pi t}{t_{ta}}\right)^{2/3}\left[1-\frac{1}{20}\left(\frac{6\pi t}{t_{ta}}\right)^{2/3}+\ldots\right]$$



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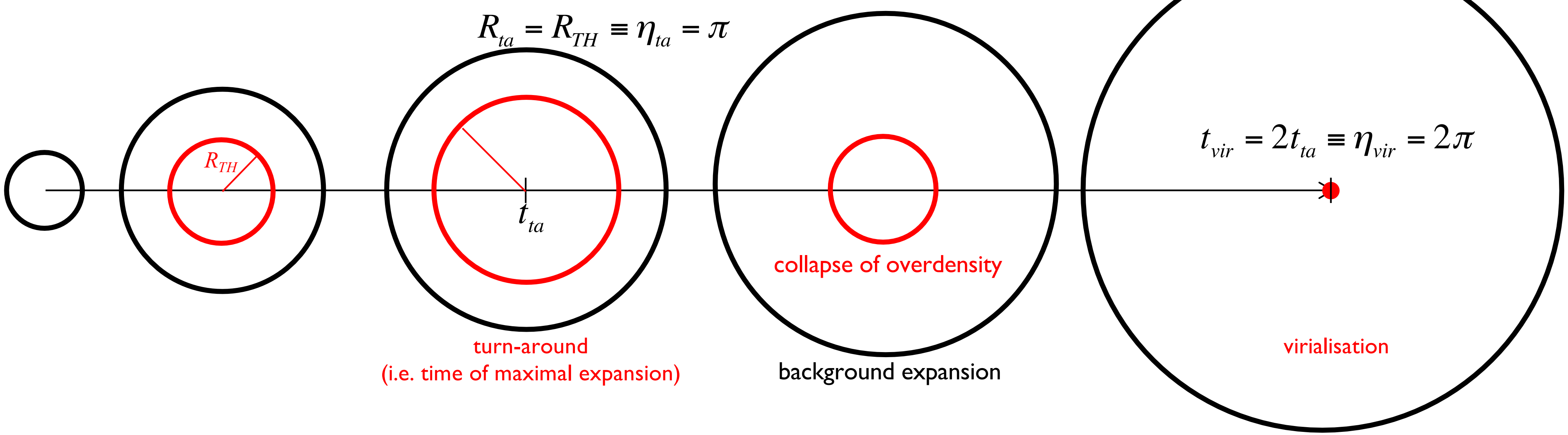

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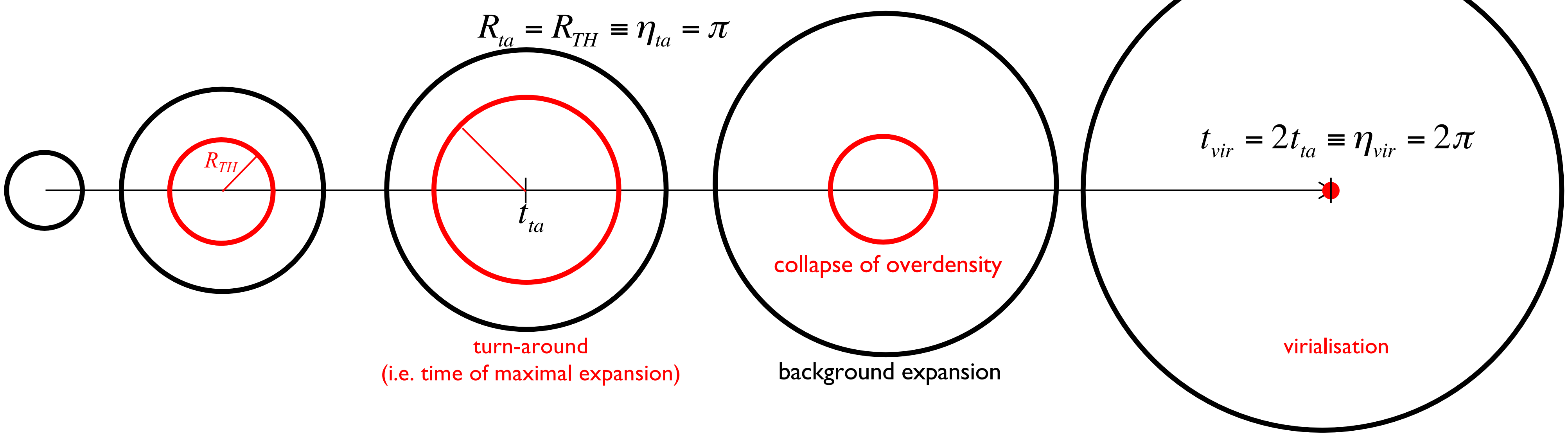

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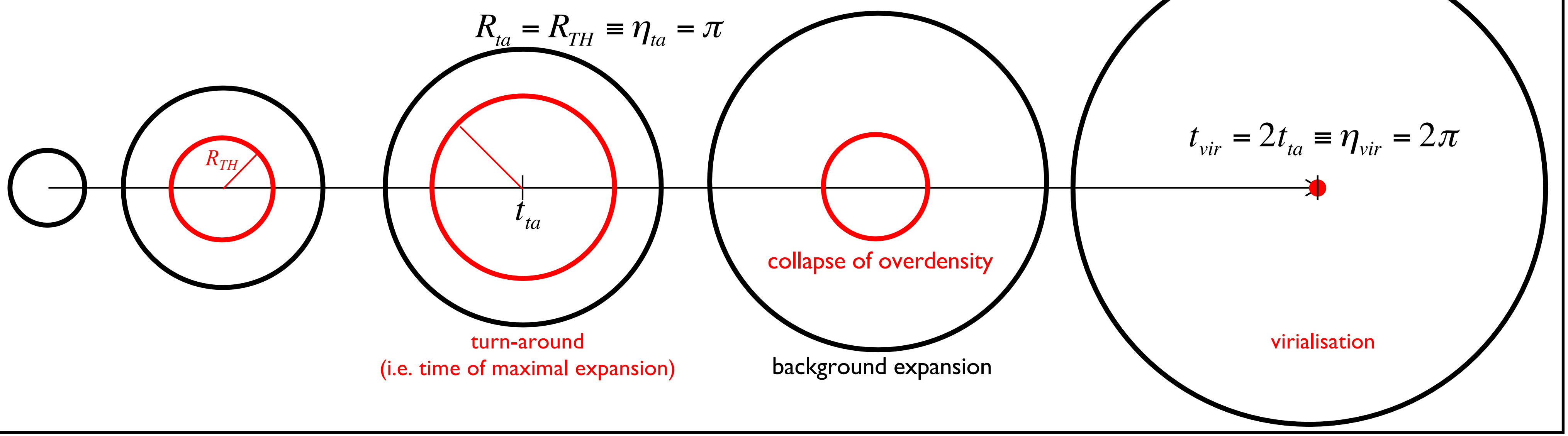

▪ Spherical Top-Hat Collapse

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- territory of computational cosmology…
- …but powerful, analytical (quasi-linear) approaches exist, too:
    - Zel'dovich approximation (1^{st} order Lagrangian perturbation theory)
    - Spherical Top-Hat Collapse
    - **Press-Schechter halo mass function**
    - …

- powerful "Press-Schechter" theory

THE ASTROPHYSICAL JOURNAL, **187**:425–438, 1974 February 1

FORMATION OF GALAXIES AND CLUSTERS OF GALAXIES BY SELF-SIMILAR GRAVITATIONAL CONDENSATION*

WILLIAM H. PRESS AND PAUL SCHECHTER
California Institute of Technology
Received 1973 August 1

ABSTRACT

We consider an expanding Friedmann cosmology containing a "gas" of self-gravitating masses. The masses condense into aggregates which (when sufficiently bound) we identify as single particles of a larger mass. We propose that after this process has proceeded through several scales, the mass spectrum of condensations becomes "self-similar" and independent of the spectrum initially assumed. Some details of the self-similar distribution, and its evolution in time, can be calculated with the linear perturbation theory. Unlike other authors, we make no ad hoc assumptions about the spectrum of long-wavelength initial perturbations: the nonlinear N-body interactions of the mass points randomize their positions and generate a perturbation to all larger scales; this should fix the self-similar distribution almost uniquely. The results of numerical experiments on 1000 bodies are presented; these appear to show new nonlinear effects: condensations can "bootstrap" their way up in size faster than the linear theory predicts. Our self-similar model predicts relations between the masses and radii of galaxies and clusters of galaxies, as well as their mass spectra. We compare the predictions with available data, and find some rather striking agreements. If the model is to explain galaxies, then isothermal "seed" masses of $\sim 3 \times 10^7\, M_\odot$ must have existed at recombination. To explain clusters of galaxies, the only necessary seeds are the galaxies themselves. The size of clusters determines, in principle, the deceleration parameter q_0; presently available data give only very broad limits, unfortunately.

Subject headings: cosmology — galaxies — galaxies, clusters of

I. INTRODUCTION

The observed matter content of the Universe is very clumpy over a range of mass exceeding 15 orders of magnitude, from stars ($\sim 1\, M_\odot$) through clusters and galaxies, to clusters of galaxies of $10^{15}\, M_\odot$. However, on progressively larger scales the evidence for clumpiness is less striking. Considerable effort has been required to demonstrate the existence of superclusters (see Bogart and Wagoner 1973 for a recent treatment). On scales larger than ~ 50 Mpc (but smaller than the present horizon of $\sim 10^3$ Mpc) the Universe is probably isotropic and homogeneous, corresponding to an expanding Friedmann cosmology. Even if the earliest epochs were characterized by chaos and large-scale inhomogeneity (Misner 1968; Rees 1972; Peebles 1972), the isotropy of the cosmic microwave background argues for a Friedmann model at recombination and subsequently. After recombination (and the roughly coincident transition to matter dominance) the Universe probably evolves according to the pressureless dynamical equations (see, e.g., Peebles 1971 or Weinberg 1972).

In this context, how are the various scales of clumpiness to be explained? Star formation from a diffuse medium of sufficient density (and suitable other parameters) may be a purely astrophysical—as opposed to cosmological—problem: stars form at various epochs, and the process can be observed and studied in the present. In contrast, the condensation of substantially larger scales, especially galaxies and clusters of galaxies, seems to be a unique cosmological event. The accepted view, convincingly stated by Peebles (1965), Silk (1968) and others, puts the formation of these large-scale objects at some epoch between recombination and the present, because only in this period have the large condensing masses been substantially smaller than the cosmological horizon but bigger than their Jeans lengths.

A linear theory of inhomogeneous perturbations has been extensively developed for both isothermal and adiabatic disturbances (Lifshitz 1946; Zel'dovich 1967; for a review see Field 1974), and much recent work has been directed toward propagating an initially postulated spectrum of matter perturbations through the complicated era of recombination, into the era where the perturbations condense into (hopefully) observed objects. This program has yielded considerable cosmological understanding in many respects, but it has not thus far been completely successful in explaining the basic observational data: the mass distribution of galaxies and their linear sizes, the masses and sizes of clusters of galaxies, the fact that there is no strong clustering on larger scales.

In the usual framework of the linear perturbation analysis, physical processes at or before recombination

* Supported in part by the National Science Foundation [GP-36687X, GP-28027].

