THE ORIGIN OF DENSITY PERTURBATIONS

If cosmological inflation made the universe so extremely flat and homogeneous, where did the galaxies and clusters of galaxies come from? One of the most astonishing predictions of inflation, one that was not even expected, is that quantum fluctuations of the inflaton field are stretched by the exponential expansion and generate large-scale perturbations in the metric. Inflaton fluctuations are small wave packets of energy that, according to general relativity, modify the space-time fabric, creating a whole spectrum of curvature perturbations. The use of the word spectrum here is closely related to the case of light waves propagating in a medium: a spectrum characterizes the amplitude of each given wavelength. In the case of inflation, the inflaton fluctuations induce waves in the space-time metric that can be decomposed into different wavelengths, all with approximately the same amplitude, that is, corresponding to a scale-invariant spectrum. These patterns of perturbations in the metric are like fingerprints that unequivocally characterize a period of inflation. When matter fell in the troughs of these waves, it created density perturbations that collapsed gravitationally to form galaxies, clusters and superclusters of galaxies, with a spectrum that is also scale invariant. Such a type of spectrum was proposed in the early 1970s (before inflation) by Harrison and Zel'dovich, to explain the distribution of galaxies and clusters of galaxies on very large scales in our observable universe. Perhaps the most interesting aspect of structure formation is the possibility that the detailed knowledge of what seeded galaxies and clusters of galaxies will allow us to test the idea of inflation.

Reparametrization invariant perturbation theory

Until now we have considered only the unperturbed FRW metric described by a scale factor $a(t)$ and a homogeneous scalar field $\phi(t)$,

$$ds^2 = a^2(\eta)[-d\eta^2 + \gamma_{ij}\, dx^i dx^j]\,, \tag{1}$$
$$\phi = \phi(\eta)\,, \tag{2}$$

where $\eta = \int dt/a(t)$ is the conformal time, under which the background equations of motion can be written as

$$\mathcal{H}^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}\phi'^2 + a^2 V(\phi)\right)\,, \tag{3}$$
$$\mathcal{H}' - \mathcal{H}^2 = -\frac{\kappa^2}{2}\phi'^2\,, \tag{4}$$
$$\phi'' + 2\mathcal{H}\phi' + a^2 V'(\phi) = 0\,, \tag{5}$$

where $\mathcal{H} = aH$ and $\phi' = a\dot{\phi}$.

During inflation, the quantum fluctuations of the scalar field will induce metric perturbations which will backreact on the scalar field. Let us consider, in linear perturbation theory, the most general line element with both scalar and tensor metric perturbations,[1] together with the scalar field perturbations

$$\begin{aligned} ds^2 &= a^2(\eta)\left[-(1+2A)\,d\eta^2 + 2B_{|i}\,dx^i d\eta \right.\\ &\qquad \left. + \Big\{(1+2\mathcal{R})\gamma_{ij} + 2E_{|ij} + 2h_{ij}\Big\}dx^i dx^j\right]\,, \end{aligned} \tag{6}$$
$$\phi = \phi(\eta) + \delta\phi(\eta, x^i)\,. \tag{7}$$

The indices $\{i, j\}$ label the three-dimensional spatial coordinates with metric γ_{ij}, and the $|i$ denotes covariant derivative with respect to that metric. The gauge invariant tensor perturbation h_{ij} corresponds to

[1] Note that inflation cannot generate, to linear order, a vector perturbation.

a transverse traceless gravitational wave, $\nabla^i h_{ij} = h^i_i = 0$. The four scalar perturbations $(A, B, \mathcal{R}, E)$ are *gauge dependent* functions of (η, x^i). Under a general coordinate (gauge) transformation

$$\tilde{\eta} = \eta + \xi^0(\eta, x^i)\,, \tag{8}$$

$$\tilde{x}^i = x^i + \gamma^{ij}\xi_{|j}(\eta, x^i)\,, \tag{9}$$

with arbitrary functions (ξ^0, ξ), the scalar and tensor perturbations transform, to linear order, as

$$\tilde{A} = A - \xi^{0\prime} - \mathcal{H}\xi^0\,, \qquad \tilde{B} = B + \xi^0 - \xi'\,, \tag{10}$$

$$\tilde{\mathcal{R}} = \mathcal{R} - \mathcal{H}\xi^0\,, \qquad \tilde{E} = E - \xi\,, \tag{11}$$

$$\tilde{h}_{ij} = h_{ij}\,, \tag{12}$$

where a prime denotes derivative with respect to conformal time. It is possible to construct, however, two gauge-invariant gravitational potentials,

$$\Phi = A + (B - E')' + \mathcal{H}(B - E')\,, \tag{13}$$

$$\Psi = \mathcal{R} + \mathcal{H}(B - E')\,, \tag{14}$$

which are related through the perturbed Einstein equations,

$$\Phi = \Psi\,, \tag{15}$$

$$\frac{k^2 - 3K}{a^2}\Psi = \frac{\kappa^2}{2}\delta\rho\,, \tag{16}$$

where $\delta\rho$ is the gauge-invariant density perturbation, and the latter expression is nothing but the Poisson equation for the gravitational potential, written in relativistic form.

During inflation, the energy density is given in terms of a scalar field, and thus the gauge-invariant equations for the perturbations on comoving hypersurfaces (constant energy density hypersurfaces) are

$$
\begin{aligned}
\Phi'' + 3\mathcal{H}\Phi' + (\mathcal{H}' + 2\mathcal{H}^2)\Phi &= \frac{\kappa^2}{2}[\phi'\delta\phi' - a^2V'(\phi)\delta\phi]\,, && (17)\\
-\nabla^2\Phi + 3\mathcal{H}\Phi' + (\mathcal{H}' + 2\mathcal{H}^2)\Phi &= -\frac{\kappa^2}{2}[\phi'\delta\phi' + a^2V'(\phi)\delta\phi]\,, && (18)\\
\Phi' + \mathcal{H}\Phi &= \frac{\kappa^2}{2}\phi'\delta\phi\,, && (19)\\
\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi = 4\phi'\Phi' &- 2a^2V'(\phi)\Phi - a^2V''(\phi)\delta\phi\,. && (20)
\end{aligned}
$$

This system of equations seem too difficult to solve at first sight. However, there is a gauge invariant combination of variables that allows one to find exact solutions. Let us define

$$
u \equiv a\delta\phi + z\Phi\,, \tag{21}
$$

$$
z \equiv a\frac{\phi'}{\mathcal{H}}\,. \tag{22}
$$

Under this redefinition, the above equations simplify enormously to just three independent equations,

$$
u'' - \nabla^2 u - \frac{z''}{z}u = 0\,, \tag{23}
$$

$$
\nabla^2\Phi = \frac{\kappa^2}{2}\frac{\mathcal{H}}{a^2}(zu' - z'u)\,, \tag{24}
$$

$$
\left(\frac{a^2\Phi}{\mathcal{H}}\right)' = \frac{\kappa^2}{2}zu\,. \tag{25}
$$

From Equation (23) we can find a solution $u(z)$, which substituted into (25) can be integrated to give $\Phi(z)$, and together with $u(z)$ allow us to obtain $\delta\phi(z)$.

