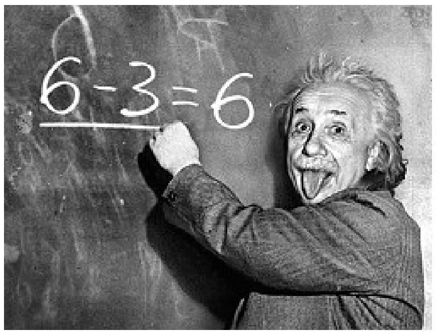
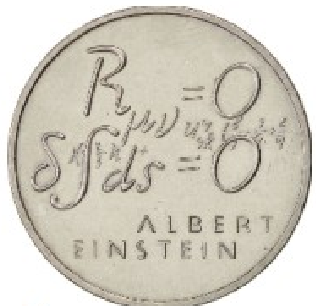
Introduction to Inflation





$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

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Main points of the lecture

- The Hot Big Bang & why we need inflation
- Inflationary observables
- Scalar field models and other curiosities
- CMB constraints
- Conclusions and literature

The Standard Cosmological model

Einstein equations in pure GR:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

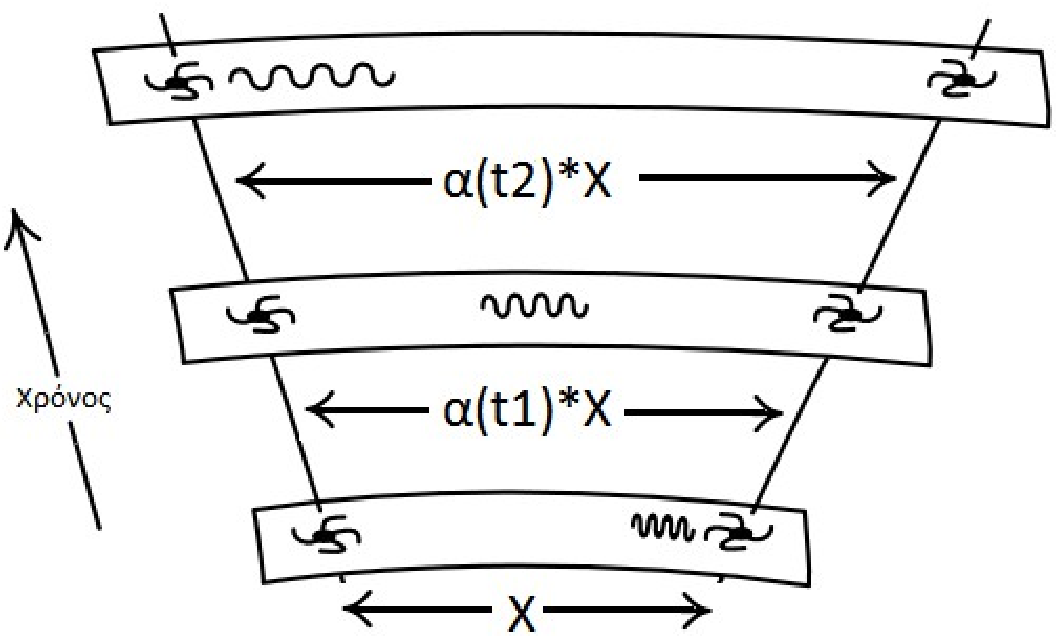
Einstein tensor

$$T^{\mu}_{\nu} = P g^{\mu}_{\nu} + (\rho + P) U^{\mu} U_{\nu}$$

Friedmann-Lemaitre-Robertson-Walker (FLRW) metric:

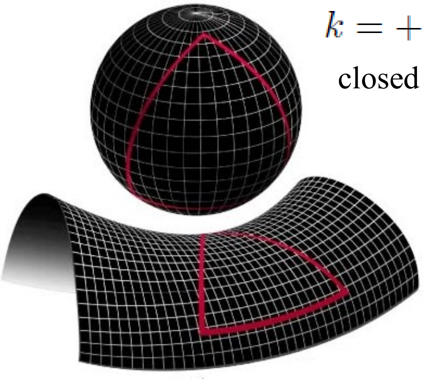
$$ds^2 = c^2 dt^2 - \alpha(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + sin(\theta)^2 d\phi^2) \right)$$

Scale factor α(t):



The Standard Cosmological model

The curvature:



Friedmann equations (1924):

$$H^2(\alpha) = \left(\frac{\dot{\alpha}}{\alpha}\right)^2 = \frac{8\pi G}{3}\rho(\alpha) - \frac{k}{\alpha^2}$$

$$\frac{\ddot{\alpha}}{\alpha} = -\frac{4\pi G}{3}(\rho(\alpha) + P(\alpha))$$

Continuity equations:

(via Bianchi identities)

$$\nabla_\nu T^{\mu\nu} = 0 \quad \Rightarrow \quad \dot{\rho} + 3H(\rho + P) = 0$$

The Standard Cosmological model

Hubble (1929): The Universe is expanding — Redshift of distant galaxies

Riess et al. (1998): ...and it's also accelerating! — Type Ia supernovae

2nd Friedmann equation: $\frac{\ddot{\alpha}}{\alpha} = -\frac{4\pi G}{3}\left(\rho(\alpha) + 3P(\alpha)\right) \Rightarrow P < -\frac{\rho}{3}$

Equation of state $P = w\,\rho$

- $w = 0$ Non-relativistic matter $P << \rho$
- $w = \frac{1}{3}$ Relativistic matter (photons etc) $P = \frac{1}{3}\rho$

$P < -\frac{\rho}{3} \Rightarrow w < -\frac{1}{3}$

The known forms of matter cannot explain the accelerated expansion of the Universe…

The Standard Cosmological model

Fractional density parameters:

$$\rho_c(t) = \frac{3H^2}{8\pi G}$$

$$\Omega(t) \equiv \frac{\rho}{\rho_c}$$

$$\Omega_{K,0} = -\frac{k}{H_0^2 a_0^2}$$

1st Friedmann equation:

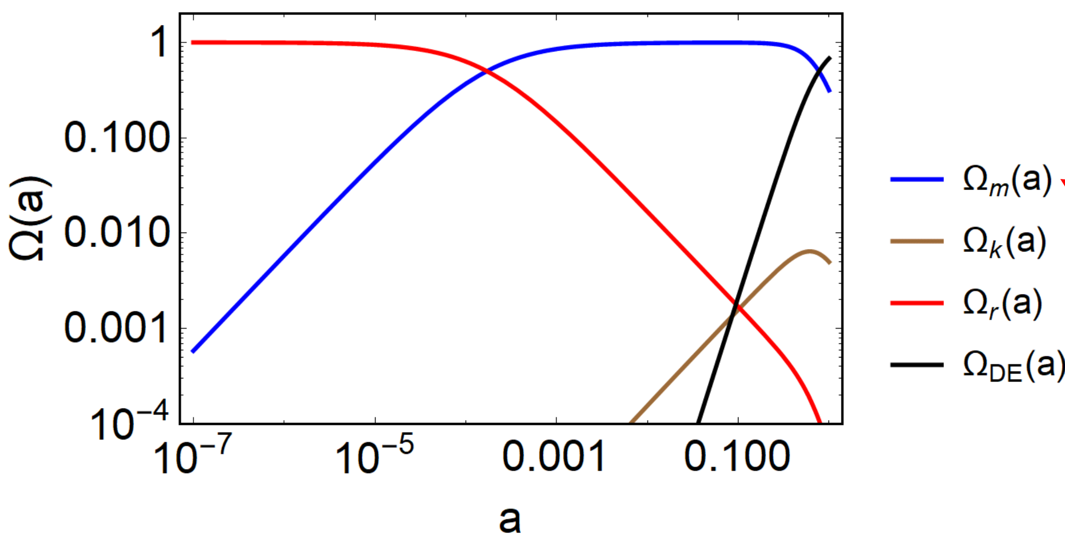
$$H(\alpha)^2 = H_0^2 \left(\Omega_{b,0}\alpha^{-3} + \Omega_{c,0}\alpha^{-3} + \Omega_{r,0}\alpha^{-4} + \Omega_{K,0}\alpha^{-2} + \Omega_{DE,0}\alpha^{-3(1+w)}\right)$$

Big Bang theory predictions

1) Initial singularity: Universe was very hot at early times!

$$\rho_r(a) \sim a^{-4} \to \infty|_{a\to 0}$$

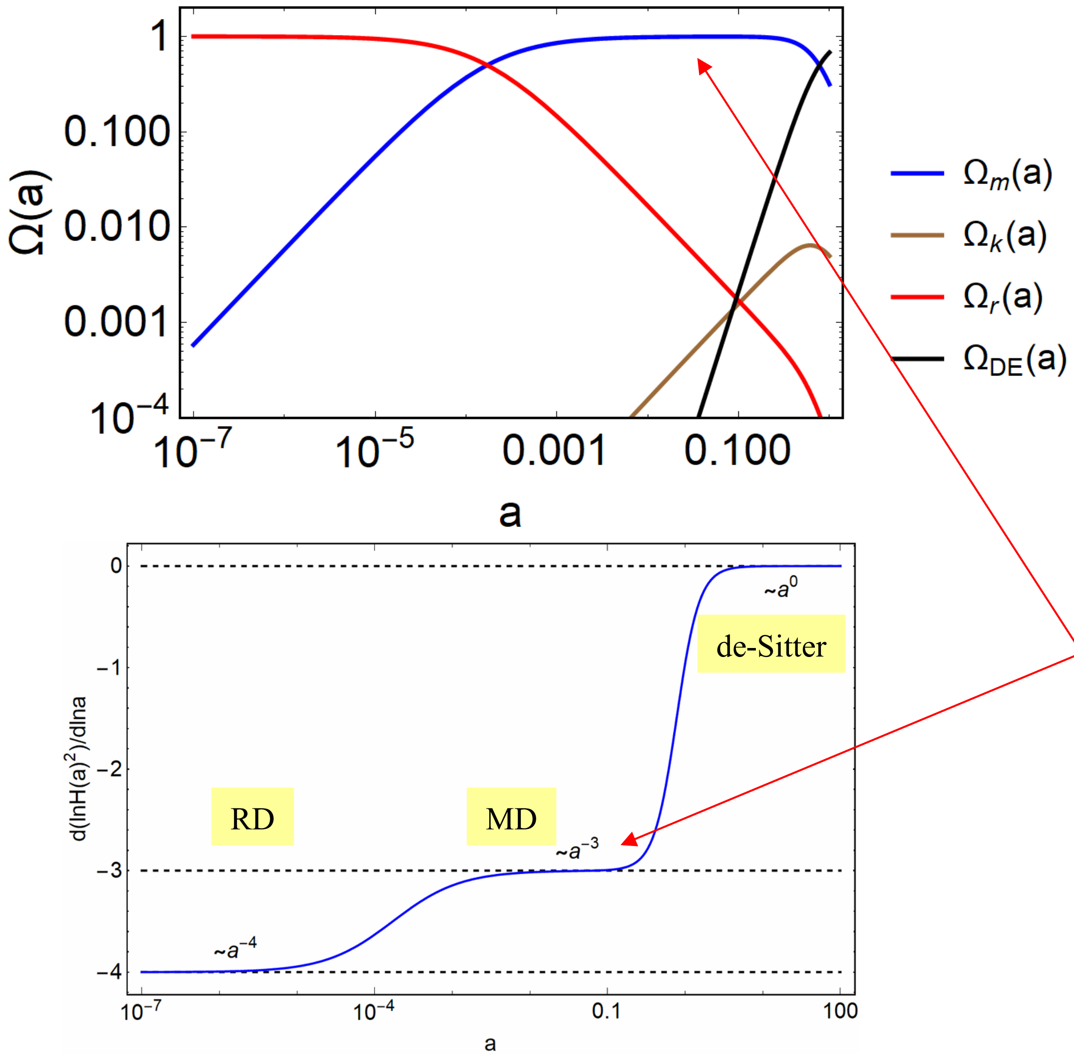
$$\rho_r \sim T^4 \quad \Rightarrow \quad T(a \to 0) \to \infty$$