- powerful "Press-Schechter" theory

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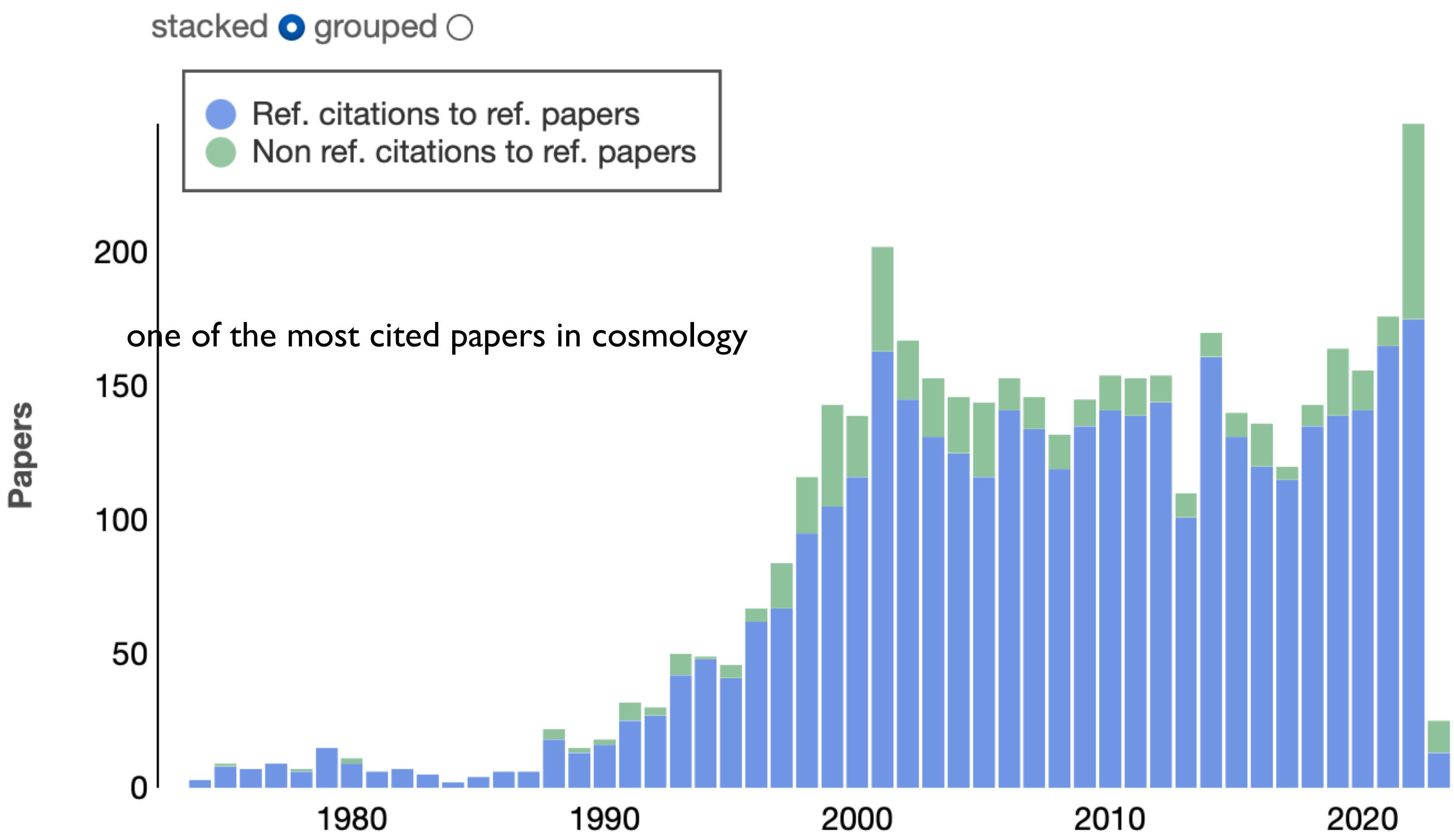
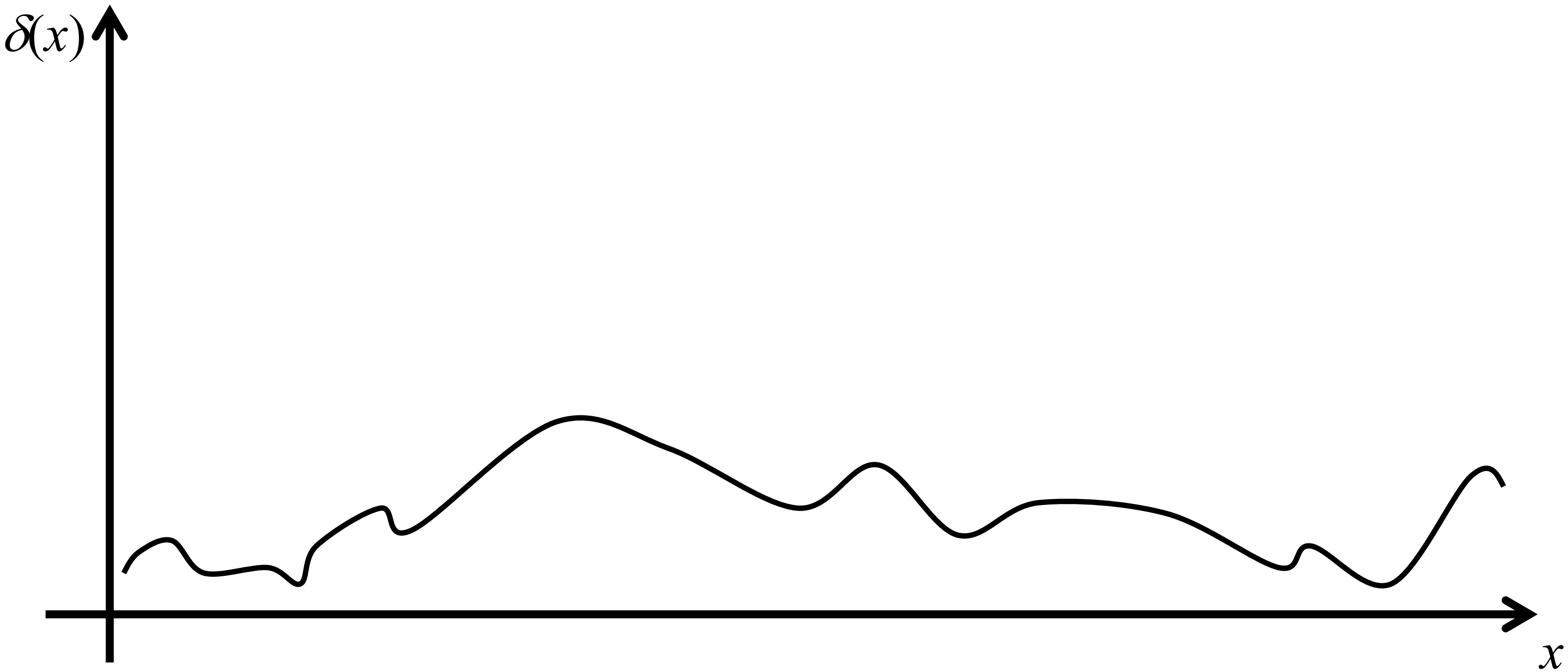


one of the most cited papers in cosmology

- Press-Schechter formula

growing density contrast



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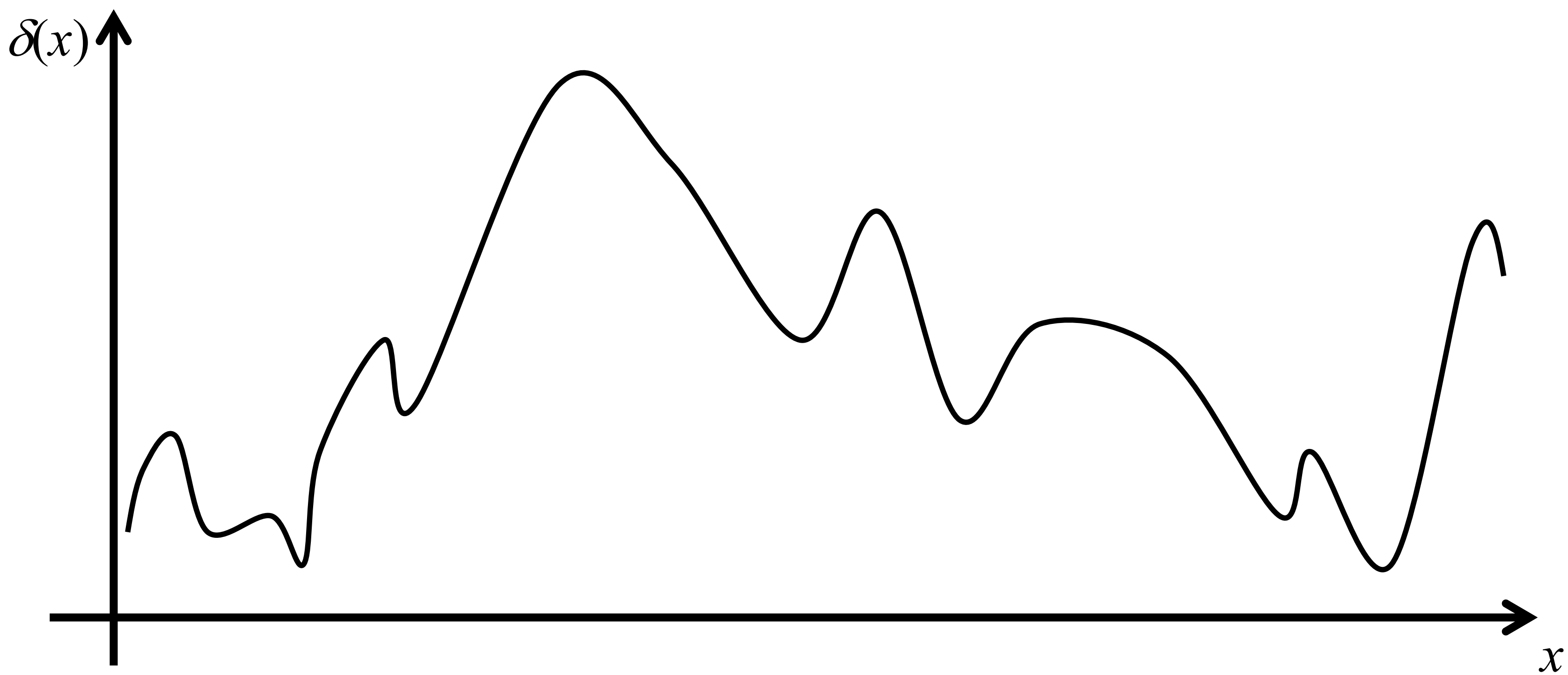
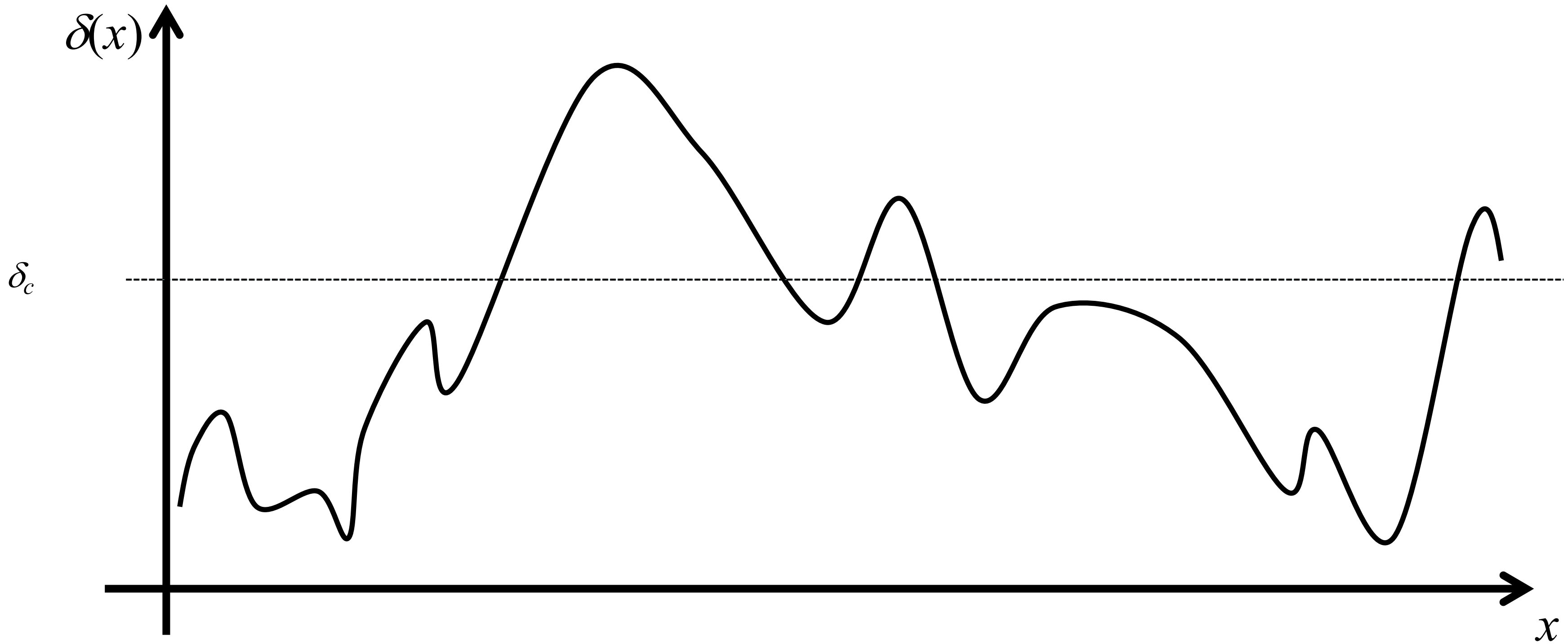