Quantum Mechanics in curved space-time

Until now we have treated the perturbations as classical, but we should in fact consider the perturbations Φ and $\delta\phi$ as quantum fields. Note that the perturbed action for the scalar mode u can be written as

$$\delta S = \frac{1}{2}\int d^3x\, d\eta \left[(u')^2 - (\nabla u)^2 + \frac{z''}{z}u^2\right]. \tag{26}$$

In order to quantize the field u in the curved background defined by the metric (1), we can write the operator

$$\hat{u}(\eta,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\left[u_k(\eta)\,\hat{a}_{\mathbf{k}}\,e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\eta)\,\hat{a}_{\mathbf{k}}^\dagger\,e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{27}$$

where the creation and annihilation operators satisfy the commutation relation of bosonic fields, and the scalar field's Fock space is defined through the vacuum condition,

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta^3(\mathbf{k}-\mathbf{k}')\,, \tag{28}$$

$$\hat{a}_{\mathbf{k}}|0\rangle = 0\,. \tag{29}$$

Note that we are not assuming that the inflaton is a fundamental scalar field, but that is can be written as a quantum field with its commutation relations (as much as a pion can be described as a quantum field).

The equations of motion for each mode $u_k(\eta)$ are decoupled in linear perturbation theory,

$$u_k'' + \left(k^2 - \frac{z''}{z}\right)u_k = 0\,. \tag{30}$$

The ratio z''/z acts like a time-dependent potential for this Schrödinger like equation. In order to find exact solutions to the mode equation, we will use the slow-roll parameters,

$$\epsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = \frac{\kappa^2}{2}\frac{z^2}{a^2}\,, \tag{31}$$

$$\delta = 1 - \frac{\phi''}{\mathcal{H}\phi'} = 1 + \epsilon - \frac{z'}{\mathcal{H}z}\,, \tag{32}$$

$$\xi = -\left(2 - \epsilon - 3\delta + \delta^2 - \frac{\phi'''}{\mathcal{H}^2\phi'}\right). \tag{33}$$

In terms of these parameters, the conformal time and the effective potential for the u_k mode can be written as

$$\eta = \frac{-1}{\mathcal{H}} + \int \frac{\epsilon da}{a\mathcal{H}}\,, \tag{34}$$

$$\frac{z''}{z} = \mathcal{H}^2 \left[(1+\epsilon-\delta)(2-\delta) + \mathcal{H}^{-1}(\epsilon'-\delta')\right]. \tag{35}$$

Note that the slow-roll parameters, (31) and (32), can be taken as *constant*,[2] to order ϵ^2,

$$\begin{aligned} \epsilon' &= 2\mathcal{H}\left(\epsilon^2 - \epsilon\delta\right) = \mathcal{O}(\epsilon^2)\,, \\ \delta' &= \mathcal{H}\left(\epsilon\delta - \xi\right) = \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{36}$$

In that case, for constant slow-roll parameters, we can write

$$\eta = \frac{-1}{\mathcal{H}}\frac{1}{1-\epsilon}\,, \tag{37}$$

$$\frac{z''}{z} = \frac{1}{\eta^2}\Big(\nu^2 - \frac{1}{4}\Big)\,, \qquad \text{where} \quad \nu = \frac{1+\epsilon-\delta}{1-\epsilon} + \frac{1}{2}\,. \tag{38}$$

We are now going to search for approximate solutions of the mode equation (30), where the effective potential (35) is of order $z''/z \simeq 2\mathcal{H}^2$ in the slow-roll approximation. In quasi-de Sitter there is a characteristic scale given by the (event) horizon size or Hubble scale during inflation, H^{-1}. There will be modes u_k with physical wavelengths much smaller than this scale, $k/a \gg H$, that are well within the de Sitter horizon and therefore do not feel the curvature of space-time. On the other hand, there will be modes with physical wavelengths much greater than the Hubble scale, $k/a \ll H$. In these two asymptotic regimes, the solutions can be written as

$$u_k = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} \qquad k \gg aH\,, \tag{39}$$

$$u_k = C_1\, z \qquad\qquad k \ll aH\,. \tag{40}$$

[2]For instance, there are models of inflation, like power-law inflation, $a(t) \sim t^p$, where $\epsilon = \delta = 1/p < 1$, that give constant slow-roll parameters.

In the limit $k \gg aH$ the modes behave like ordinary quantum modes in Minkowsky space-time, appropriately normalized, while in the opposite limit, u/z becomes constant on superhorizon scales. For approximately constant slow-roll parameters one can find exact solutions to (30), with the effective potential given by (38), that interpolate between the two asymptotic solutions,

$$u_k(\eta) = \frac{\sqrt{\pi}}{2}\, e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}\,(-\eta)^{1/2}\, H_\nu^{(1)}(-k\eta)\,, \tag{41}$$

where $H_\nu^{(1)}(z)$ is the Hankel function of the first kind, and ν is given by (38) in terms of the slow-roll parameters. In the limit $k\eta \to 0$, the solution becomes

$$|u_k| = \frac{2^{\nu-\frac{3}{2}}}{\sqrt{2k}}\,\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\,(-k\eta)^{\frac{1}{2}-\nu} \equiv \frac{C(\nu)}{\sqrt{2k}}\left(\frac{k}{aH}\right)^{\frac{1}{2}-\nu}\,, \tag{42}$$

$$C(\nu) = 2^{\nu-\frac{3}{2}}\,\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\,(1-\epsilon)^{\nu-\frac{1}{2}} \;\simeq 1 \qquad \text{for} \quad \epsilon, \delta \ll 1\,. \tag{43}$$

We can now compute Φ and $\delta\phi$ from the super-Hubble-scale mode solution (40), for $k \ll aH$. Substituting into Eq. (25), we find

$$\Phi = C_1\left(1 - \frac{\mathcal{H}}{a^2}\int a^2 d\eta\right) + C_2\,\frac{\mathcal{H}}{a^2}\,, \tag{44}$$

$$\delta\phi = \frac{C_1}{a^2}\int a^2 d\eta - \frac{C_2}{a^2}\,. \tag{45}$$

The term proportional to C_1 corresponds to the growing solution, while that proportional to C_2 corresponds to the decaying solution, which can soon be ignored. These quantities are gauge invariant but evolve with time outside the horizon, during inflation, and before entering again the horizon during the radiation or matter eras. We would like to write an expression for a gauge invariant quantity that is also *constant* for superhorizon modes. Fortunately, in the case of adiabatic perturbations, there is such a quantity:

$$\zeta \equiv \Phi + \frac{1}{\epsilon\mathcal{H}}\left(\Phi' + \mathcal{H}\Phi\right) = \frac{u}{z}\,, \tag{46}$$

which is constant, see Eq. (40), for $k \ll aH$. In fact, this quantity ζ is identical, for superhorizon modes, to the gauge invariant curvature metric perturbation $\mathcal{R}_c$ on comoving (constant energy density) hypersurfaces,

$$\zeta = \mathcal{R}_c + \frac{1}{\epsilon\mathcal{H}^2}\,\nabla^2\Phi\,. \tag{47}$$