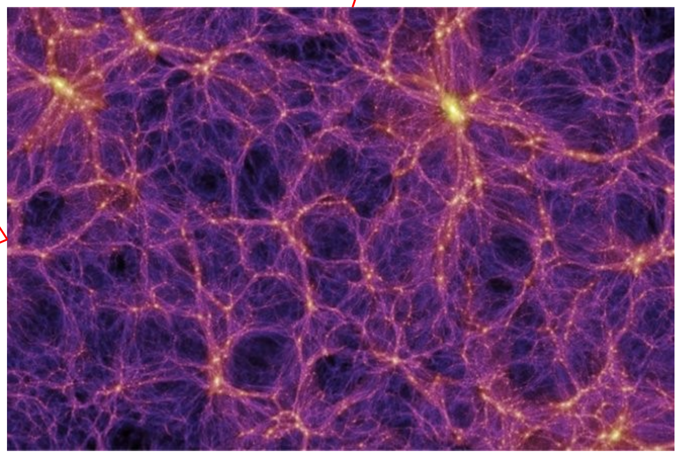
$$\Omega_i(a) = \frac{\Omega_{i,0}a^{-3(1+w)}}{H(a)^2}$$

Big Bang theory predictions

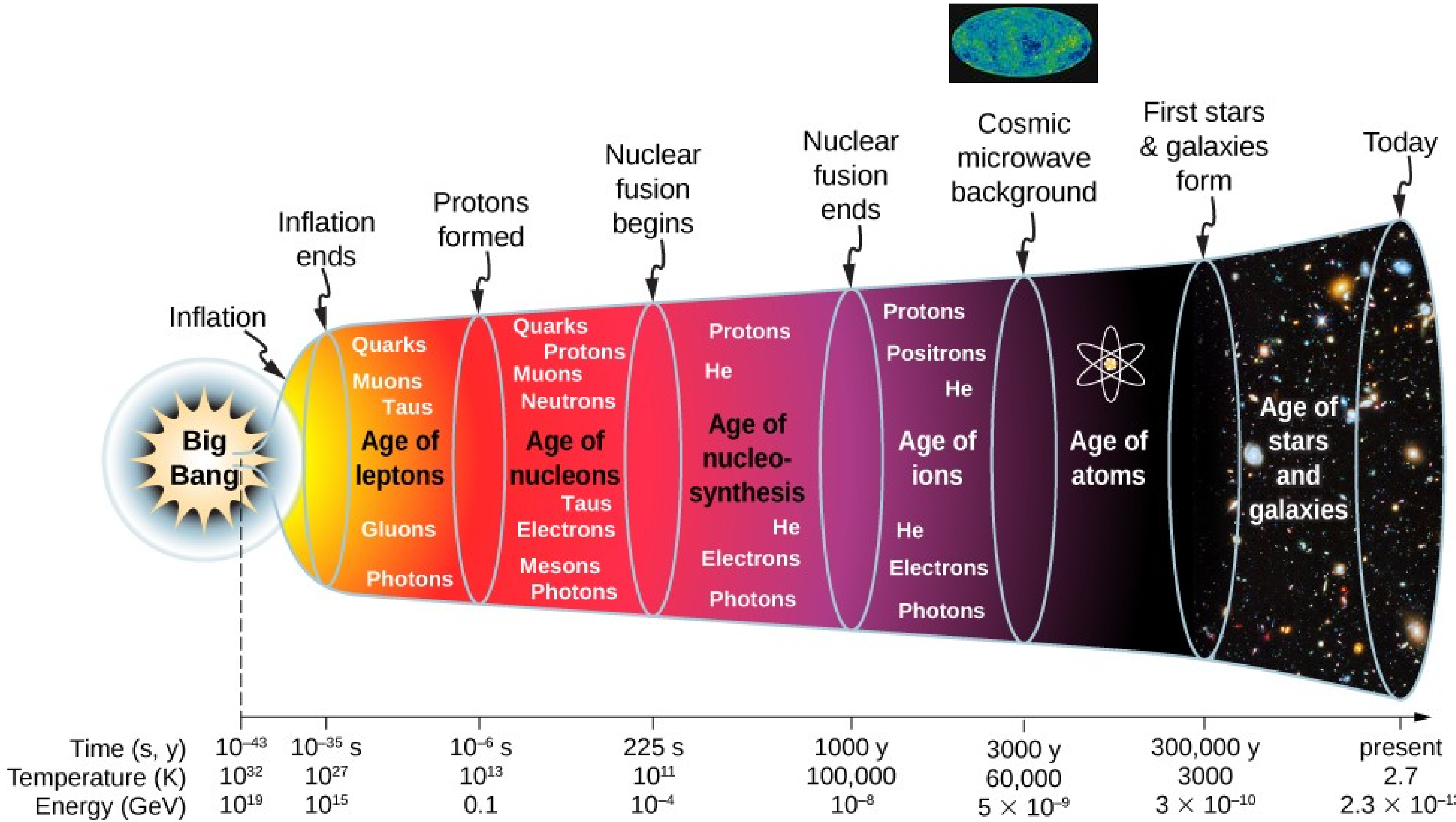
2) Structure formation happens during the matter dominated era:



Need something to act as seeds.
See next lecture!



Big Bang theory timeline



Problems in the Big Bang Theory

1) Horizon problem: why is the Universe homogeneous?