- Press-Schechter formula

  - a halo has formed when its *linear* density contrast $\delta(x, a)$ has reached δ_c



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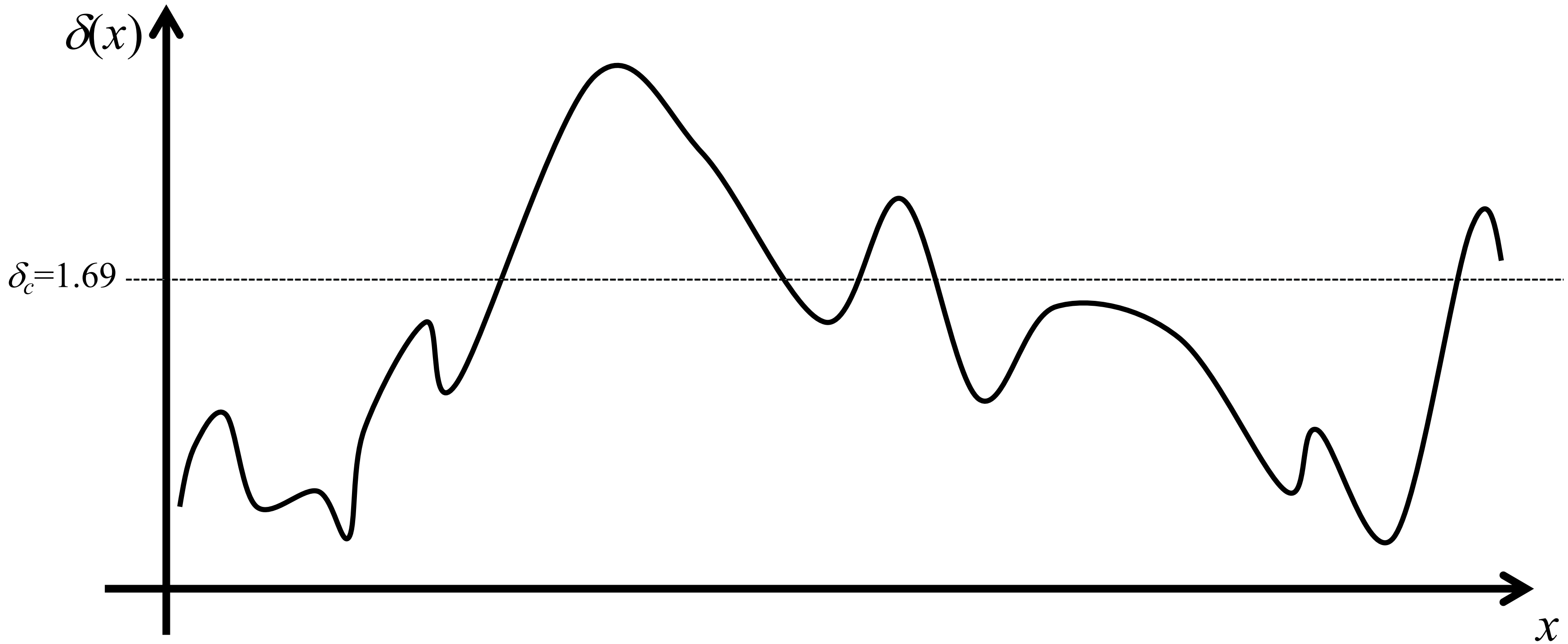
  - a halo has formed when its *linear* density contrast $\delta(x, a)$ has reached δ_c=1.69



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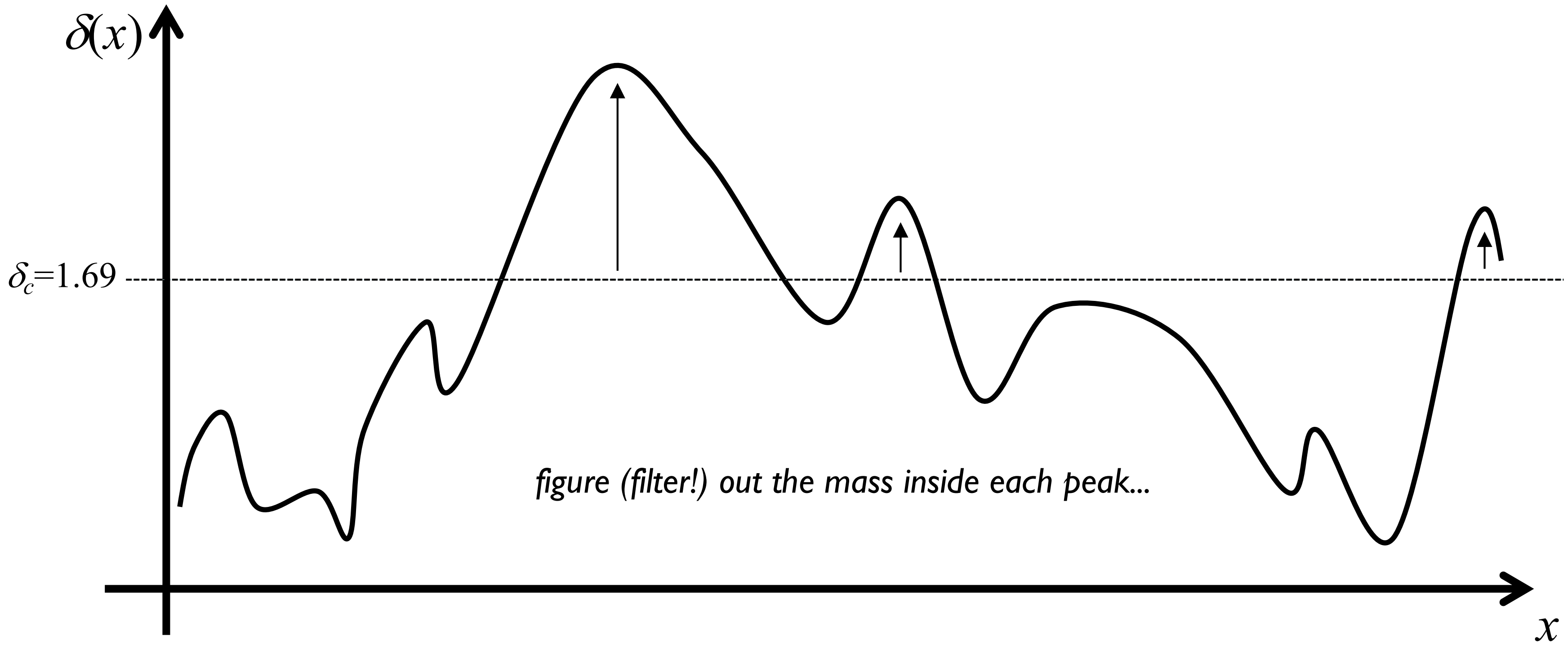

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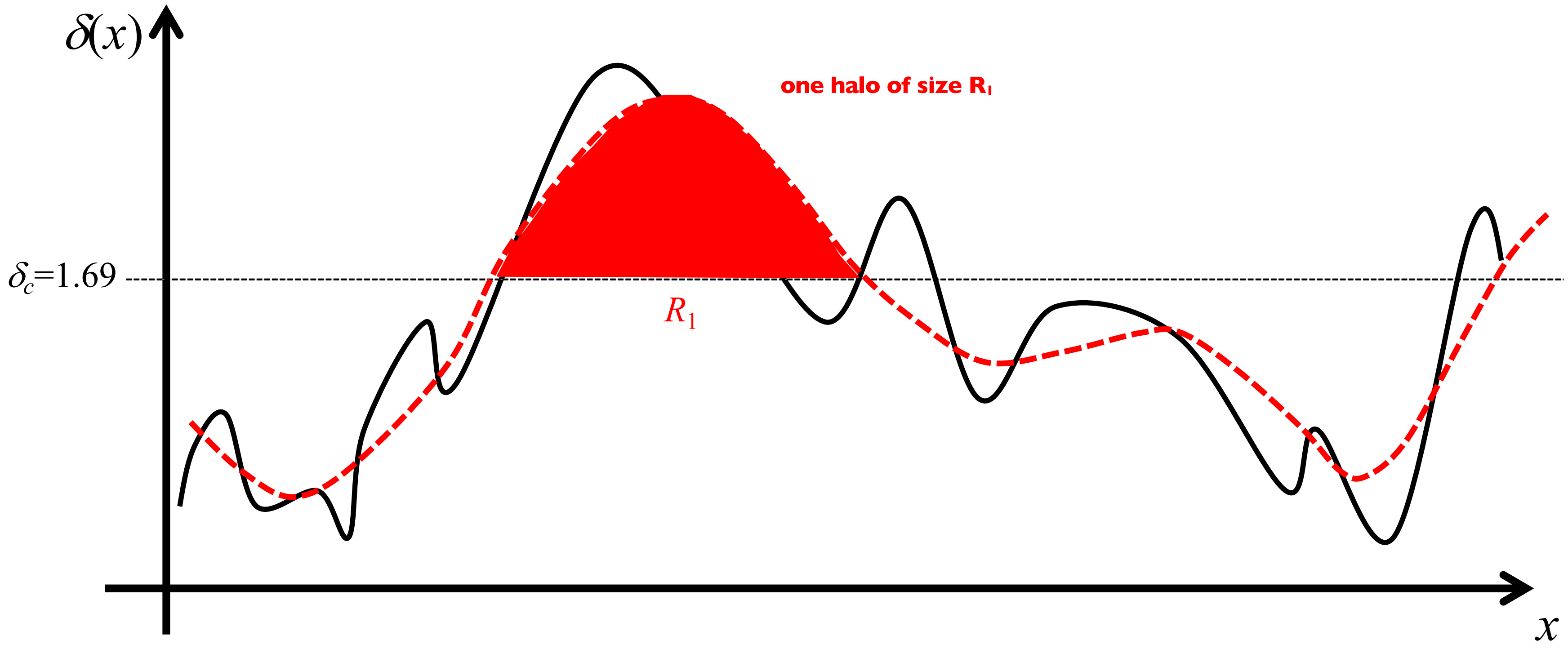

we consider perturbations on a certain scale R^*

$$\delta_R(\vec{x},a) = \int \delta(\vec{x}',a) W_R(\vec{x}-\vec{x}') d^3x'$$

*which can be related to halo mass via $M = \Omega_m \rho_{crit} \frac{4\pi}{3} R^3$