Using Eq. (24) we can write the evolution equation for $\zeta = \frac{u}{z}$ as $\zeta' = \frac{1}{\epsilon\mathcal{H}}\,\nabla^2\Phi$, which confirms that ζ is constant for (adiabatic[3]) superhorizon modes, $k \ll aH$. Therefore, we can evaluate the Newtonian potential Φ_k when the perturbation reenters the horizon during radiation/matter eras in terms of the curvature perturbation $\mathcal{R}_k$ when it left the Hubble scale during inflation,

$$\Phi_k = \left(1 - \frac{\mathcal{H}}{a^2}\int a^2 d\eta\right)\mathcal{R}_k = \frac{3+3\omega}{5+3\omega}\,\mathcal{R}_k = \begin{cases} \frac{2}{3}\,\mathcal{R}_k & \text{radiation era}\,, \\ \frac{3}{5}\,\mathcal{R}_k & \text{matter era}\,. \end{cases} \tag{48}$$

Let us now compute the tensor or gravitational wave metric perturbations generated during inflation. The perturbed action for the tensor mode can be written as

$$\delta S = \frac{1}{2}\int d^3x\, d\eta\, \frac{a^2}{2\kappa^2}\left[(h'_{ij})^2 - (\nabla h_{ij})^2\right], \tag{49}$$

with the tensor field h_{ij} considered as a quantum field,

$$\hat{h}_{ij}(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \sum_{\lambda=1,2}\left[h_k(\eta)\, e_{ij}(\mathbf{k},\lambda)\, \hat{a}_{\mathbf{k},\lambda}\, e^{i\mathbf{k}\cdot\mathbf{x}} + h.c.\right], \tag{50}$$

where $e_{ij}(\mathbf{k},\lambda)$ are the two polarization tensors, satisfying symmetric,

[3]This conservation fails for entropy or isocurvature perturbations

transverse and traceless conditions

$$e_{ij} = e_{ji}\,, \quad k^i e_{ij} = 0\,, \quad e_{ii} = 0\,, \tag{51}$$

$$e_{ij}(-\mathbf{k},\lambda) = e^*_{ij}(\mathbf{k},\lambda)\,, \quad \sum_\lambda e^*_{ij}(\mathbf{k},\lambda)e^{ij}(\mathbf{k},\lambda) = 4\,, \tag{52}$$

while the creation and annihilation operators satisfy the usual commutation relation of bosonic fields, Eq. (28). We can now redefine our gauge invariant tensor amplitude as

$$v_k(\eta) = \frac{a}{\sqrt{2}\kappa}\, h_k(\eta)\,, \tag{53}$$

which satisfies the following evolution equation, decoupled for each mode $v_k(\eta)$ in linear perturbation theory,

$$v''_k + \Big(k^2 - \frac{a''}{a}\Big) v_k = 0\,. \tag{54}$$

The ratio a''/a acts like a time-dependent potential for this Schrödinger like equation, analogous to the term z''/z for the scalar metric perturbation. For constant slow-roll parameters, the potential becomes

$$\frac{a''}{a} = 2\mathcal{H}^2\Big(1 - \frac{\epsilon}{2}\Big) = \frac{1}{\eta^2}\Big(\mu^2 - \frac{1}{4}\Big)\,, \tag{55}$$

$$\mu = \frac{1}{1-\epsilon} + \frac{1}{2}\,. \tag{56}$$

We can solve equation (54) in the two asymptotic regimes,

$$v_k = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} \qquad k \gg aH\,, \tag{57}$$

$$v_k = C\,a \qquad\qquad k \ll aH\,. \tag{58}$$

In the limit $k \gg aH$ the modes behave like ordinary quantum modes in Minkowsky space-time, appropriately normalized, while in the opposite limit, the metric perturbation h_k becomes *constant* on superhorizon scales. For constant slow-roll parameters one can find exact solutions to (54), with effective potential given by (55), that interpolate between the two asymptotic solutions. These are identical to Eq. (41) except for the substitution $\nu \to \mu$. In the limit $k\eta \to 0$, the solution becomes

$$|v_k| = \frac{C(\mu)}{\sqrt{2k}} \left(\frac{k}{aH}\right)^{\frac{1}{2}-\mu} . \tag{59}$$

Since the mode h_k becomes constant on superhorizon scales, we can evaluate the tensor metric perturbation when it reentered during the radiation or matter era directly in terms of its value during inflation.

Power spectrum of scalar and tensor metric perturbations

Not only do we expect to measure the amplitude of the metric perturbations generated during inflation and responsible for the anisotropies in the CMB and density fluctuations in LSS, but we should also be able to measure its power spectrum, or two-point correlation function in Fourier space. Let us consider first the scalar metric perturbations $\mathcal{R}_k$, which enter the horizon at $a = k/H$. Its correlator is given by

$$\langle 0|\mathcal{R}_k^* \mathcal{R}_{k'}|0\rangle = \frac{|u_k|^2}{z^2}\, \delta^3(\mathbf{k}-\mathbf{k}') \equiv \frac{\mathcal{P}_\mathcal{R}(k)}{4\pi k^3}\, (2\pi)^3\, \delta^3(\mathbf{k}-\mathbf{k}')\,, \tag{60}$$

$$\mathcal{P}_\mathcal{R}(k) = \frac{k^3}{2\pi^2}\,\frac{|u_k|^2}{z^2} = \frac{\kappa^2}{2\epsilon}\left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu} \equiv A_S^2 \left(\frac{k}{aH}\right)^{n_s-1} \tag{61}$$

where we have used $\mathcal{R}_k = \zeta_k = \frac{u_k}{z}$ and Eq. (42). This last equation determines the power spectrum in terms of its amplitude at horizon-crossing, A_S, and a tilt,

$$n_s - 1 \equiv \frac{d\ln \mathcal{P}_\mathcal{R}(k)}{d\ln k} = 3 - 2\nu = 2\Big(\frac{\delta - 2\epsilon}{1-\epsilon}\Big) \simeq 2\eta_V - 6\epsilon_V\,. \tag{62}$$

Note that it is possible, in principle, to obtain from inflation a scalar tilt which is either positive ($n > 1$) or negative ($n < 1$). Furthermore, depending on the particular inflationary model, we can have significant departures from scale invariance.

Note that at horizon entry $k\eta = -1$, and thus we can alternatively evaluate the tilt as

$$n_s - 1 \equiv -\frac{d\ln \mathcal{P}_\mathcal{R}}{d\ln\eta} = -2\eta\mathcal{H}\Big[(1-\epsilon) - (\epsilon-\delta) - 1\Big] = 2\Big(\frac{\delta - 2\epsilon}{1-\epsilon}\Big) \simeq 2\eta_V - 6\epsilon_V\,, \tag{63}$$

and the running of the tilt

$$\frac{dn_s}{d\ln k} = -\frac{dn_s}{d\ln\eta} = -\eta\mathcal{H}\Big(2\xi + 8\epsilon^2 - 10\epsilon\delta\Big) \simeq 2\xi_V + 24\epsilon_V^2 - 16\eta_V\epsilon_V\,, \tag{64}$$

where we have used Eqs. (36).