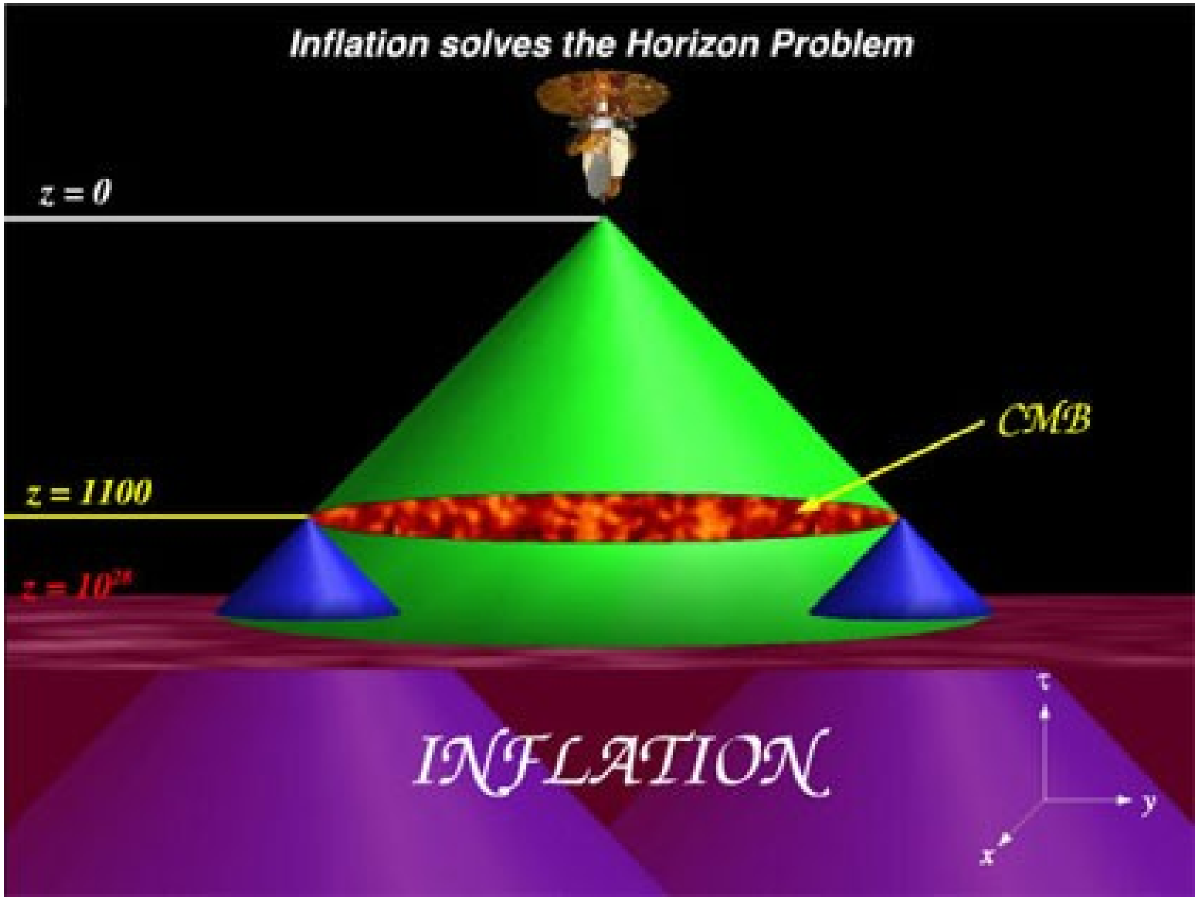
2) Flatness problem: why is the Universe flat?

3) Monopole problem: what happened to the monopoles?

4) Origin of large scale structure: what was the seed of LSS?

Why we need inflation

1a) Horizon problem: causally disconnected regions of space (or of the CMB) appear to be homogeneous ($\sim 10^{-5}$!) today



Credit: Will Kinney

Why we need inflation

1b) Horizon problem in numbers:

The Standard Cosmological model contains a (comoving) particle horizon:

$$r_H = \int_0^t \frac{c\, dt}{R(t)}$$

In radiation domination $a(t) \propto t^{1/2}$, so at late times:

$$D_H = R_0 r_H \simeq \frac{6000}{\sqrt{\Omega z}} h^{-1}\mathrm{Mpc}$$

At $z \sim 1000$ the horizon was roughly $\sim 100\mathrm{Mpc}$ or ~1 degree. Then, why do we live in a nearly homogeneous Universe?

Why we need inflation

1c) Solution to Horizon problem:

Assume a phase of exponential accelerated expansion (w→ -1),

then the scale factor is $a(t) \simeq e^{H_I(t-t_{end})}$

and the horizon is $D_H \simeq e^N$

where the e-folds are given by $N = -\int d\ln a = \int H dt$

The horizon now is exponentially large, for N~60 problem solved!

Why we need inflation

2) Flatness problem.

Consider the sum of densities today: $1 - \Omega_0 = -\frac{kc^2}{R_0^2 H_0^2} < 0.01$

All other matter+radiation components etc

In general: $1 - \Omega = -\frac{kc^2}{R_0^2 a(t)^2 H(t)^2}$ $\Rightarrow$ $1 - \Omega(t) = \frac{H_o^2(1-\Omega_0)}{H^2(t)a^2(t)}$

Assuming matter domination: $a \sim t^{2/3}$

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} \Rightarrow 1 - \Omega(t) = \frac{(1-\Omega_0)a^2}{\Omega_{r,0} + \Omega_{m,0}a} = \frac{1-\Omega_0}{\Omega_{m,0}}\left(\frac{t}{t_0}\right)^{2/3}$$

Increasing function of time, causes severe fine-tuning!

Why we need inflation

Inflation to the rescue!

$$H = const\,; \quad a \sim \exp(H_I t) \quad \Rightarrow \quad \frac{a(t_f)}{a(t_i)} = e^N \quad N = H_i(t_f - t_i)$$

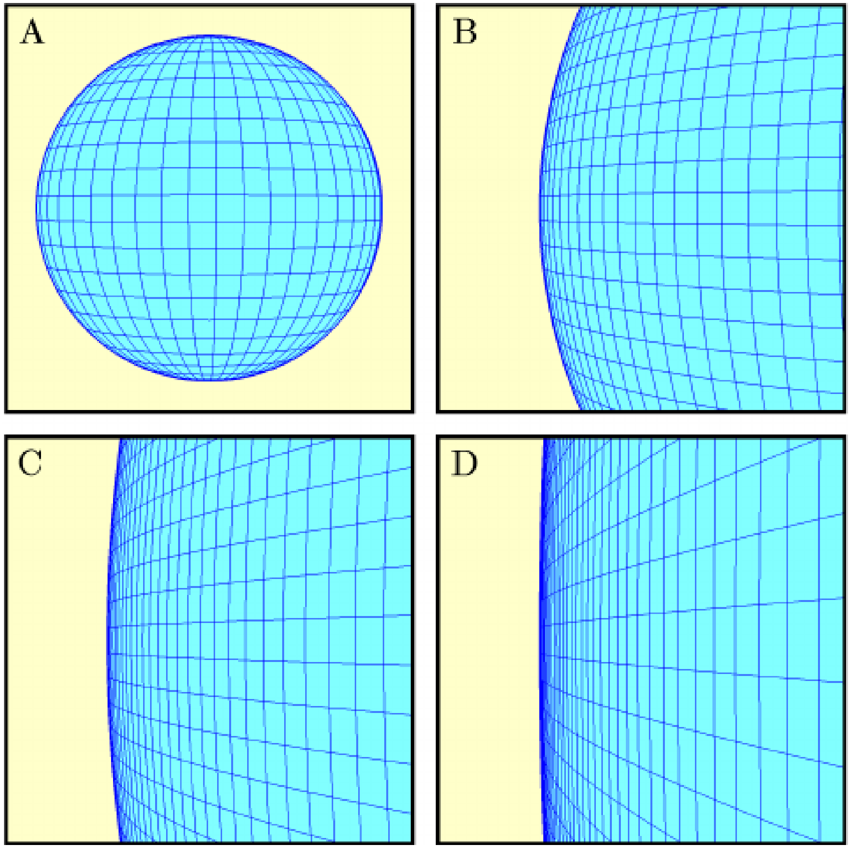
Then

$$1 - \Omega(t) = \frac{c^2}{R_0^2 a^2 H^2}$$

$$\Downarrow$$

$$|1 - \Omega(t_f)| = \exp(-2N)|1 - \Omega(t_i)|$$

Curvature is "inflated" away!