▪ Press-Schechter formula

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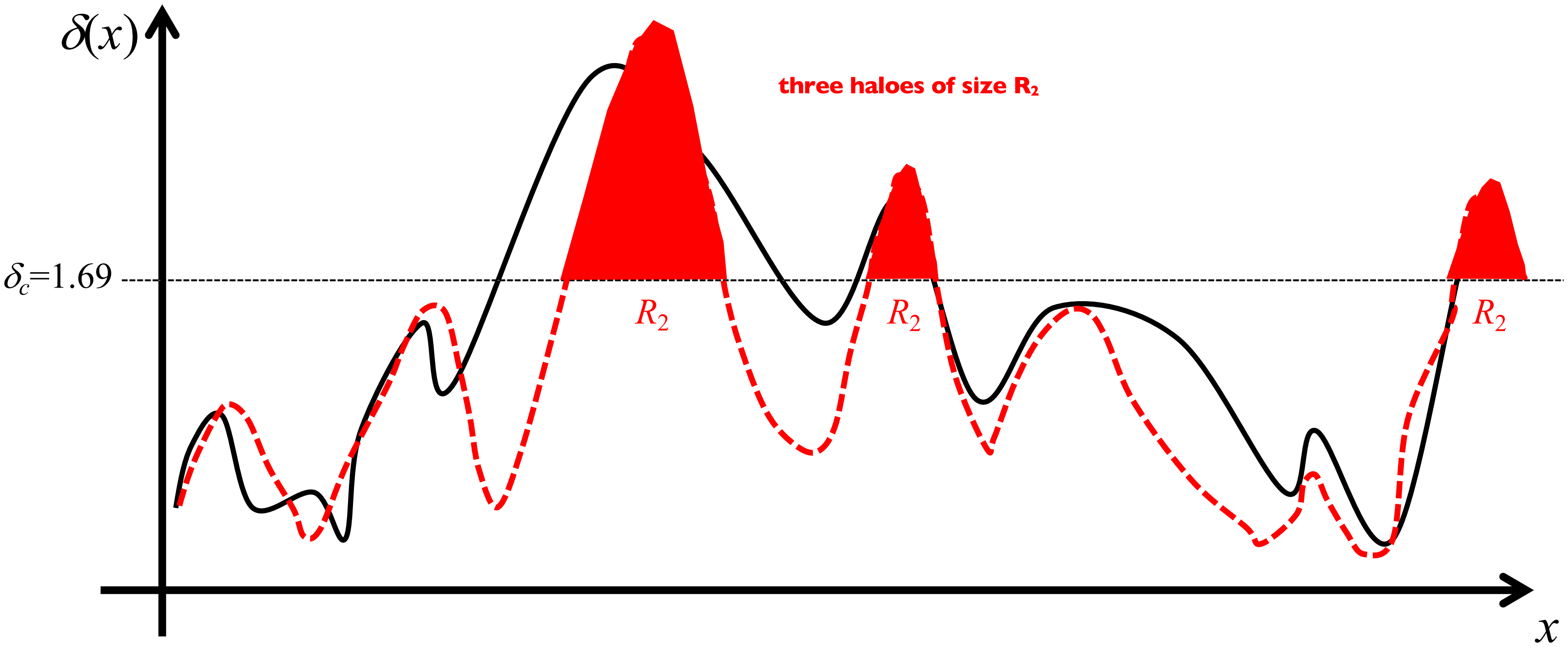

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$$\delta_R(\vec{x},a) = \int \delta(\vec{x}',a) W_R(\vec{x}-\vec{x}') d^3x'$$

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  - a halo of size R has formed when its *linear* density contrast $\delta_R(x, a)$ has reached δ_c=1.69

$$\delta_R(\vec{x},a) = \int \delta(\vec{x}',a) W_R(\vec{x}-\vec{x}') d^3x'$$

→ we need to count the number of haloes of size R

▪ Press-Schechter formula

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$$\delta_R(\vec{x},a) = \int \delta(\vec{x}',a) W_R(\vec{x}-\vec{x}') d^3x'$$

- the density contrast $\delta_R(x)$ is a Gaussian random field with variance σ_R

$$p(\delta_R) = \frac{1}{\sqrt{2\pi\sigma_R^2}} e^{-\frac{1}{2}\left(\frac{\delta_R}{\sigma_R}\right)^2}$$

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power spectrum of density fluctuations:
(all waves inside R-window affect σ_R)

$$P(k) = \left(\frac{D(a)}{D(a_0)}\right)^2 P_0(k)$$

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- probability to have $\delta_R > \delta_c$

$$F_{>\delta_c}(R) = \int_{\delta_c}^{\infty} p(\delta_R)\, d\delta_R$$

→ we need to count the number of haloes of size R

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- number of peaks in range $[R, R+dR]$

$$dN \propto F_{>\delta_c}(R) - F_{>\delta_c}(R+dR)$$

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- number of peaks in range [R, R+dR]

$$dN \propto F_{>\delta_c}(R) - F_{>\delta_c}(R+dR)$$

- relate scale R to mass M

$$M = \Omega_m \rho_{crit} \frac{4\pi}{3} R^3$$

- Press-Schechter formula

$$\frac{dn}{dM}dM = \sqrt{\frac{2}{\pi}}\,\frac{\bar{\rho}}{M}\,\frac{\delta_c}{\sigma_M}\left|\frac{d\ln\sigma_M}{d\ln M}\right|\exp\left(\frac{-\delta_c^2}{2\sigma_M^2}\right)\frac{dM}{M}$$

$$\sigma_M^2 = \frac{1}{2\pi^2}\int_0^{+\infty} P(k)\hat{W}^2(kR)k^2\,dk \qquad P(k) = \left(\frac{D(a)}{D(a_0)}\right)^2 P_0(k)$$

$$\hat{W}(x) = \frac{3}{x^3}\left(\sin(x) - x\cos(x)\right)$$

$\overline{\rho}$: mean density of Universe

δ_c : density contrast of collapsed structure according to linear perturbation theory

σ_M : variance of mass on scale corresponding to $M = (4\pi/3)\Omega_m\rho_{crit}R^3$

- Press-Schechter formula

$$\frac{dn}{dM}dM = \sqrt{\frac{2}{\pi}}\,\frac{\bar{\rho}}{M}\,\frac{\delta_c}{\sigma_M}\left|\frac{d\ln\sigma_M}{d\ln M}\right|\exp\left(\frac{-\delta_c^2}{2\sigma_M^2}\right)\frac{dM}{M}$$

$$\sigma_M^2 = \frac{1}{2\pi^2}\int_0^{+\infty} P(k)\hat{W}^2(kR)k^2\,dk$$

$$P(k) = \left(\frac{D(a)}{D(a_0)}\right)^2 P_0(k)$$

$$\hat{W}(x) = \frac{3}{x^3}\left(\sin(x) - x\cos(x)\right)$$

side note:

$$\xi_2(R) = \frac{1}{2\pi^2}\int_0^{+\infty} P(k)\hat{W}^2(kR)k^2\,dk$$

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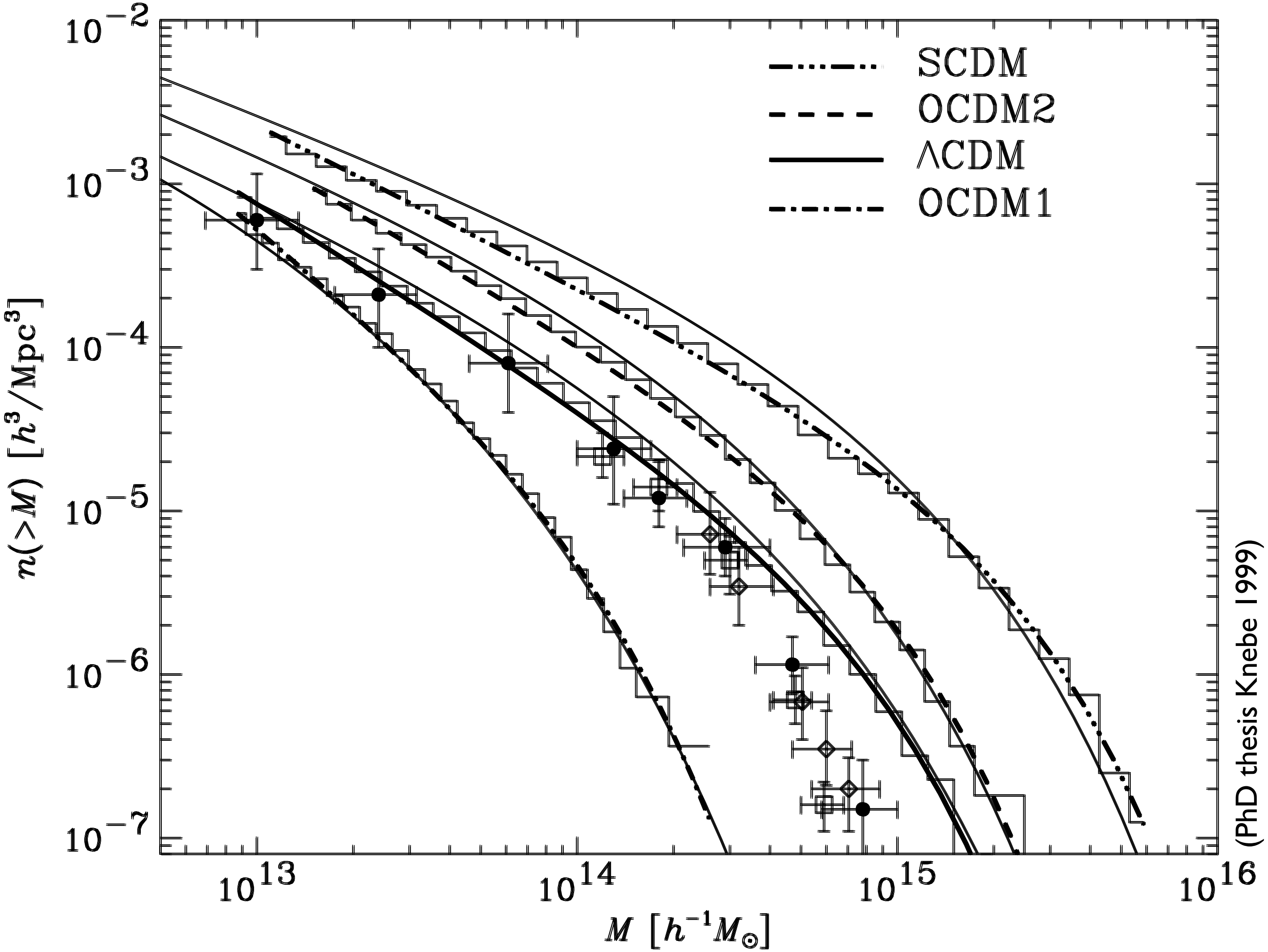
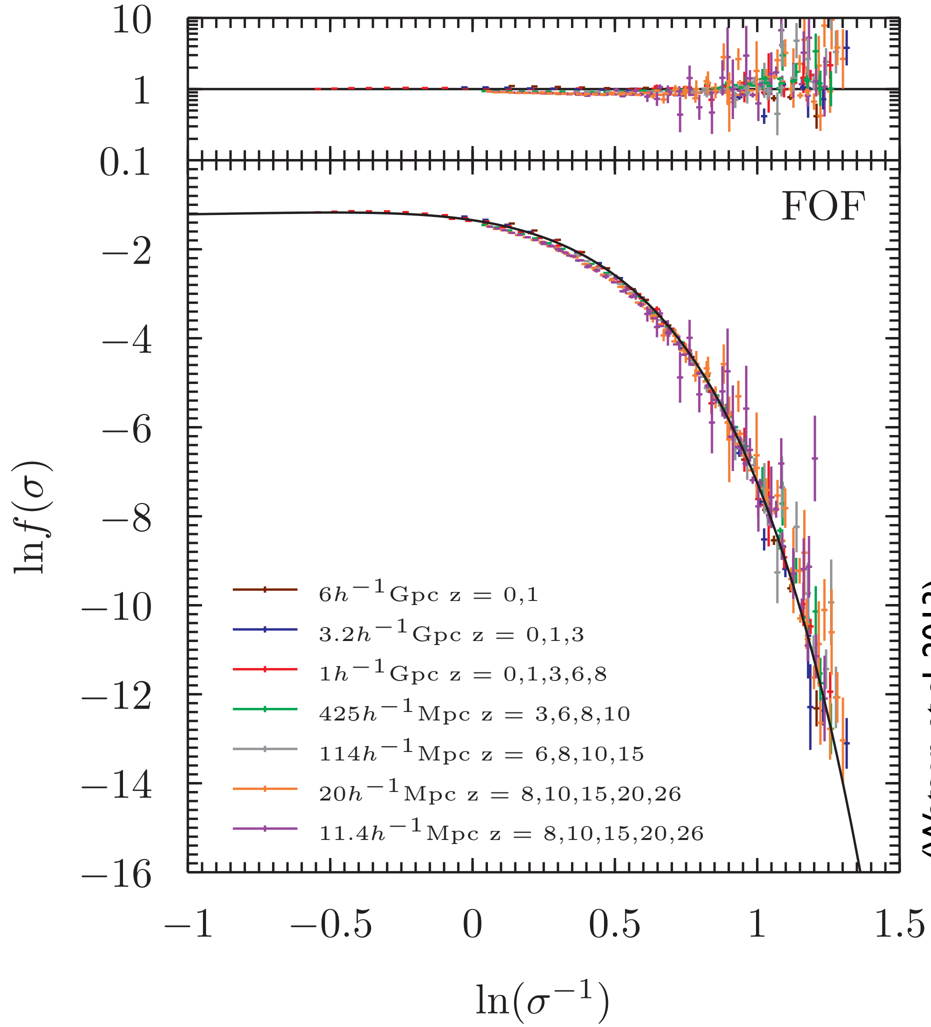
  - very good agreement with cosmological simulations