Let us consider now the tensor (gravitational wave) metric perturbation, which enter the horizon at $a = k/H$,

$$\sum_\lambda \langle 0|h^*_{k,\lambda} h_{k',\lambda}|0\rangle = \frac{8\kappa^2}{a^2}\,|v_k|^2\delta^3(\mathbf{k}-\mathbf{k}') \equiv \frac{\mathcal{P}_g(k)}{4\pi k^3}\,(2\pi)^3\,\delta^3(\mathbf{k}-\mathbf{k}')\,,$$

$$\mathcal{P}_g(k) = 8\kappa^2\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\mu} \equiv A_T^2\left(\frac{k}{aH}\right)^{n_T}\,, \tag{65}$$

where we have used Eqs. (53) and (59). Therefore, the power spectrum can be approximated by a power-law expression, with amplitude A_T and tilt

$$n_T \equiv \frac{d\ln\mathcal{P}_g(k)}{d\ln k} = 3 - 2\mu = \frac{-2\epsilon}{1-\epsilon} \simeq -2\epsilon_V < 0\,, \tag{66}$$

which is always negative. In the slow-roll approximation, $\epsilon \ll 1$, the tensor power spectrum is scale invariant.

Alternatively, we can evaluate the tensor tilt by

$$n_T \equiv -\frac{d\ln\mathcal{P}_g}{d\ln\eta} = -2\eta\mathcal{H}\Big[(1-\epsilon) - 1\Big] = \frac{-2\epsilon}{1-\epsilon} \simeq -2\epsilon_V\,, \tag{67}$$

and its running by

$$\frac{dn_T}{d\ln k} = -\frac{dn_T}{d\ln\eta} = -\eta\mathcal{H}\Big(4\epsilon^2 - 4\epsilon\delta\Big) \simeq 8\epsilon_V^2 - 4\eta_V\epsilon_V\,, \tag{68}$$

where we have used Eqs. (36).

Massless minimally coupled scalar field fluctuations

The fluctuations of a massless minimally-coupled scalar field ϕ during inflation (quasi de Sitter) are quantum fields in a curved background. We will redefine $y(\mathbf{x},t) = a(t)\,\delta\phi(\mathbf{x},t)$, whose action is

$$\mathcal{S} = \int d^4\mathbf{x}\,\mathcal{L}(y,y') = \frac{1}{2}\int d^3\mathbf{x}\,d\eta\,\left[(y')^2 - (\nabla y)^2 + \frac{a''}{a}\,y^2\right]\,, \quad (69)$$

where primes denote derivatives w.r.t. conformal time $\eta = \int dt/a(t) = -1/(aH)$, with H the constant rate of expansion during inflation. Now using the identity $(y')^2 + \frac{a''}{a}\,y^2 = (y' - \frac{a'}{a}\,y)^2 + (\frac{a'}{a}\,y^2)'$, which gives a total derivative in the Lagrangian, we can define the conjugate momentum as $p = \frac{\partial\mathcal{L}}{\partial y'} = y' - \frac{a'}{a}\,y$, and write the corresponding Hamiltonian as

$$\mathcal{H} = p\,y' - \mathcal{L}(y,y') = \frac{1}{2}\left[p^2 + (\nabla y)^2 + 2\,\frac{a'}{a}\,p\,y\right]\,. \quad (70)$$

We can now Fourier transform: $\Phi(\mathbf{k},\eta) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}}\,\Phi(\mathbf{x},\eta)\,e^{-i\,\mathbf{x}\cdot\mathbf{k}}$ all the fields and momenta. Since the scalar field is assumed real, we have: $y(\mathbf{k},\eta) = y^\dagger(-\mathbf{k},\eta)$ and $p(\mathbf{k},\eta) = p^\dagger(-\mathbf{k},\eta)$, and the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2}\Big[p(\mathbf{k},\eta)\,p^\dagger(\mathbf{k},\eta) + k^2\,y(\mathbf{k},\eta)\,y^\dagger(\mathbf{k},\eta) \quad (71)$$

$$+ \frac{a'}{a}\Big(y(\mathbf{k},\eta)\,p^\dagger(\mathbf{k},\eta) + p(\mathbf{k},\eta)\,y^\dagger(\mathbf{k},\eta)\Big)\Big]\,. \quad (72)$$

As we will see later, it is the last term, proportional to a'/a, which is responsible for squeezing.

The Euler-Lagrange equations for this field can be written in terms of the field eigenmodes as a series of uncoupled oscillator equations,

$$\left.\begin{aligned} p' &= -i\,[p,\mathcal{H}] = -k^2\,y - \frac{a'}{a}\,p \\ y' &= -i\,[y,\mathcal{H}] = p + \frac{a'}{a}\,y\,. \end{aligned}\right\} \quad y''(\mathbf{k},\eta) + \left(k^2 - \frac{a''}{a}\right)\,y(\mathbf{k},\eta) = 0\,, \tag{73}$$

where we have used the commutation relation ($\hbar = 1$)

$$\Big[y(\mathbf{k},\eta), p^\dagger(\mathbf{k},\eta)\Big] = i\,\delta^3(\mathbf{k}-\mathbf{k}')\,. \tag{74}$$

Heisenberg picture: The field operators

We can now treat each mode as a quantum oscillator, and introduce the corresponding creation and annihilation operators:

$$a(\mathbf{k},\eta) \;=\; \sqrt{\frac{k}{2}}\,y(\mathbf{k},\eta) + i\,\frac{1}{\sqrt{2k}}\,p(\mathbf{k},\eta)\,, \tag{75}$$

$$a^\dagger(-\mathbf{k},\eta) \;=\; \sqrt{\frac{k}{2}}\,y(\mathbf{k},\eta) - i\,\frac{1}{\sqrt{2k}}\,p(\mathbf{k},\eta)\,, \tag{76}$$

which can be inverted to give

$$y(\mathbf{k},\eta) \;=\; \frac{1}{\sqrt{2k}}\,\left[a(\mathbf{k},\eta) + a^\dagger(-\mathbf{k},\eta)\right]\,, \tag{77}$$

$$p(\mathbf{k},\eta) \;=\; -i\,\sqrt{\frac{k}{2}}\,\left[a(\mathbf{k},\eta) - a^\dagger(-\mathbf{k},\eta)\right]\,. \tag{78}$$

The usual equal-time commutation relations for fields ($\hbar = 1$ here and throughout),

$$\Big[y(\mathbf{x},\eta),\; p(\mathbf{x}',\eta)\Big] = i\,\delta^3(\mathbf{x}-\mathbf{x}')\,, \tag{79}$$

becomes a commutation relation for the creation and annihilation operators,

$$\Big[y(\mathbf{k},\eta),\ p^\dagger(\mathbf{k}',\eta)\Big] = i\,\delta^3(\mathbf{k}-\mathbf{k}') \Rightarrow \Big[a(\mathbf{k},\eta),\ a^\dagger(\mathbf{k}',\eta)\Big] = \delta^3(\mathbf{k}-\mathbf{k}')\,. \tag{80}$$

In terms of these operators, the Hamiltonian becomes:

$$\mathcal{H} \;=\; \frac{1}{2}\Big[k\,\Big(a(\mathbf{k},\eta)\,a^\dagger(\mathbf{k},\eta) + a^\dagger(-\mathbf{k},\eta)\,a(-\mathbf{k},\eta)\Big) \tag{81}$$

$$+i\,\frac{a'}{a}\Big(a^\dagger(-\mathbf{k},\eta)\,a^\dagger(\mathbf{k},\eta) - a(\mathbf{k},\eta)\,a(-\mathbf{k},\eta)\Big)\Big]\ . \tag{82}$$

It is the last (non-diagonal) term which is responsible for squeezing.