Why we need inflation

3) Topological defects, eg monopoles, cosmic strings, domain walls, coming from GUTs (~10^16 GeV) are created **before** inflation, usually ~1/horizon.

If M~GUT, should be dominant contribution!

$$\Omega^{(0)}_{\rm mon} = \frac{M}{3H_0^2 M_{\rm Pl}^2 \left[D^{(0)}_{\rm mon}\right]^3} \simeq 10^{15}$$

One per horizon (before inflation) → Horizon expands by e^60~10^27 → monopoles diluted!

Basics of Inflation model-building

Guth 1981, Linde 1982,
Albrecht and Steinhardt 1982

Introduce phase of exponential expansion = aka de-Sitter

2nd Friedmann equation: $\frac{\ddot{\alpha}}{\alpha} = -\frac{4\pi G}{3}(\rho(\alpha) + 3P(\alpha))$ ⇒ $P < -\frac{\rho}{3}$

Equation of state: $P = w\,\rho$

- $w = 0$ Non-relativistic matter $P << \rho$
- $w = \frac{1}{3}$ Relativistic matter (photons etc) $P = \frac{1}{3}\rho$

$P < -\frac{\rho}{3}$ ⇒ $w < -\frac{1}{3}$

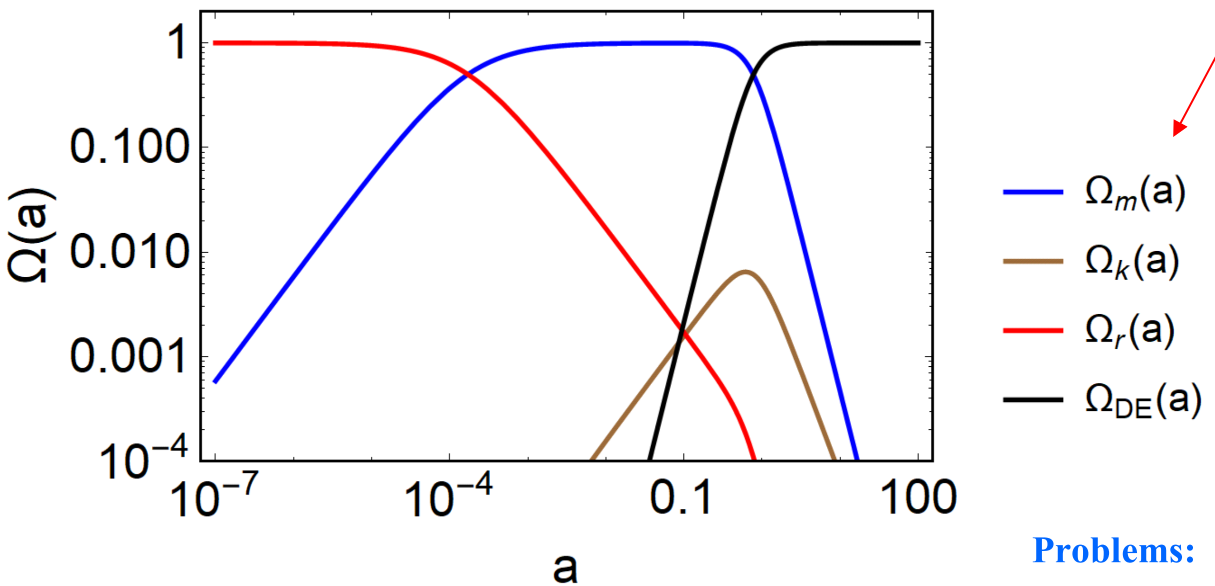
A cosmological constant can do that!

Basics of Inflation model-building

Pure cosmological constant is problematic due to constant expansion!

$$\Rightarrow \quad a(t) = e^{H_0(t-t_0)}$$

$$\Omega_i(a) = \frac{\Omega_{i,0}\, a^{-3(1+w)}}{H(a)^2}$$



Problems:

1) de-Sitter never ends & dilutes everything.

2) Results in empty Universe, need reheating.

Let's see the details more carefully!

Basics of Inflation model-building

Possible inflationary models must:

1) Solve "classical" problems (horizon, flatness, monopoles) etc.

2) End before radiation epoch and be followed by reheating to create particles.

3) Set the initial conditions for LSS.

4) Make unique & testable predictions (see next lecture for criticisms).

5) Be motivated from high-energy physics (stand. model or quantum gravity).

Scalar field inflation

1) Simplest thing we can add to GR Lagrangian (on RHS!) is a scalar field

i) Scalar fields (bosons with spin 0) have been observed (Higgs)!
ii) Already used in Dark Energy (also an accelerating phase).
iii) Dynamics are very well understood.

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R + S_m \quad \Longrightarrow \quad S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R + \mathcal{L}_\phi\right] + S_M$$

$$\mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$$

2) Energy momentum tensor:

$$T^{(\phi)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right]$$

Scalar field inflation

3) Effective density and pressure and equation of state w:

$$P_\phi = \frac{1}{3}T_i^{i(\phi)} = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$$\rho_\phi = -T_0^{0(\phi)} = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$\Longrightarrow \quad w_\phi \equiv \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}$$

4) In quintessence, w(z) cannot cross -1! Use continuity equation:

Nesseris et al, astro-ph/0610092

$$\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0 \qquad \dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p(\rho))$$

when w⟶ -1 $\Longrightarrow$ $p(\rho) \rightarrow -\rho$ $\Longrightarrow$ $\dot{\rho} \rightarrow 0$ and $\lim_{w\rightarrow -1} \frac{d^n\rho(t)}{dt^n} = 0$

So w(z) goes asymptotically to w⟶ -1_+!