(PhD thesis Knebe 1999)

- Press-Schechter formula

  - very good agreement with cosmological simulations, though improvements match simulations even better…

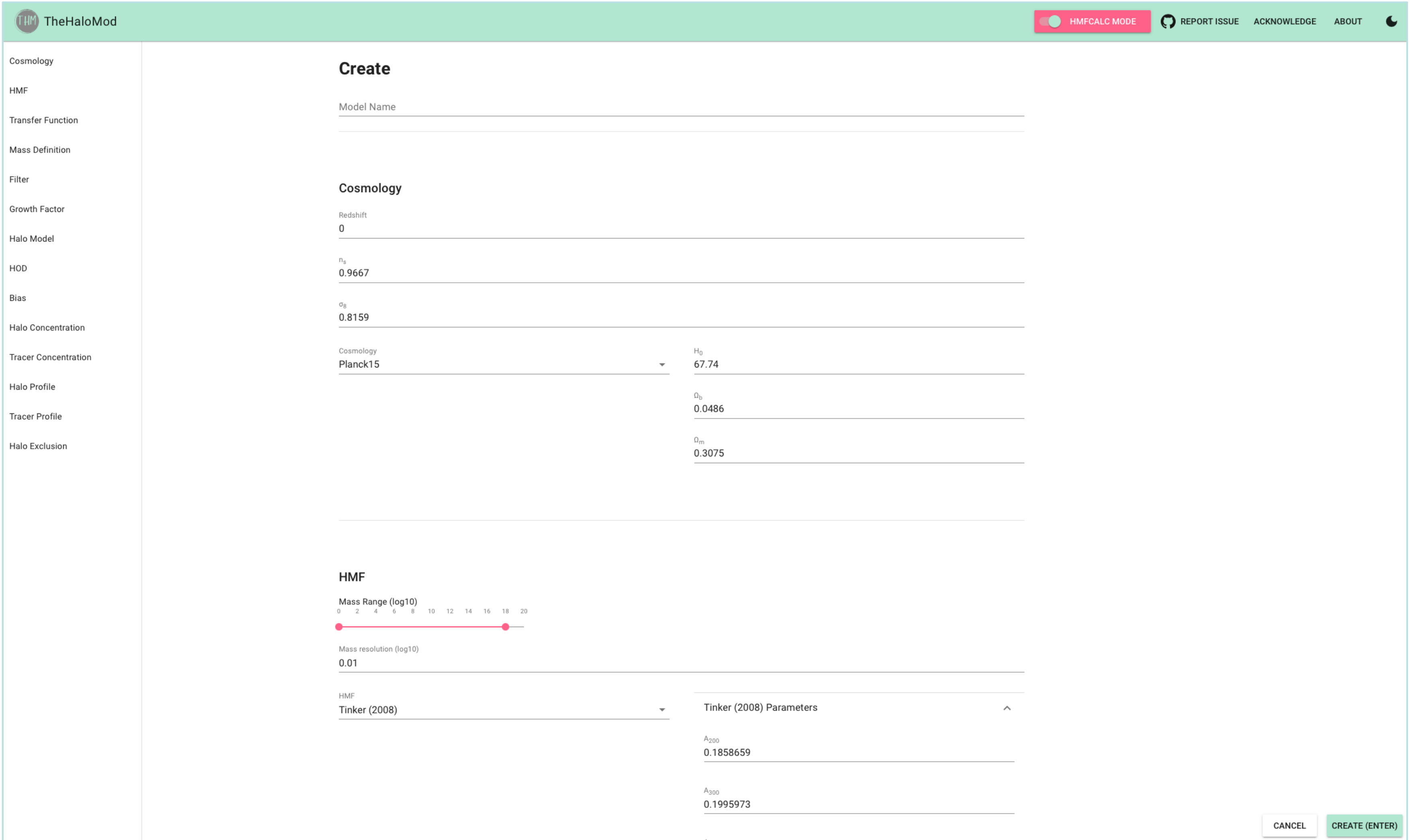


(Watson et al. 2013)

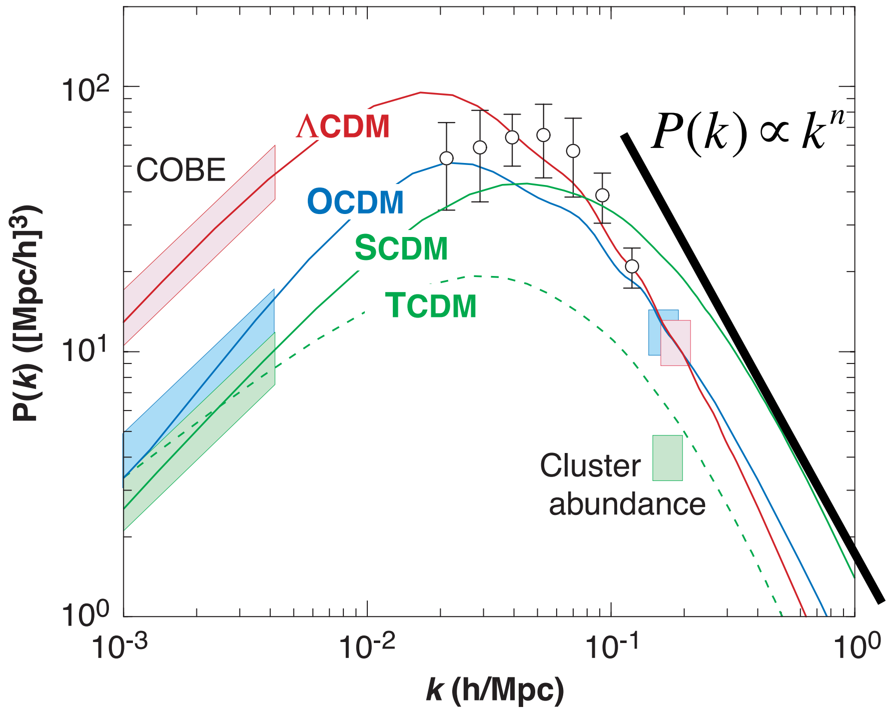
$$\frac{dn}{dM}dM = f(\sigma_M)\frac{\bar{\rho}}{M}\frac{d\ln\sigma_M^{-1}}{dM}dM$$

$$f(\sigma_M) = A\left[\left(\frac{\beta}{\sigma_M}\right)^{\alpha} + 1\right]\exp(-\gamma/\sigma_M^2)$$

- mass function calculator – https://thehalomod.app/



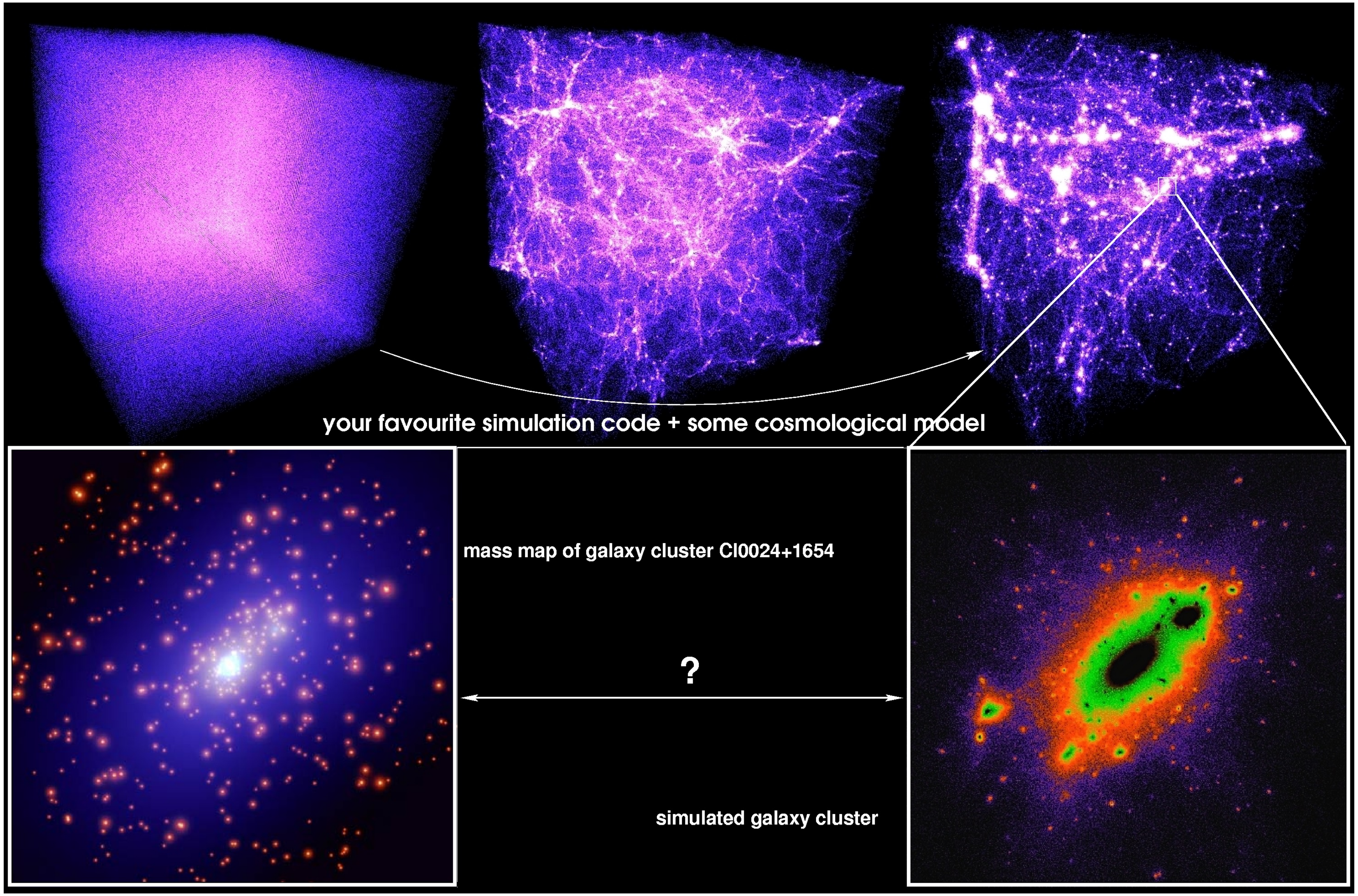
- Press-Schechter formula – scale-free cosmology



$$\sigma_M^2 = \sigma_0^2 \left(\frac{4\pi\bar{\rho}}{3}\right)^{\frac{n+3}{3}} M^{-\frac{n+3}{3}} = \left(\frac{M}{M_*}\right)^{-\frac{n+3}{3}}$$

$$\frac{dn}{dM}dM = \frac{n+3}{\sqrt{2\pi}} \frac{\bar{\rho}}{M^2} \delta_c \left(\frac{M}{M_*}\right)^{\frac{n+3}{6}} \exp\left[-\frac{1}{2}\delta_c^2 \left(\frac{M}{M_*}\right)^{\frac{n+3}{3}}\right] dM$$