The evolution equations, $a' = -i[a,\mathcal{H}]$, can be written as

$$\begin{pmatrix} a'(\mathbf{k}) \\ a^{\dagger\prime}(-\mathbf{k}) \end{pmatrix} = \begin{pmatrix} -ik & \frac{a'}{a} \\ \frac{a'}{a} & ik \end{pmatrix} \begin{pmatrix} a(\mathbf{k}) \\ a^\dagger(-\mathbf{k}) \end{pmatrix}, \tag{83}$$

whose general solution is, in terms of the initial conditions $a(\mathbf{k},\eta_0)$,

$$a(\mathbf{k},\eta) = u_k(\eta)\,a(\mathbf{k},\eta_0) + v_k(\eta)\,a^\dagger(-\mathbf{k},\eta_0)\,, \tag{84}$$

$$a^\dagger(-\mathbf{k},\eta) = u_k^*(\eta)\,a^\dagger(-\mathbf{k},\eta_0) + v_k^*(\eta)\,a(\mathbf{k},\eta_0)\,, \tag{85}$$

which correspond to a Bogoliubov transformation of the creation and annihilation operators, and characterizes the time evolution of the system of harmonic oscillators in the Heisenberg representation.

The commutation relation (80) is preserved under the unitary evolution if

$$|u_k(\eta)|^2 - |v_k(\eta)|^2 = 1\,, \tag{86}$$

which gives a normalization condition for these functions.

We can write the quantum fields y and p in terms of these as,

$$y(\mathbf{k},\eta) \;=\; f_k(\eta)\,a(\mathbf{k},\eta_0) + f_k^*(\eta)\,a^\dagger(-\mathbf{k},\eta_0)\,, \tag{87}$$

$$p(\mathbf{k},\eta) \;=\; -i\,\left[g_k(\eta)\,a(\mathbf{k},\eta_0) - g_k^*(\eta)\,a^\dagger(-\mathbf{k},\eta_0)\right]\,, \tag{88}$$

where the functions

$$f_k(\eta) \;=\; \frac{1}{\sqrt{2k}}\,[u_k(\eta) + v_k^*(\eta)]\,, \tag{89}$$

$$g_k(\eta) \;=\; \sqrt{\frac{k}{2}}\,[u_k(\eta) - v_k^*(\eta)]\,, \tag{90}$$

are the field and momentum modes, respectively, satisfying the following equations and initial conditions,

$$f_k'' + \left(k^2 - \frac{a''}{a}\right) f_k = 0\,, \qquad f_k(\eta_0) = \tfrac{1}{\sqrt{2k}}\,, \tag{91}$$

$$g_k = i\,\left(f_k' - \frac{a'}{a}\,f_k\right)\,, \qquad g_k(\eta_0) = \sqrt{\tfrac{k}{2}}\,, \tag{92}$$

as well as the Wronskian condition,

$$i\,(f_k'\,f_k^* - f_k'^*\,f_k) = g_k\,f_k^* + g_k^*\,f_k = 1\,. \tag{93}$$

Squeezing parameters

Since we have two complex functions, f_k and g_k, plus a constraint (93), we can write these in terms of three real functions in the standard parametrization for squeezed states,

$$u_k(\eta) \;=\; e^{-i\,\theta_k(\eta)}\,\cosh\, r_k(\eta)\,, \tag{94}$$

$$v_k(\eta) \;=\; e^{i\,\theta_k(\eta)+2i\,\phi_k(\eta)}\,\sinh\, r_k(\eta)\,, \tag{95}$$

where r_k is the squeezing parameter, ϕ_k the squeezing angle, and θ_k the phase.

We can also write its relation to the usual Bogoliubov formalism in terms of the functions $\{\alpha_k,\ \beta_k\}$,

$$u_k = \alpha_k\,e^{-ik\eta}\,, \qquad v_k^* = \beta_k\,e^{ik\eta}\,, \tag{96}$$

which is useful for the adiabatic expansion, and allows one to write the average number of particles and other quantities,

$$
\begin{aligned}
n_k &= |\beta_k|^2 = |v_k|^2 = \frac{1}{2k}\left|g_k - k\, f_k\right|^2 = \sinh^2\, r_k\,, &(97)\\
\sigma_k &= 2\mathrm{Re}\left(\alpha_k^*\beta_k\, e^{2ik\eta}\right) = 2\mathrm{Re}\,(u_k^*\, v_k^*) = \cos 2\phi_k\, \sinh 2r_k\,, &(98)\\
\tau_k &= 2\mathrm{Im}\left(\alpha_k^*\beta_k\, e^{2ik\eta}\right) = 2\mathrm{Im}\,(u_k^*\, v_k^*) = -\sin 2\phi_k\, \sinh 2r_k\,. &(99)
\end{aligned}
$$

We can invert these expressions to give $(r_k,\ \theta_k,\ \phi_k)$ as a function of u_k and v_k,

$$
\begin{aligned}
\sinh r_k &= \sqrt{\mathrm{Re}v_k^2 + \mathrm{Im}v_k^2}\,, \qquad \cosh r_k = \sqrt{\mathrm{Re}u_k^2 + \mathrm{Im}u_k^2}\,, &(100)\\
\tan\theta_k &= -\frac{\mathrm{Im}u_k}{\mathrm{Re}u_k}\,, \qquad \tan 2\phi_k = \frac{\mathrm{Im}v_k\mathrm{Re}u_k + \mathrm{Im}u_k\mathrm{Re}v_k}{\mathrm{Re}v_k\mathrm{Re}u_k - \mathrm{Im}u_k\mathrm{Im}v_k}\,. &(101)
\end{aligned}
$$

We can now write Eqs. (87) and (88) in terms of the initial values,

$$
\begin{aligned}
y(\mathbf{k},\eta) &= \sqrt{2k}\, f_{k1}(\eta)\, y(\mathbf{k},\eta_0) - \sqrt{\frac{2}{k}}\, f_{k2}(\eta)\, p(\mathbf{k},\eta_0)\,, &(102)\\
p(\mathbf{k},\eta) &= \sqrt{\frac{2}{k}}\, g_{k1}(\eta)\, p(\mathbf{k},\eta_0) + \sqrt{2k}\, g_{k2}(\eta)\, y(\mathbf{k},\eta_0)\,, &(103)
\end{aligned}
$$

where subindices 1 and 2 correspond to real and imaginary parts, $f_{k1} \equiv \mathrm{Re}\, f_k$ and $f_{k2} \equiv \mathrm{Im}\, f_k$, and similarly for the momentum mode g_k.