Scalar field inflation

5) Equations of motion:

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_M\right],$$

$$\dot{H} = -\frac{\kappa^2}{2}\left(\dot{\phi}^2 + \rho_M + P_M\right),$$

and

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

$$V_{,\phi} \equiv \mathrm{d}V/\mathrm{d}\phi$$

$$\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0$$

6) Example models (all high energy physics inspired):

Freezing

$$V(\phi) = M^{4+n}\phi^{-n} \quad (n > 0),$$
$$V(\phi) = M^{4+n}\phi^{-n}\exp(\alpha\phi^2/m_{\rm pl}^2).$$

Thawing

$$V(\phi) = V_0 + M^{4-n}\phi^n \quad (n > 0),$$
$$V(\phi) = M^4\cos^2(\phi/f).$$

Scalar field inflation

7) Potential reconstruction (E(z)=H(z)/H0):

Dark Energy,
L.A. and S.T.

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_M\right],$$

$$\dot{H} = -\frac{\kappa^2}{2}\left(\dot{\phi}^2 + \rho_M + P_M\right),$$

$$\Rightarrow$$

$$\frac{\kappa^2}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2 = \frac{1}{1+z}\frac{\mathrm{d}\ln E(z)}{\mathrm{d}z} - \frac{3\Omega_m^{(0)}}{2}\frac{1+z}{E^2(z)} \geq 0$$

$$\frac{\kappa^2 V}{3H_0^2} = E(z) - \frac{1+z}{6}\frac{\mathrm{d}E^2(z)}{\mathrm{d}z} - \frac{1}{2}\Omega_m^{(0)}(1+z)^3.$$

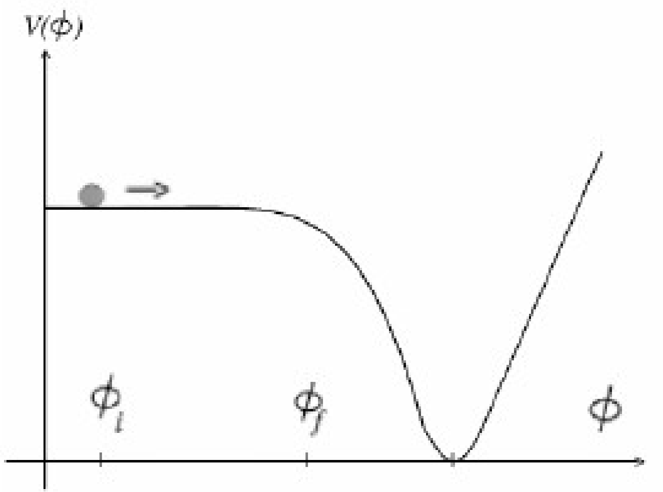
8) Condition for reconstruction

$$\frac{\mathrm{d}H^2}{\mathrm{d}z} \geq 3\Omega_m^{(0)} H_0^2 (1+z)^2 \quad \Rightarrow \quad \rho_\phi + P_\phi \geq 0$$

(weak energy condition)

Slow roll inflation

1) Generic potential that satisfies previous constraints



2) Slow rolling……….

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_M\right],$$

$$\dot{H} = -\frac{\kappa^2}{2}\left(\dot{\phi}^2 + \rho_M + P_M\right),$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

$\Rightarrow$

$$3H^2 = \rho \simeq V,$$

$$-2\dot{H} = (p + \rho) = \dot{\phi}^2,$$

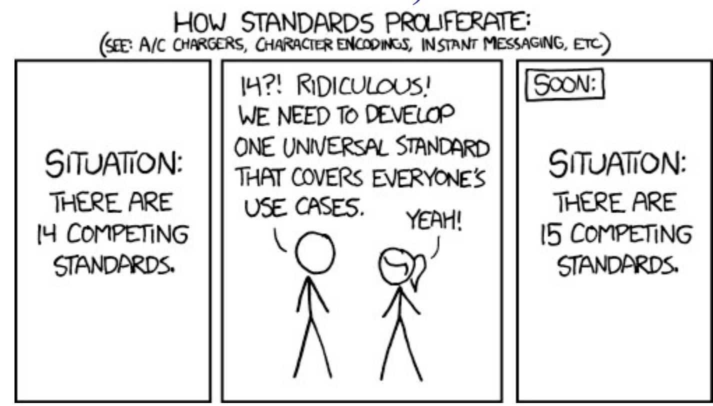
$$3H\dot{\phi} \simeq -\frac{\partial V}{\partial \phi}.$$

Slow roll inflation

3) Introduce slow roll parameters (note: various notations in literature!)

$$\epsilon_1 \equiv -\frac{\dot{H}}{H^2} \simeq \frac{1}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \equiv \epsilon_V \,,$$

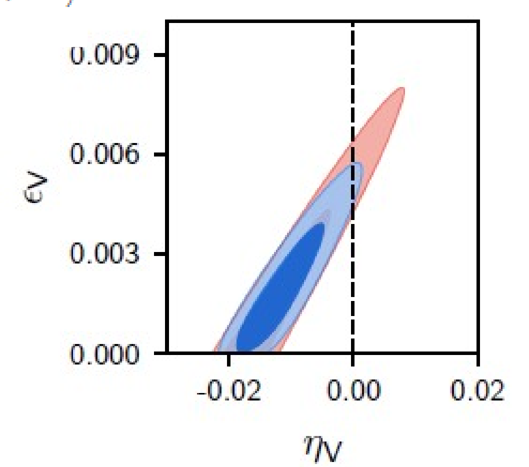
$$\epsilon_2 \equiv \frac{\mathrm{d}\ln(\epsilon_1)}{\mathrm{d}\ln a} \simeq -2\frac{V_{,\phi\phi}}{V} + 2\left(\frac{V_{,\phi}}{V}\right)^2 \equiv -2\eta_V + 4\epsilon_V \,,$$



or

$$\epsilon = \frac{2}{\kappa^2}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 \simeq \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \equiv \epsilon_V \ll 1 \,,$$

$$\delta = \frac{2}{\kappa^2}\frac{H''(\phi)}{H(\phi)} \simeq \frac{1}{\kappa^2}\frac{V''(\phi)}{V(\phi)} - \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \equiv \eta_V - \epsilon_V \ll 1 \,,$$



4) Number of e-folds (until the end of inflation → ε=1)

$$N(t) \equiv -\int_{a_f}^{a} \mathrm{d}\ln\hat{a} = -\int_{t_f}^{t} H(\hat{t})\mathrm{d}\hat{t} \simeq \int_{\phi_f}^{\phi} \frac{V(\hat{\phi})}{V_{,\phi}(\hat{\phi})}\mathrm{d}\hat{\phi}$$

Slow roll inflation

5) Spectrum of perturbations (see next lecture!)