- territory of computational cosmology…



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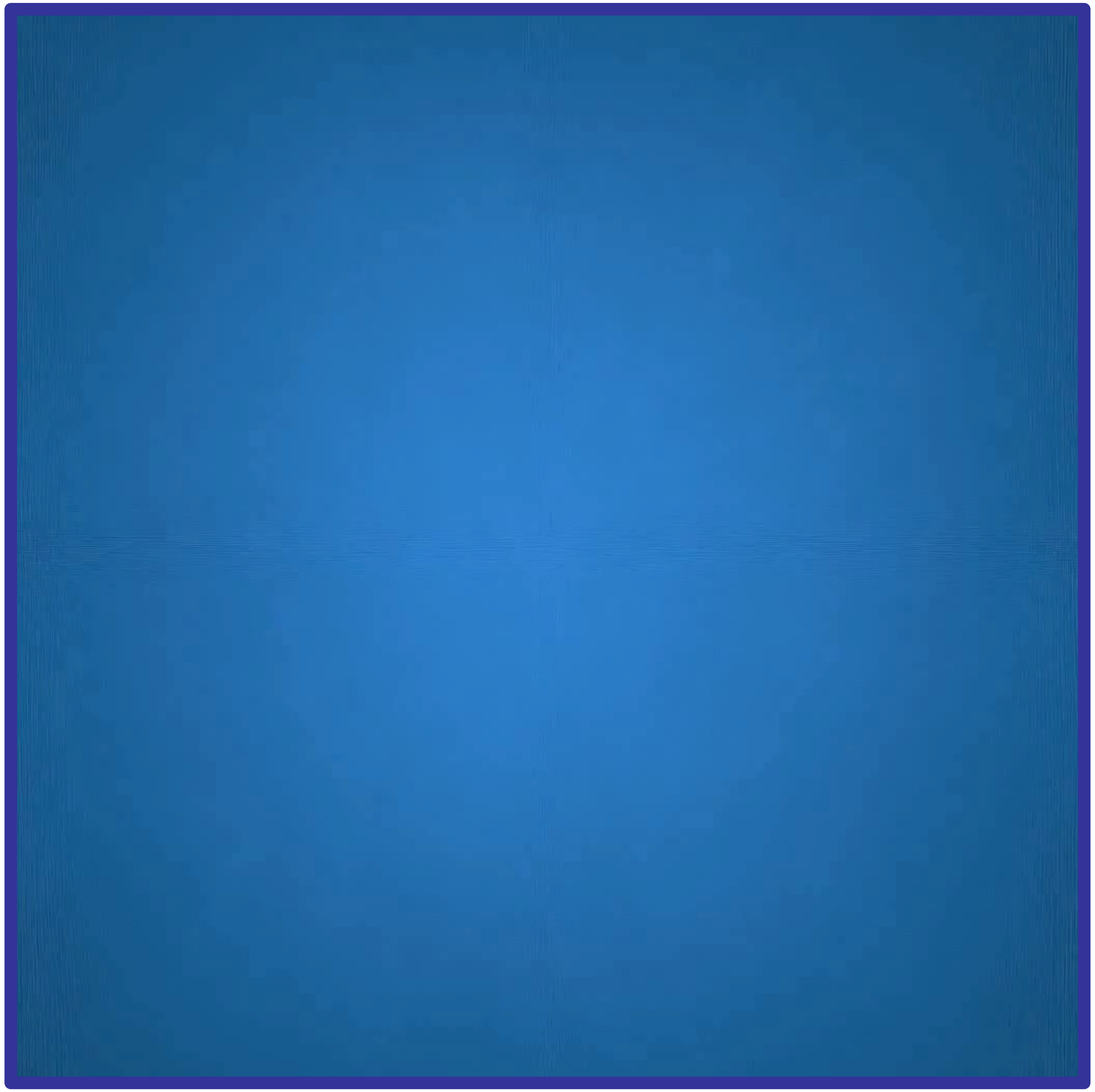
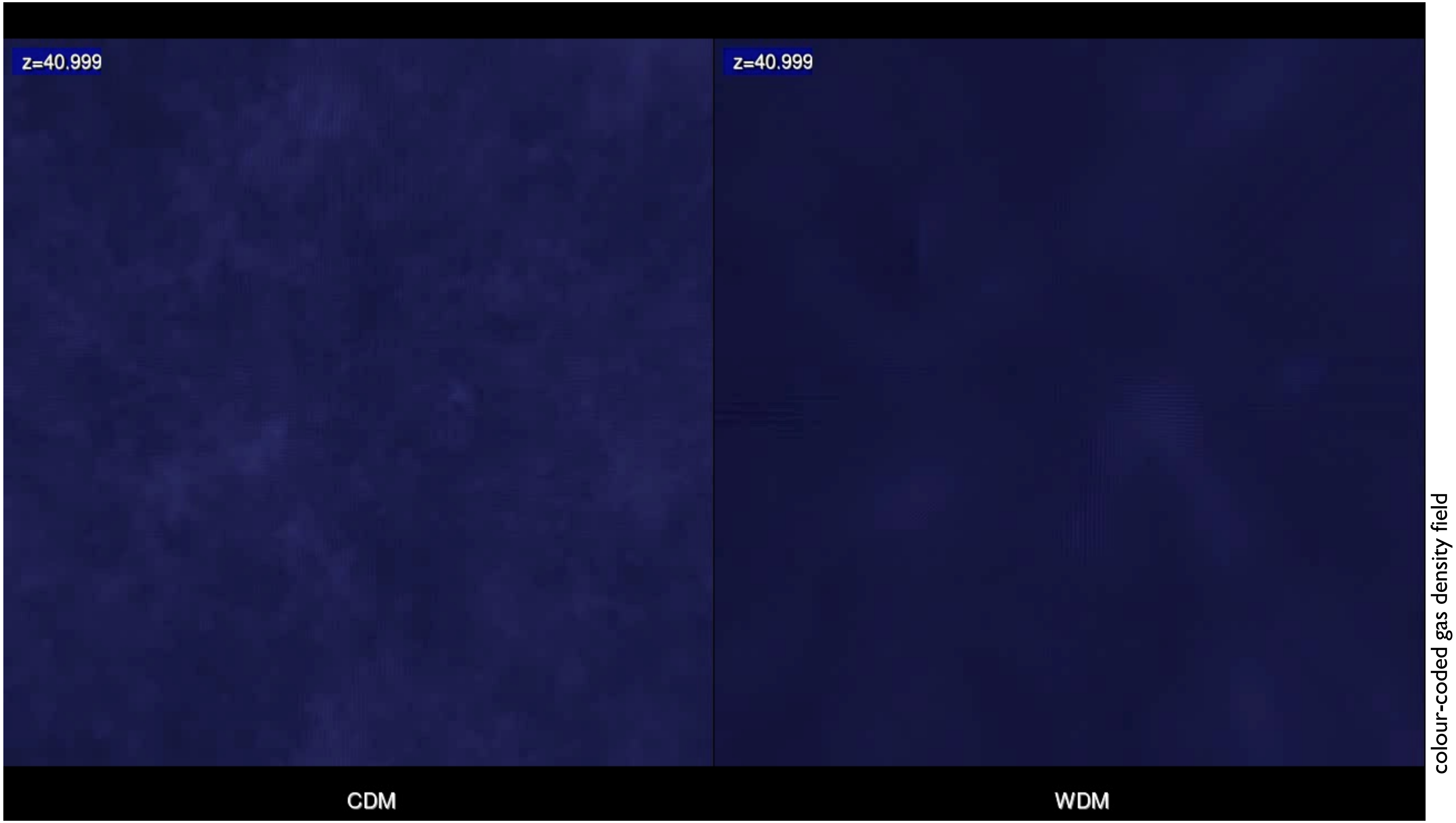


colour-coded dark matter density field

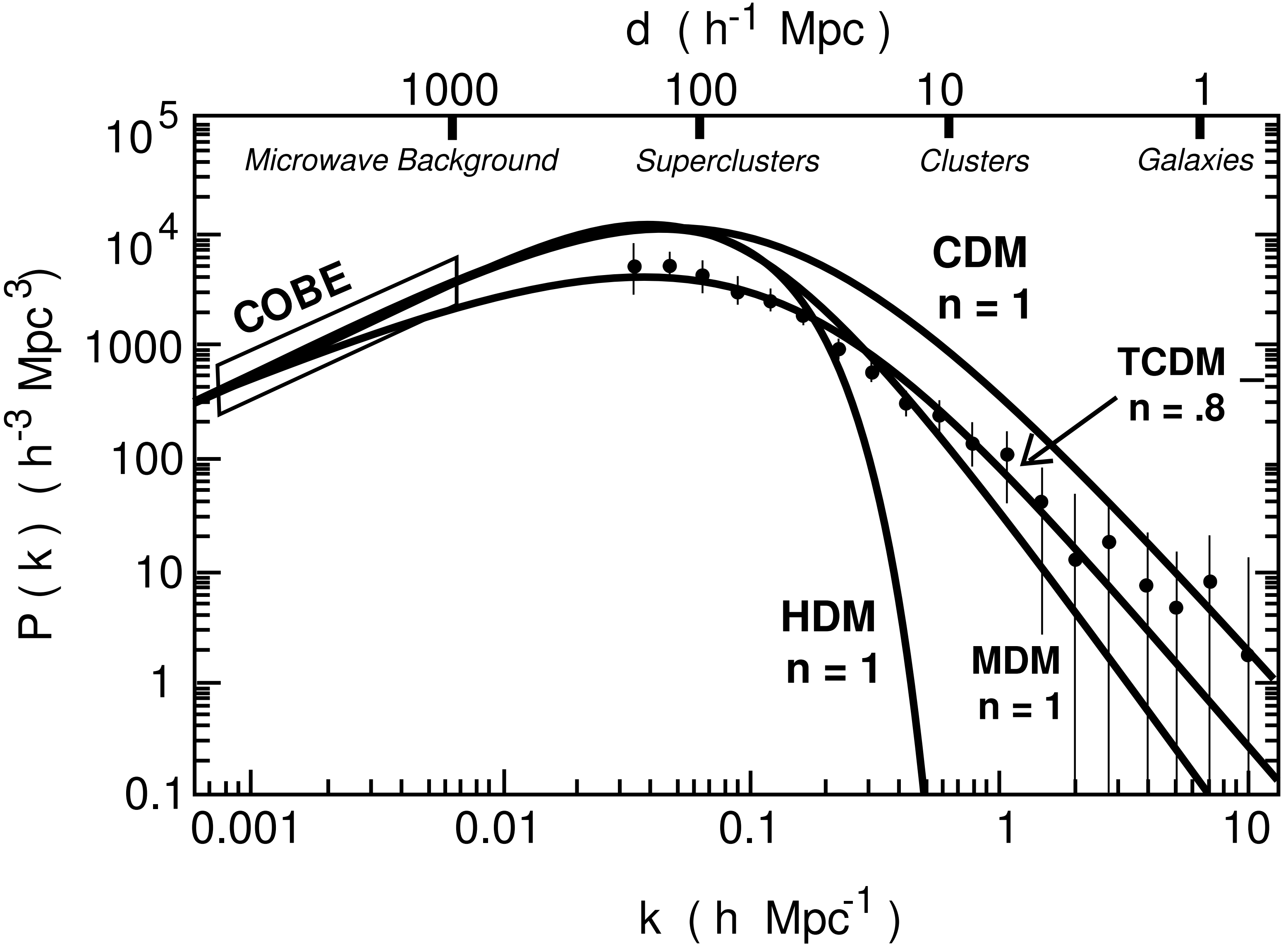
Simulation of actual Local Universe using observationally constrained $P(k)$