The squeezing formalism

Let us now use the squeezing formalism to describe the evolution of the wave function. The equations of motion for the squeezing parameters follow from those of the field and momentum modes,

$$r_k' = \frac{a'}{a}\cos 2\phi_k\,, \tag{104}$$

$$\phi_k' = -k - \frac{a'}{a}\coth 2r_k\,\sin 2\phi_k\,, \tag{105}$$

$$\theta_k' = k + \frac{a'}{a}\tanh 2r_k\,\sin 2\phi_k\,. \tag{106}$$

As we will see, the evolution is driven towards large $r_k \propto N \gg 1$, the number of e-folds during inflation. Thus, in that limit,

$$(\theta_k + \phi_k)' = -\frac{a'}{a}\frac{\sin 2\phi_k}{\sinh 2r_k} \to 0\,,$$

and therefore $\theta_k + \phi_k \to$ const. We can always choose this constant to be zero, so that the real and imaginary components of the field and momentum modes become

$$f_{k1} = \frac{1}{\sqrt{2k}}\,e^{r_k}\,\cos\phi_k\,, \qquad f_{k2} = \frac{1}{\sqrt{2k}}\,e^{-r_k}\,\sin\phi_k\,, \tag{107}$$

$$g_{k1} = \sqrt{\frac{k}{2}}\,e^{-r_k}\,\cos\phi_k\,, \qquad g_{k2} = \sqrt{\frac{k}{2}}\,e^{r_k}\,\sin\phi_k\,. \tag{108}$$

It is clear that, in the limit of large squeezing ($r_k \to \infty$), the field mode f_k becomes purely real, while the momentum mode g_k becomes pure imaginary.

This means that the field (102) and momentum (103) operators become, in that limit,

$$\left.\begin{array}{rcl} \hat{y}(\mathbf{k},\eta) & \to & \sqrt{2k}\,f_{k1}(\eta)\,\hat{y}(\mathbf{k},\eta_0) \\ \hat{p}(\mathbf{k},\eta) & \to & \sqrt{2k}\,g_{k2}(\eta)\,\hat{y}(\mathbf{k},\eta_0) \end{array}\right\} \Rightarrow \hat{p}(\mathbf{k},\eta) \simeq \frac{g_{k2}(\eta)}{f_{k1}(\eta)}\,\hat{y}(\mathbf{k},\eta)\,. \tag{109}$$

As a consequence of this squeezing, information about the initial momentum $\hat{p}_0$ distribution is lost, and the positions (or field amplitudes) at different times commute,

$$\Big[\hat{y}(\mathbf{k},\eta_1),\ \hat{y}(\mathbf{k},\eta_2)\Big] \simeq \frac{1}{2}e^{-2r_k}\,\cos^2\phi_k \approx 0\,. \tag{110}$$

This result defines what is known as a quantum non-demolition (QND) variable, which means that one can perform succesive measurements of this variable with arbitrary precision without modifying the wave function. Note that $y = a\delta\phi$ is the amplitude of fluctuations produced during inflation, so what we have found is: first, that the amplitude is distributed as a classical Gaussian random field with probability (115); and second that we can measure its amplitude at any time, and as much as we like, without modifying the distribution function.

In a sense, this problem is similar to that of a free non-relativistic quantum particle, described initially by a minimum wave packet, with initial expectation values $\langle x\rangle_0 = x_0$ and $\langle p\rangle_0 = p_0$, which becomes broader by its unitary evolution, and at late times ($t \gg mx_0/p_0$) this Gaussian state becomes an exact WKB state,

$$\Psi(x) = \Omega_R^{-1/2}\,\exp(-\Omega\,x^2/2),$$

with Im$\Omega \gg$ ReΩ (i.e. high squeezing limit). In that limit, $[\hat{x},\ \hat{p}] \approx 0$, and we have lost information about the initial position x_0 (instead of the initial momentum like in the inflationary case), $\hat{x}(t) \simeq \hat{p}(t)\,t/m = p_0\,t/m$ and $\hat{p}(t) = p_0$. Therefore, not only $[\hat{p}(t_1),\ \hat{p}(t_2)] = 0$, but also, at late times, $[\hat{x}(t_1),\ \hat{x}(t_2)] \approx 0$. This explains why we can make subsequent measurements of a particle's position and momentum in a particle physics detector (e.g. a bubble chamber) and still retain all its quantum properties like spin, etc.

The Schrödinger picture: The vacuum wave function

Let us go now from the Heisenberg to the Schrödinger picture, and compute the initial state vacuum eigenfunction $\Psi_0(\eta = \eta_0)$. The initial vacuum state $|0, \eta_0\rangle$ is defined through the condition

$$\forall \mathbf{k}\,, \qquad \hat{a}(\mathbf{k}, \eta_0)|0, \eta_0\rangle = \left[\sqrt{\frac{k}{2}}\,\hat{y}_{\mathbf{k}}(\eta_0) + i\,\frac{1}{\sqrt{2k}}\,\hat{p}_{\mathbf{k}}(\eta_0)\right]|0, \eta_0\rangle = 0\,,$$

$$\left[y_k^0 + \frac{1}{k}\,\frac{\partial}{\partial y_k^{0*}}\right]\Psi_0\left(y_k^0, y_k^{0*}, \eta_0\right) = 0 \;\Rightarrow\; \Psi_0\left(y_k^0, y_k^{0*}, \eta_0\right) = N_0\, e^{-k\,|y_k^0|^2}$$

where we have used the position representation, $\hat{y}_{\mathbf{k}}(\eta_0) = y_k^0\,,\ \hat{p}_{\mathbf{k}}(\eta_0) = -i\,\frac{\partial}{\partial y_k^{0*}}$, and N_0 gives the corresponding normalization.

We will now study the time evolution of this initial wave function using the unitary evolution operator $S = S(\eta, \eta_0)$, i.e. the state evolves in the Schrödinger picture as $|0, \eta\rangle = S|0, \eta_0\rangle$. Now, inverting (87) and (88)

$$\hat{a}(\mathbf{k}, \eta_0) = g_k^*(\eta)\,\hat{y}(\mathbf{k}, \eta) + i\, f_k^*(\eta)\,\hat{p}(\mathbf{k}, \eta)\,, \tag{111}$$

which, acting on the initial state becomes, $\forall \mathbf{k}\,, \forall \eta$,

$$\begin{aligned}
S\left[\hat{y}(\mathbf{k}, \eta) + i\,\frac{f_k^*(\eta)}{g_k^*(\eta)}\,\hat{p}(\mathbf{k}, \eta)\right] S^{-1}S|0, \eta_0\rangle &= 0\\
\Rightarrow \qquad \left[\hat{y}_{\mathbf{k}}(\eta_0) + i\,\frac{f_k^*(\eta)}{g_k^*(\eta)}\,\hat{p}_{\mathbf{k}}(\eta_0)\right]|0, \eta\rangle &= 0\,,
\end{aligned}$$

$$\Rightarrow \qquad \Psi_0\left(y_{\mathbf{k}}^0, y_{\mathbf{k}}^{0*}, \eta\right) = \frac{1}{\sqrt{\pi}\,|f_k(\eta)|}\, e^{-\Omega_k(\eta)\,|y_k^0|^2}\,, \tag{112}$$

where

$$\Omega_k(\eta) = \frac{g_k^*(\eta)}{f_k^*(\eta)} = k\,\frac{u_k^* - v_k}{u_k^* + v_k} = \frac{1 - 2i\,F_k(\eta)}{2|f_k(\eta)|^2}\,, \tag{113}$$

$$F_k(\eta) = \mathrm{Im}(f_k^*\, g_k) = \mathrm{Im}(u_k\, v_k) = \frac{1}{2}\,\sin 2\phi_k\,\sinh 2r_k\,. \tag{114}$$

We see that the unitary evolution preserves the Gaussian form of the wave functional. The wave function (112) is called a 2-mode squeezed state.