Scalar field seeds scalar metric perturbations

$$\langle 0|\mathcal{R}_k^* \mathcal{R}_{k'}|0\rangle \equiv \frac{\mathcal{P}_\mathcal{R}(k)}{4\pi k^3}(2\pi)^3\,\delta^3(\mathbf{k}-\mathbf{k}')$$

$$\mathcal{P}_\mathcal{R}(k) = \frac{\kappa^2}{2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \equiv A_S^2\left(\frac{k}{aH}\right)^{n_S-1}$$

$$\nu = \frac{1+\epsilon-\delta}{1-\epsilon}+\frac{1}{2}$$

6) Spectral index n_s is prediction of inflation!

$$n_s - 1 \equiv \frac{d\ln \mathcal{P}_\mathcal{R}(k)}{d\ln k} = 3-2\nu = 2\left(\frac{\delta-2\epsilon}{1-\epsilon}\right) \simeq 2\eta_V - 6\epsilon_V$$

Slow roll inflation

7) Primordial power spectrum might have a "running", ie higher order corrections

$$\frac{dn_s}{d\ln k} = -\frac{dn_s}{d\ln\eta} = -\eta\mathcal{H}\left(2\xi + 8\epsilon^2 - 10\epsilon\delta\right) \simeq 2\xi_V + 24\epsilon_V^2 - 16\eta_V\epsilon_V$$

8) Scalar field also seeds tensor metric perturbations (see next lecture)

$$\sum_\lambda \langle 0|h^*_{k,\lambda}h_{k',\lambda}|0\rangle = \frac{8\kappa^2}{a^2}|v_k|^2\delta^3(\mathbf{k}-\mathbf{k}') \equiv \frac{\mathcal{P}_g(k)}{4\pi k^3}(2\pi)^3\,\delta^3(\mathbf{k}-\mathbf{k}')$$

$$\mathcal{P}_g(k) = 8\kappa^2\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\mu} \equiv A_T^2\left(\frac{k}{aH}\right)^{n_T}$$

$$n_T \equiv \frac{d\ln\mathcal{P}_g(k)}{d\ln k} = 3-2\mu = \frac{-2\epsilon}{1-\epsilon} \simeq -2\epsilon_V < 0$$

Slow roll inflation

9) Primordial power spectrum for tensors might also have a "running":

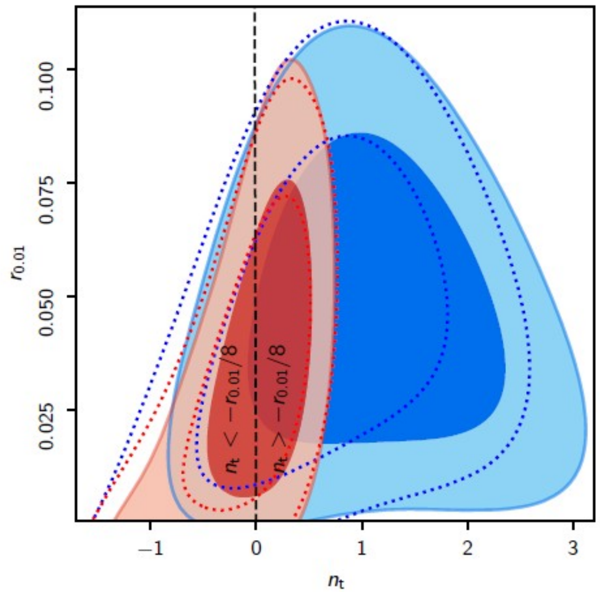
$$\frac{dn_T}{d\ln k} = -\frac{dn_T}{d\ln \eta} = -\eta\mathcal{H}\left(4\epsilon^2 - 4\epsilon\delta\right) \simeq 8\epsilon_V^2 - 4\eta_V\epsilon_V$$

10) In single-field slow-roll models $n_t \sim -r/8$, $r = P_{tensor}/P_{scalar}$

astro-ph/9303019

$$\ln \mathcal{P}_s(k) = \ln \mathcal{P}_0(k) + \frac{1}{2}\frac{d\ln n_s}{d\ln k}\ln(k/k_*)^2 + \frac{1}{6}\frac{d^2\ln n_s}{d\ln k^2}\ln(k/k_*)^3 + \ldots,$$

$$\ln \mathcal{P}_t(k) = \ln(rA_s) + n_t\ln(k/k_*) + \ldots,$$



Slow roll inflation

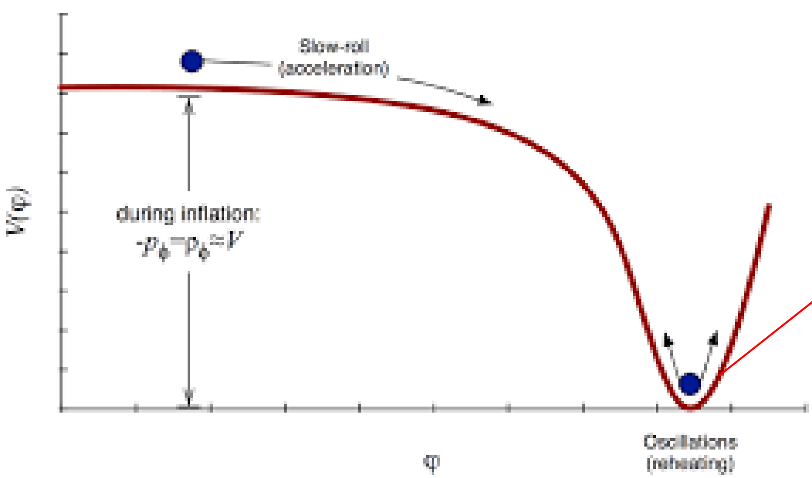
11) After inflation Universe is empty and cold, we need to “reheat it”. This can be done by friction term in Klein-Gordon:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0$$