- territory of computational cosmology…



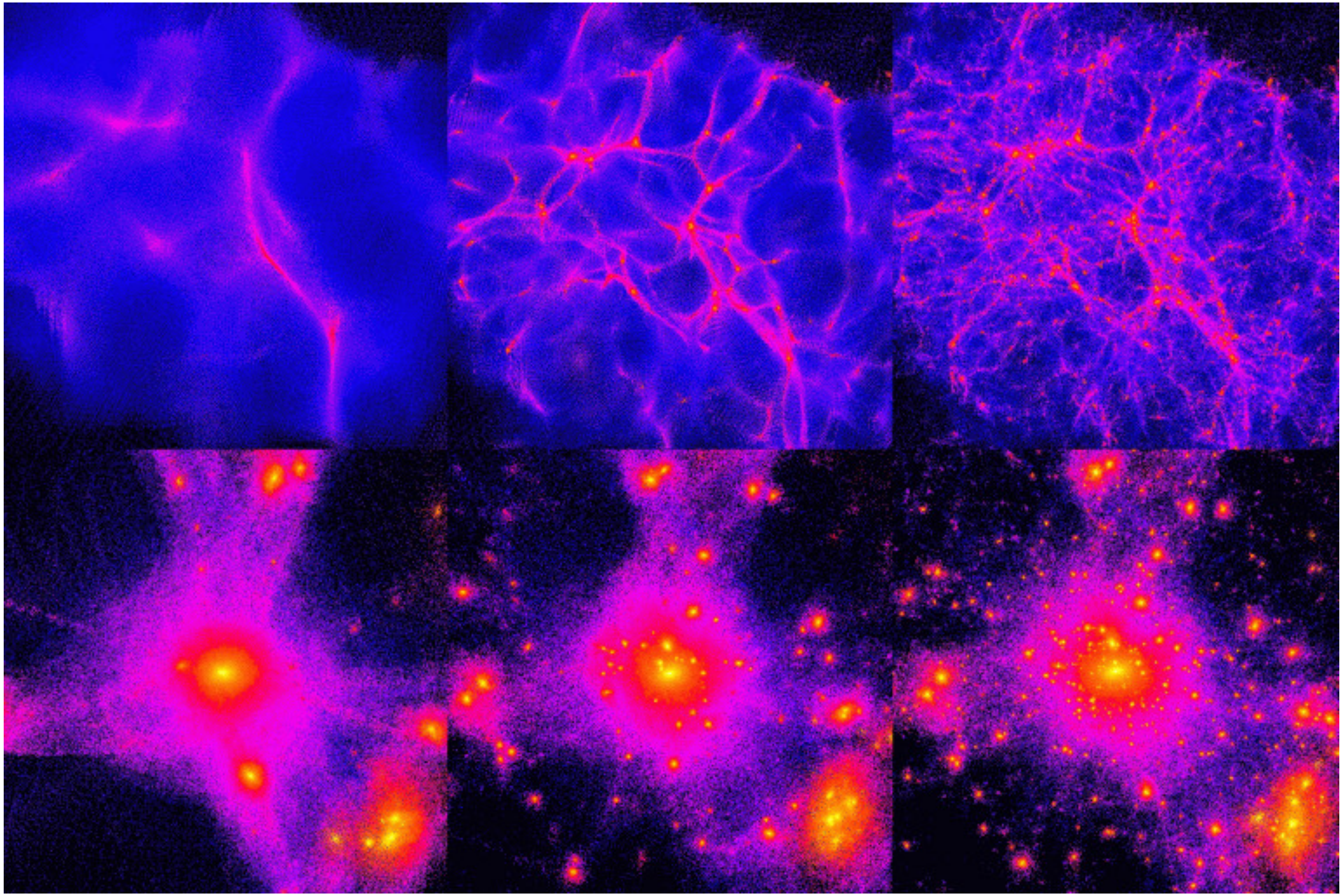
Simulation of actual Local Group using observationally constrained $P(k)$