The normalized probability distribution,

$$P_0\left(y(\mathbf{k},\eta_0), y(-\mathbf{k},\eta_0),\eta\right) = \frac{1}{\pi\,|f_k(\eta)|^2}\exp\left(-\frac{|y(\mathbf{k},\eta_0)|^2}{|f_k(\eta)|^2}\right)\,, \quad (115)$$

is a Gaussian distribution, with dispersion given by $|f_k|^2$.

In fact, we can compute the vacum expectation values,

$$\langle\Delta y(\mathbf{k},\eta)\,\Delta y^\dagger(\mathbf{k}',\eta)\rangle \equiv \Delta y^2(k)\,\delta^3(\mathbf{k}-\mathbf{k}') = |f_k|^2\,\delta^3(\mathbf{k}-\mathbf{k}')\,, \quad (116)$$
$$\langle\Delta p(\mathbf{k},\eta)\,\Delta p^\dagger(\mathbf{k}',\eta)\rangle \equiv \Delta p^2(k)\,\delta^3(\mathbf{k}-\mathbf{k}') = |g_k|^2\,\delta^3(\mathbf{k}-\mathbf{k}')\,, \quad (117)$$

and therefore the Heisenberg uncertainty principle reads

$$\Delta y^2(k)\,\Delta p^2(k) = |f_k|^2\,|g_k|^2 = F_k^2(\eta) + \frac{1}{4} \geq \frac{1}{4}\,. \quad (118)$$

It is clear that for $\eta=\eta_0$, $\Omega_k(\eta_0)=k$ and $F_k(\eta_0)=0$, and thus we have initially a minimum wave packet, $\Delta y\,\Delta p = \frac{1}{2}$. However, through its unitary evolution, the function F_k grows exponentially, see (114), and we quickly find $\Delta y\,\Delta p \gg \frac{1}{2}$, corresponding to the semiclassical regime, as we will soon demonstrate rigorously.

The Wigner function

The Wigner function is the best candidate for a probability density of a quantum mechanical system in phase-space. Of course, we know from QM that such a probability distribution function cannot exist, but the Wigner function is just a good approximation to that distribution. Furthermore, for a Gaussian state, this function is in fact positive definite.

Consider a quantum state described by a density matrix $\hat{\rho}$. Then the Wigner function can be written as

$$W(y_k^0, y_k^{0*}, p_k^0, p_k^{0*}) = \int\int \frac{dx_1\,dx_2}{(2\pi)^2}\, e^{-i(p_1\,x_1+p_2\,x_2)}\,\left\langle y-\frac{x}{2},\eta\right|\hat{\rho}\left|y+\frac{x}{2},\eta\right\rangle\,.$$

If we substitute for the state our vacuum initial condition $\hat{\rho} = |\Psi_0\rangle\langle\Psi_0|$, with Ψ_0 given by (112), we can perform the integration explicitly to obtain

$$\begin{aligned} W_0(y_k^0, y_k^{0*}, p_k^0, p_k^{0*}) &= \frac{1}{\pi^2}\,\exp\left(-\frac{|y|^2}{|f_k|^2} - 4|f_k|^2\left|p-\frac{F_k}{|f_k|^2}\,y\right|^2\right) \\ &\equiv \Phi(y_1,\,p_1)\,\Phi(y_2,\,p_2) \qquad (119) \\ \Phi(y_1,\,p_1) &= \frac{1}{\pi}\,\exp\left\{-\left(\frac{y_1^2}{|f_k|^2}+4|f_k|^2\,\bar{p}_1^2\right)\right\}\,, \qquad (120) \\ \bar{p}_1 &\equiv p_1 - \frac{F_k}{|f_k|^2}\,y_1\,. \end{aligned}$$

In general, W_0 describes an asymmetric Gaussian in phase space, whose 2σ contours satisfy

$$\frac{y_1^2}{|f_k|^2} + 4|f_k|^2\,\bar{p}_1^2 \;\leq\; 1\,. \qquad (121)$$

For instance, at time $\eta = \eta_0$, we have $y_1^0 = \frac{1}{\sqrt{2k}} = |f_k(\eta_0)|\,,\ p_1^0 = \sqrt{\frac{k}{2}} = 1/2|f_k(\eta_0)|$, and $F_k(\eta_0) = 0$, so that $\bar{p}_1^0 = p_1^0$, and the 2σ contours become

$$\frac{y_1^2}{y_1^{0\,2}} + \frac{p_1^2}{p_1^{0\,2}} \leq 1\,,$$

which is a circle in phase space.

On the other hand, for time $\eta \gg \eta_0$, we have

$$|f_k| \;\rightarrow\; \frac{1}{\sqrt{2k}}\,e^{r_k} \sim y_k^0\,e^N\,, \qquad \text{growing mode}\,, \qquad (122)$$

$$\frac{1}{2|f_k|} \;\rightarrow\; \sqrt{\frac{k}{2}}\,e^{-r_k} \sim p_k^0\,e^{-N}\,, \qquad \text{decaying mode}\,, \qquad (123)$$

so that the ellipse (121) becomes highly "squeezed".

Note that Liouville's theorem implies that the volume of phase space is conserved under Hamiltonian (unitary) evolution, so that the area within the ellipse should be conserved. As the probability distribution compresses (squeezes) along the p-direction, it expands along the y-direction. At late times, the Wigner function is highly concentrated around the region

$$\bar{p}^2 = \left(p - \frac{F_k}{|f_k|^2}\, y\right)^2 < \frac{1}{4|f_k|^2} \sim e^{-2N} \ll 1\,. \tag{124}$$

We can thus take the above *squeezing limit* in the Wigner function (119) and write the exponential term as a Dirac delta function,

$$W_0(y,p) \stackrel{r_k\to\infty}{\longrightarrow} \frac{1}{\pi^2}\,\exp\left\{-\frac{|y|^2}{|f_k|^2}\right\}\,\delta\left(p - \frac{F_k}{|f_k|^2}\, y\right)\,. \tag{125}$$

In this limit we have

$$\hat{p}_k(\eta) = \frac{F_k}{|f_k|^2}\,\hat{y}_k(\eta) \;\simeq\; \frac{g_{k2}(\eta)}{f_{k1}(\eta)}\,\hat{y}_k(\eta)\,, \tag{126}$$

so we recover the previous result (109). This explains why we can treat the system as a classical Gaussian random field: the amplitude of the field y is uncertain with probability distribution (115), but once a measurement of y is performed, we can automatically asign to it a *definite* value of the momentum, according to (109).

Note that the condition $F_k^2 \gg 1$ is actually a condition between operators and their commutators/anticommutators. The Heisenberg uncertainty principle states that

$$\Delta_\Psi A\,\Delta_\Psi B \geq \frac{1}{2}\Big|\langle\Psi|[A,\ B]|\Psi\rangle\Big|,$$

for any two hermitian operators (observables) in the Hilbert space of the wave function Ψ. In our case, and in Fourier space, this corresponds to (118)

$$\Delta y^2(k)\,\Delta p^2(k) = F_k^2(\eta) + \frac{1}{4} \geq \frac{1}{4}\left|\langle\Psi|[y_k(\eta),\ p_k^\dagger(\eta)]|\Psi\rangle\right|^2, \qquad (127)$$

with $|\Psi\rangle = |0,\eta\rangle$ the evolved wave function.

On the other hand, the phase F_k can be written as

$$\begin{aligned} F_k &= -\frac{i}{2}\,(g_k\, f_k^* - f_k\, g_k^*) = -\frac{i}{2}\left(\frac{g_k}{f_k}\,|f_k|^2 - |f_k|^2\,\frac{g_k^*}{f_k^*}\right) = \\ &= \frac{1}{2}\,\langle\Psi|p(\mathbf{k},\eta)\,y^\dagger(\mathbf{k},\eta) + y(\mathbf{k},\eta)\,p^\dagger(\mathbf{k},\eta)|\Psi\rangle\,, \end{aligned} \qquad (128)$$

and we have used that, in the semiclassical limit, we can write $\langle\Psi||y_k(\eta)|^2|\Psi\rangle = |f_k|^2$, as well as $p(\mathbf{k},\eta) = -\,i\,\frac{g_k}{f_k}\,y(\mathbf{k},\eta)$, see (109).

The above relation just indicates that, for any state Ψ, the condition of classicality ($F_k \gg 1$) is satisfied whenever, for that state,

$$\{y_k(\eta),\ p_k^\dagger(\eta)\} \gg |[y_k(\eta),\ p_k^\dagger(\eta)]| = \hbar\,,$$

which is an interesting condition.

Massless scalar field fluctuations on superhorizon scales

The gauge invariant tensor fluctuations (gravitational waves) act as a minimally-coupled massless scalar field during inflation, so we will study here the generation of its fluctuations during quasi de Sitter.

Let us consider here the exact solutions to the equation of motion of a minimally-coupled massless scalar field during inflation or quasi de

Sitter, with scale factor $a = -1/H\eta$,

$$f_k = \frac{1}{\sqrt{2k}} e^{-ik\eta} \left(1 - \frac{i}{k\eta}\right), \tag{129}$$

$$g_k = i\left(f_k' - \frac{a'}{a} f_k\right) = \sqrt{\frac{k}{2}}\, e^{-ik\eta}, \tag{130}$$

which satisfy the Wronskian condition, $g_k f_k^* + g_k^* f_k = 1$. The eigenmodes become

$$u_k = e^{-ik\eta} \left(1 - \frac{i}{2k\eta}\right) = e^{-ik\eta - i\delta_k} \cosh r_k\,, \tag{131}$$

$$v_k = e^{ik\eta} \frac{i}{2k\eta} = e^{ik\eta + i\frac{\pi}{2}} \sinh r_k\,, \tag{132}$$

which comparing with (94) and (95) provides the squeezing parameter, the angle and the phase, as inflation proceeds towards $k\eta \to 0^-$,

$$\sinh r_k = \tan\delta_k = \frac{1}{2k\eta} \to -\infty\,, \tag{133}$$

$$\theta_k = k\eta + \arctan\frac{1}{2k\eta} \to -\frac{\pi}{2}, \qquad \phi_k = \frac{\pi}{4} - \frac{1}{2}\arctan\frac{1}{2k\eta} \to \frac{\pi}{2}, \tag{134}$$

while the imaginary part of the phase of the wave function becomes

$$F_k(\eta) = \frac{1}{2}\sin 2\phi_k \sinh 2r_k = \frac{1}{2k\eta} \to -\infty\,. \tag{135}$$

The number of scalar field particles produced during inflation grow exponentially, $n_k = |\beta_k|^2 = \sinh^2 r_k = (2k\eta)^{-2} \to \infty$.

Thus, through unitary evolution, the fluctuations will very soon enter the semiclasical regime due to a highly squeezed wave function. The question which remains is when do fluctuations become classical?

Hubble crossing

As we will see, the field fluctuation modes will become semiclassical as their wavelength becomes larger than the only physical scale in the problem, the de Sitter horizon scale, $\lambda_{\rm phys} = 2\pi a/k \gg H^{-1}$.

Therefore, let us consider the general solution to Eq. (91) for the superhorizon modes ($k \ll aH$),

$$f_k(\eta) = C_1(k)\,a + C_2(k)\,a \int^\eta \frac{d\eta'}{a^2(\eta')} = C_1(k)\,a - C_2(k)\,\frac{1}{a^2 H}\,. \quad (136)$$

We can always choose $C_1(k)$ to be real, while $C_2(k)$ will be complex in general. The first term corresponds to the growing mode, while the second term is the decaying mode.

Integrating out g_k from (92), one finds

$$g_k(\eta) = i\,C_2(k)\,\frac{1}{a} - i\,C_1(k)\,k^2\,\frac{1}{a}\int a^2\,d\eta = i\,C_2(k)\,\frac{1}{a} - i\,C_1(k)\,\frac{k^2}{H}\,, \quad (137)$$

where we have added a k^2 term for completeness. To second order in k^2, the Wronskian becomes

$$C_1(k)\,{\rm Im}C_2(k)\,\left(1 + \frac{k^2}{a^2H^2}\right) \simeq C_1(k)\,{\rm Im}C_2(k) = -\frac{1}{2}\,. \quad (138)$$

Comparing with the exact solutions (129), we find, to first order,

$$C_1(k) = \frac{H_k}{\sqrt{2k^3}}\,, \qquad C_2(k) = -\,\frac{i\,k^{3/2}}{\sqrt{2}\,H_k}\,, \quad (139)$$

where H_k is the Hubble rate at horizon crossing, $k\eta = -1$, i.e. when the perturbation's physical wavelength becomes of the same order as the de Sitter horizon size, $k = aH = \mathcal{H}$.

We are now prepared to answer the question of classicality of the modes. Let us compute the wave function phase shift

$$|F_k| = |{\rm Im}(f_k^*\,g_k)| \;=\; \Bigg|\,C_1^2(k)\,\frac{k^2 a}{H} + |C_2(k)|^2\,\frac{1}{a^3 H} \quad (140)$$

$$-\;C_1(k)\,{\rm Re}C_2(k)\,\left(1 + \frac{k^2}{a^2H^2}\right)\Bigg|\,. \quad (141)$$

Since only the first term remains after $k\eta \to 0$, we see that $|F_k| \gg 1$ whenever

$$C_1^2(k) = \frac{H_k^2}{2k^3} \gg \frac{H}{k^2 a} \qquad \Rightarrow \qquad \lambda_{\rm phys} = \frac{2\pi a}{k} \gg \lambda_{\rm HC} = \frac{2\pi}{H_k}\,. \quad (142)$$

Therefore, we confirm that modes that start as Minkowski vacuum well inside the de Sitter horizon are stretched by the expansion and become semiclassical soon after horizon crossing, and their amplitude can be described as a classical Gaussian random variable.

Furthermore, the fact that the momentum is immediately defined once the amplitude for a given wavelength is known, implies that there is a fixed temporal phase coherence for all perturbations with the same wavelength. As we know, this implies that inflationary perturbations will induce coherent acoustic oscillations in the plasma just before decoupling, which should be seen in the microwave background anisotropies as acoustic peaks in the angular power spectrum.