- influence of nature of matter via $P(k)$



- influence of nature of matter via $P(k)$

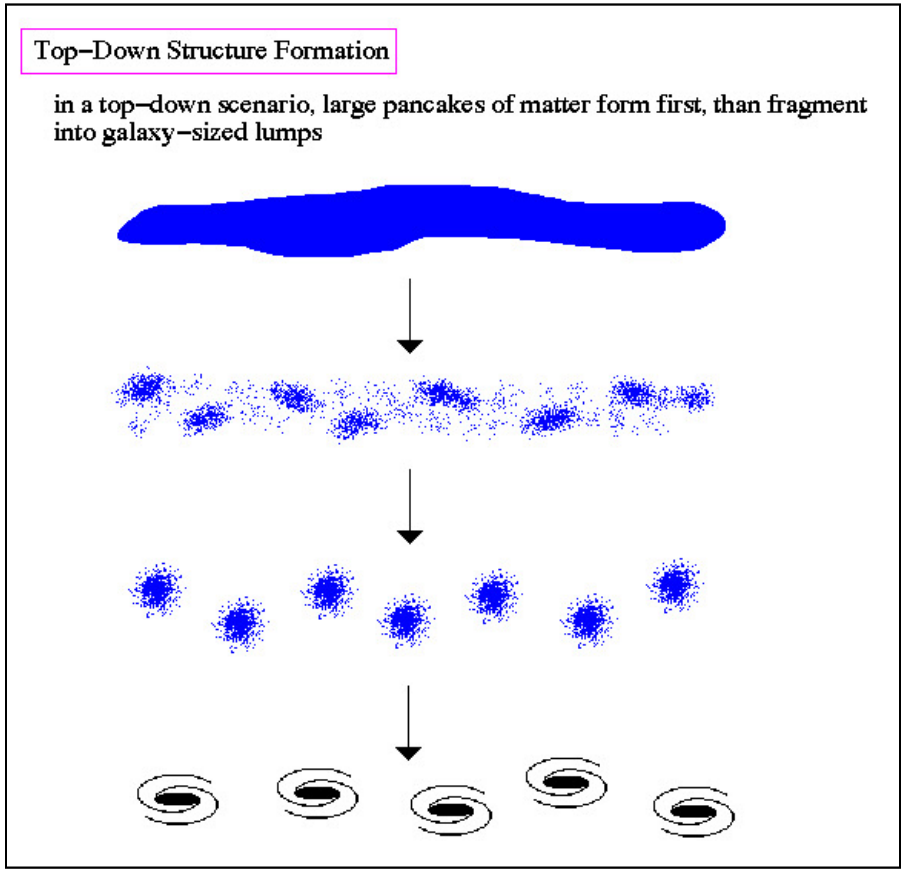
HDM WDM CDM



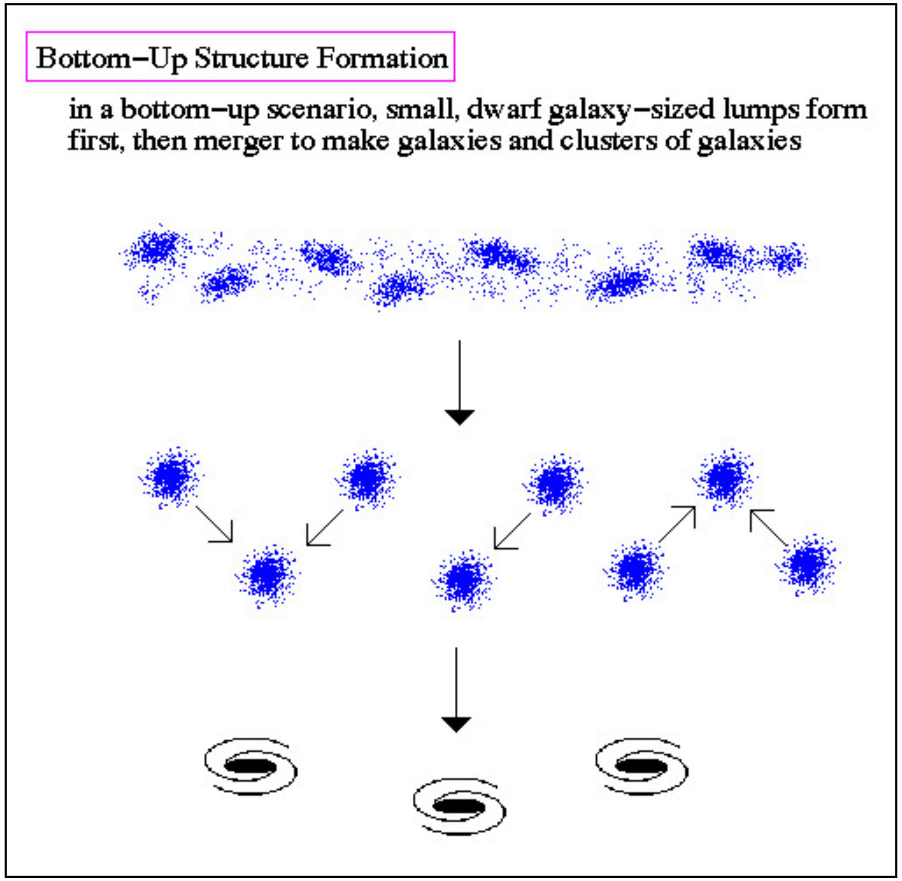
http://www.nbody.net

- influence of nature of matter via $P(k)$

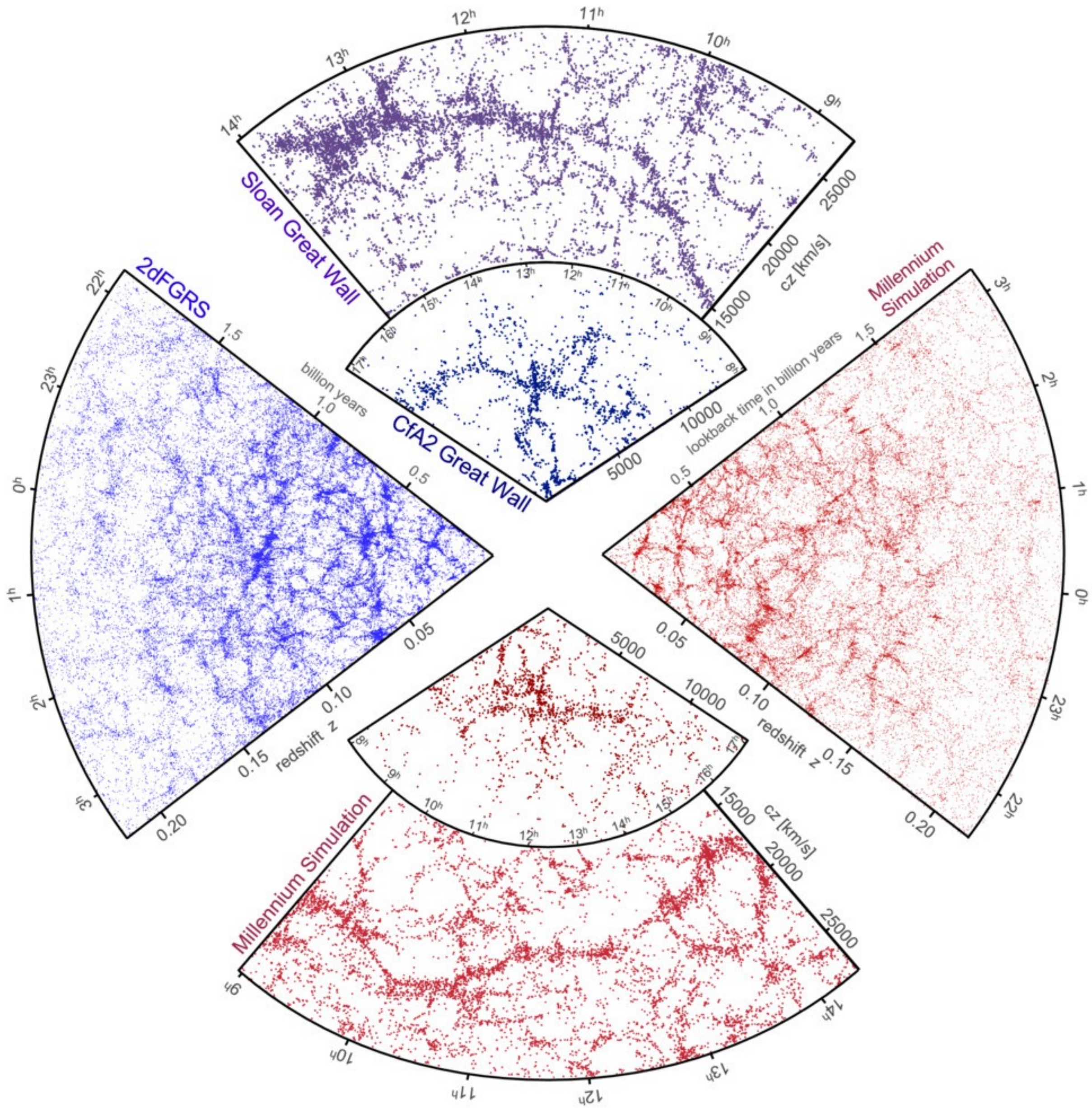
HDM



CDM



- simulations vs. observations



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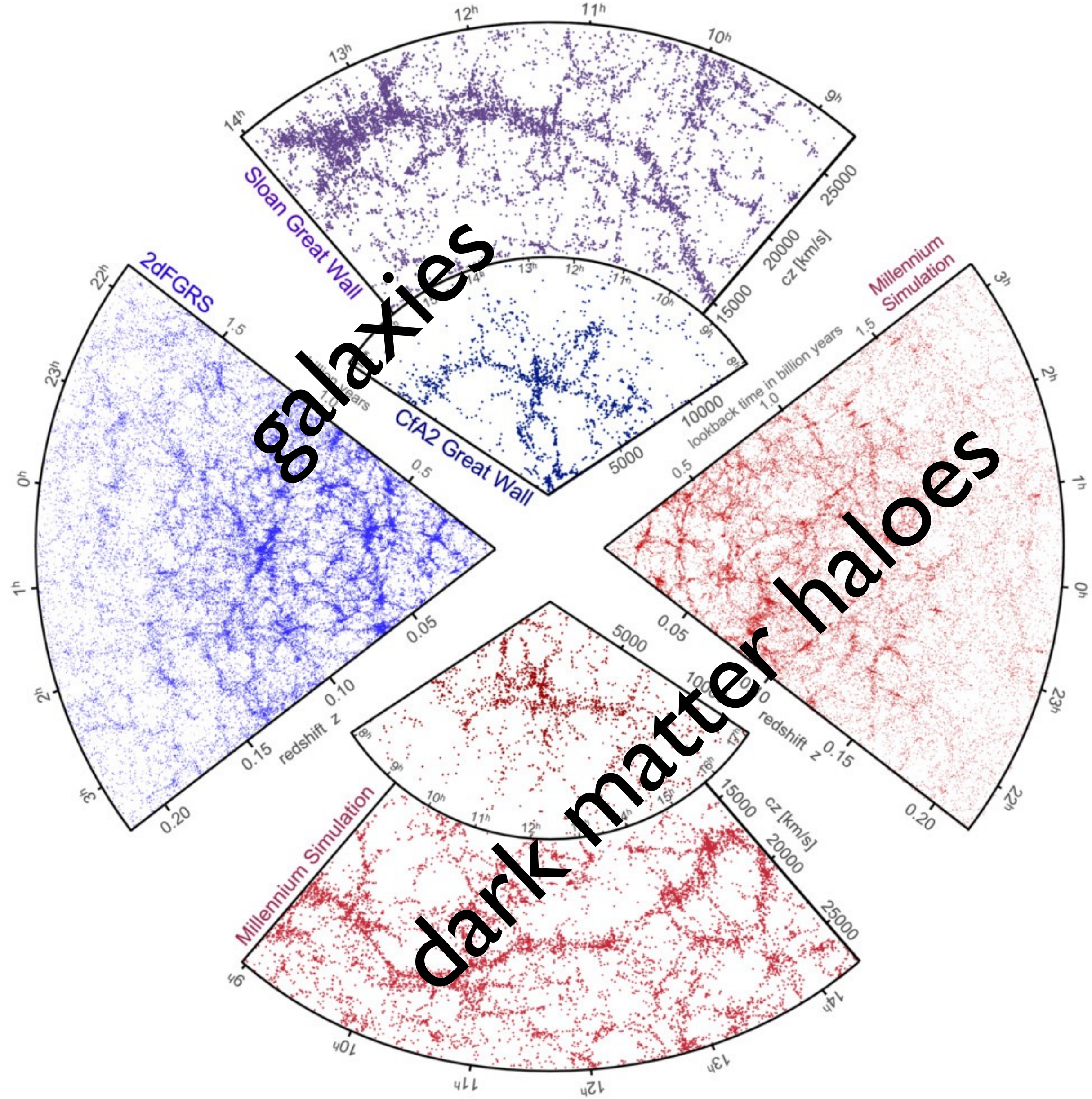

- simulations vs. observations
  - galaxy redshift surveys cover Gpc^3 volumes
  - simulations cannot follow full set of physics in such large volumes

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    - CLF: Conditional Luminosity Function
    - (S)HAM: (Subhalo) Halo Abundance Matching
    - biasing model: $P_{\mathrm{gal}}(k) = b^2(k)\, P_{\mathrm{DM}}(k)$
    - …

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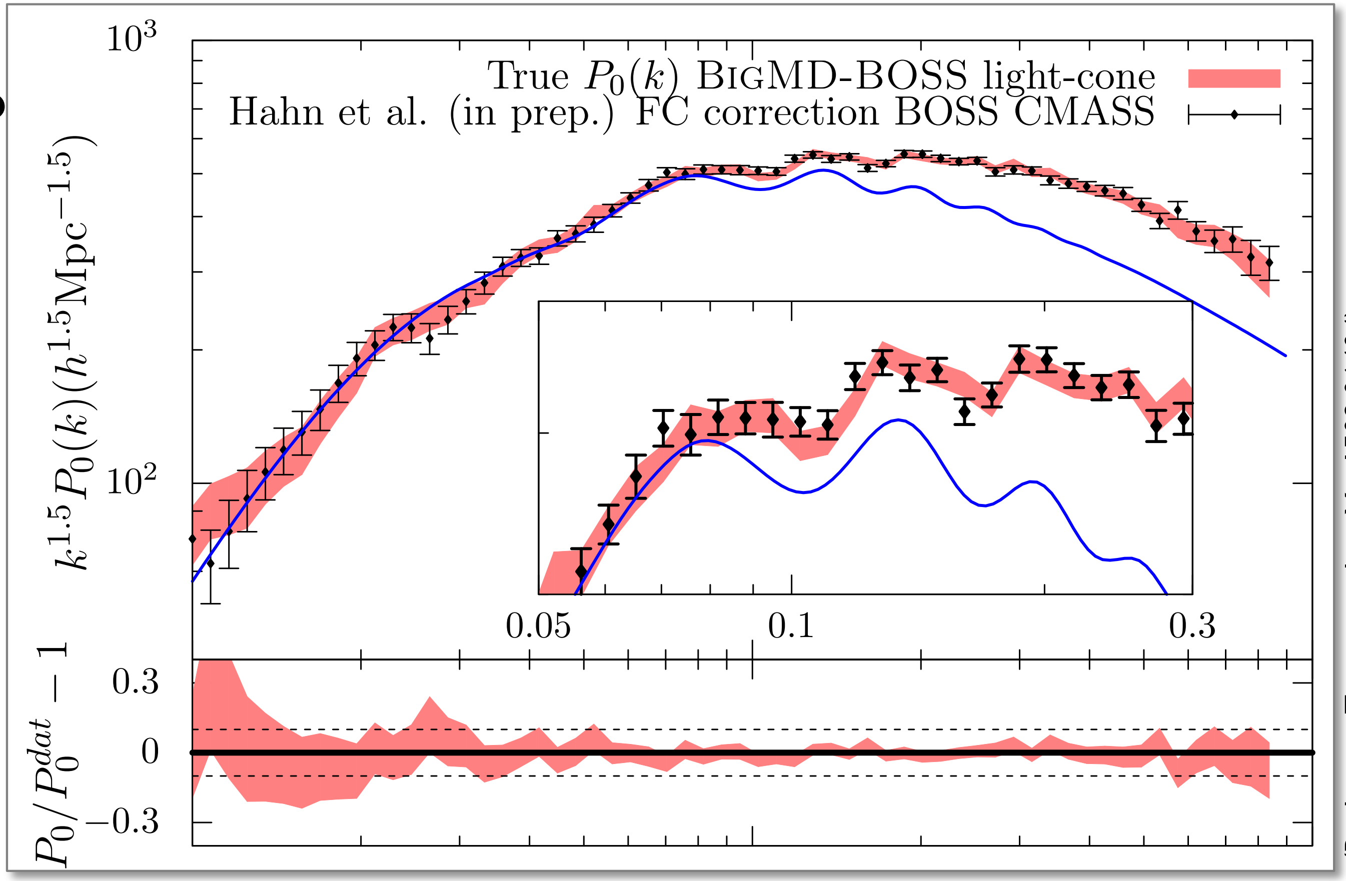

(Rodriguez-Torres et al., arXiv:1509.06404)

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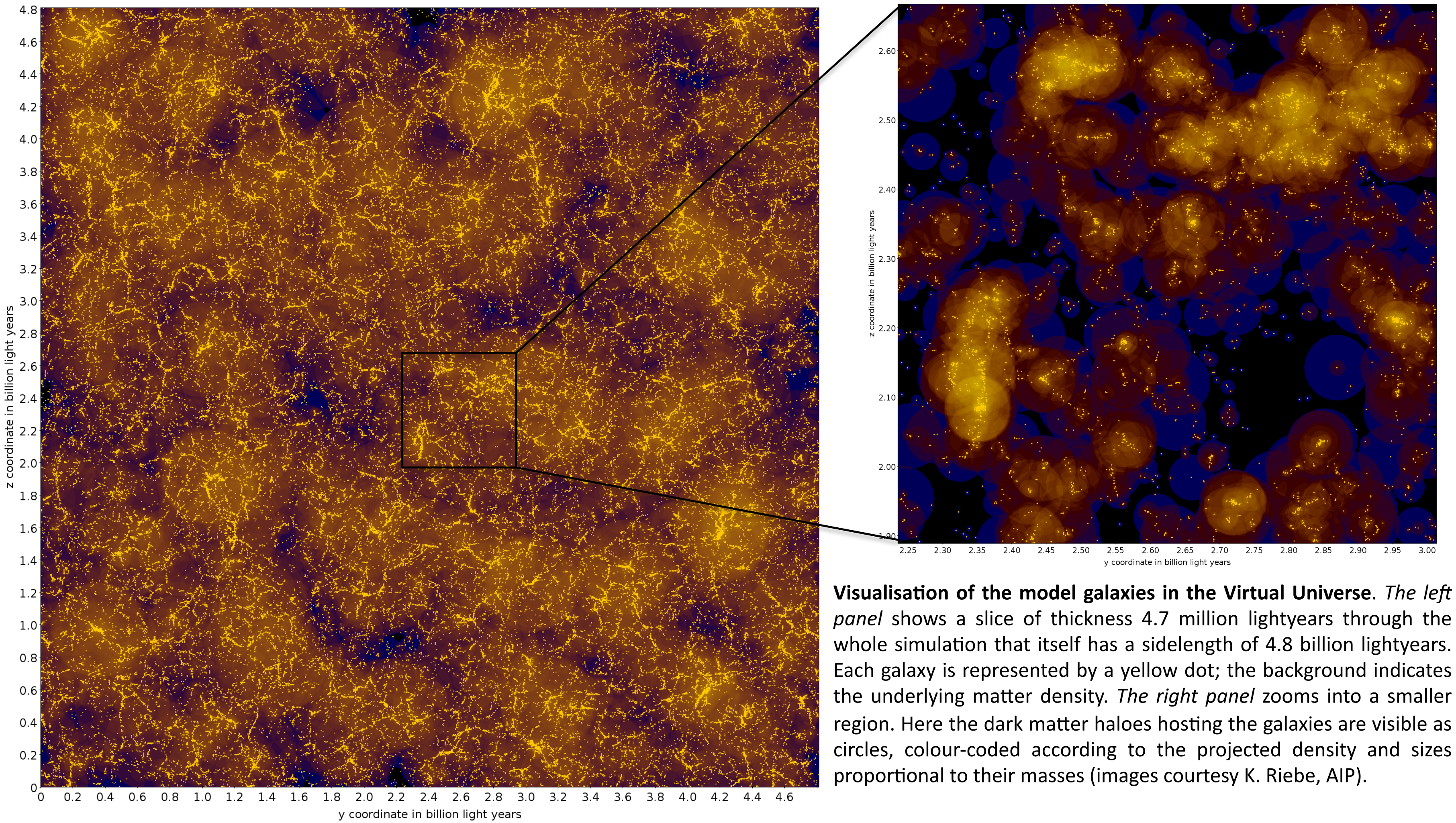

Visualisation of the model galaxies in the Virtual Universe. *The left panel* shows a slice of thickness 4.7 million lightyears through the whole simulation that itself has a sidelength of 4.8 billion lightyears. Each galaxy is represented by a yellow dot; the background indicates the underlying matter density. *The right panel* zooms into a smaller region. Here the dark matter haloes hosting the galaxies are visible as circles, colour-coded according to the projected density and sizes proportional to their masses (images courtesy K. Riebe, AIP).

"MultiDark Galaxies" (Knebe et al. 2018)

- simulations vs. observations

  - galaxy redshift surveys cover Gpc^3 volumes
  - simulations cannot follow full set of physics in such large volumes

  - but simulations are nevertheless quite spectacular…

galaxy formation simulations (incl. baryonic physics):

Illustris: https://www.youtube.com/watch?v=NjSFR40SY58
Eagle: https://www.youtube.com/watch?v=5F6bDRcy-mA
CLUES: https://www.cosmosim.org/cms/images-and-movies

LSS simulations (DM only):

Horizon: http://www.horizon-simulation.org/movies/horizonAGN-all.avi
DEUS: https://www.youtube.com/watch?v=kJ1r5NG5YGs
MultiDark: https://www.cosmosim.org/cms/images-and-movies/

- simulations database