$$V_{,\phi} \equiv \mathrm{d}V/\mathrm{d}\phi$$

Friction term

12) Pictorial view of reheating



Releases energy in Universe and converts energy of inflaton to SM particles

Example calculation of slow roll parameters

1) Consider simple exponential (toy) model:

$$V(\phi) = V_0 e^{\lambda\kappa^2\phi^2}$$

2) The slow roll parameters are:

$$\begin{aligned}\epsilon &= \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \\ &= 2\kappa^2\lambda^2\phi^2\end{aligned}$$

$$\begin{aligned}\eta &= \frac{1}{\kappa^2}\left(\frac{V''(\phi)}{V(\phi)}\right) \\ &= 2\lambda\left(2\kappa^2\lambda\phi^2 + 1\right)\end{aligned}$$

Example calculation of slow roll parameters

3) At the end of inflation (ε=1) we have

$$\phi_{end} = \frac{1}{\sqrt{2}\kappa\lambda}$$

4) Then, the number of e-folds are

$$N_{inf} = \int_{\phi_{end}}^{\phi} \kappa \frac{1}{\sqrt{2\epsilon}} \mathrm{d}\phi = \frac{\log\left(2(\kappa\lambda\phi)^2\right)}{4\lambda} \Rightarrow \phi = \frac{e^{2\lambda N_{inf}}}{\sqrt{2}\kappa\lambda}$$

Example calculation of slow roll parameters

5) The slow-roll params can be written as a function of the number of e-folds

$$\epsilon = e^{4\lambda N_{inf}}$$
$$\eta = 2\left(\lambda + e^{4\lambda N_{inf}}\right)$$

6) Inflation predictions are

$$n_s = 2\eta - 6\epsilon + 1 = 4\lambda - 2e^{4\lambda N_{inf}} + 1$$
$$r = 16\epsilon = 16e^{4\lambda N_{inf}}$$

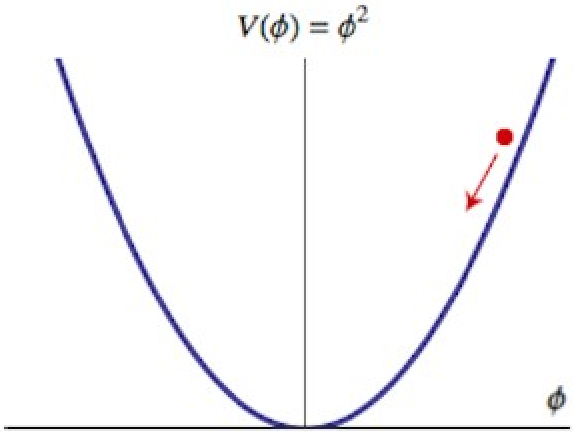
Cosmological model ΛCDM+r	Parameter	*Planck* TT,TE,EE +lowEB+lensing
	r	< 0.11
	$r_{0.002}$	< 0.10
	n_s	0.9659 ± 0.0041

Exercise B2 for V~Φ^n!

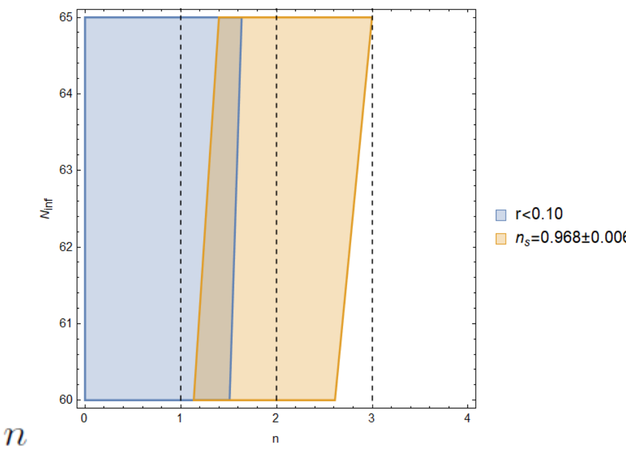
Specific models

1) Chaotic inflation:
One of the simplest potentials out there:

Andrei D. Linde. Chaotic Inflation. Phys. Lett., B129:177181, 1983.



$$V(\phi) = \Lambda^{4-n}\phi^n$$



2) Slow roll parameters

$$N_{inf} = \int_{\phi_{end}}^{\phi} \kappa \frac{1}{\sqrt{2\epsilon}} = \frac{\kappa^2\phi^2}{2n} - \frac{n}{4}$$

$$\epsilon = \frac{n^2}{2\kappa^2\phi^2} = \frac{n}{n+4N_{inf}}$$

$$\eta = \frac{(n-1)n}{\kappa^2\phi^2} = \frac{2(n-1)}{n+4N_{inf}}$$

$$\Rightarrow$$

$$n_s = 1 - \frac{2(n+2)}{n+4N_{inf}}$$

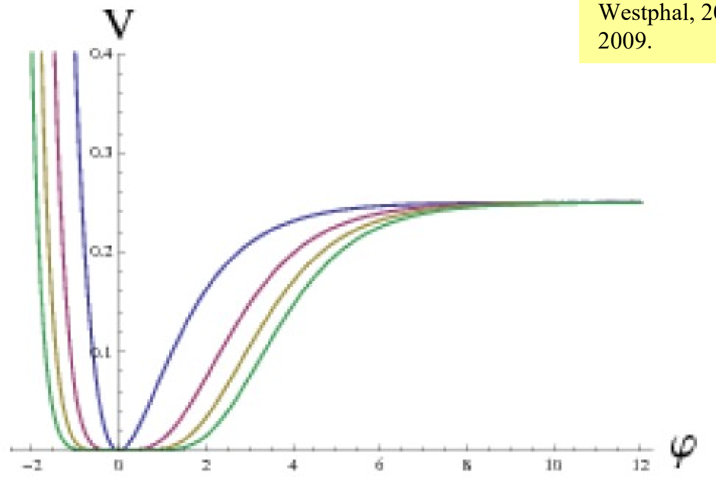
$$r = \frac{16n}{n+4N_{inf}}$$

Specific models

Ewan D. Stewart, 1995, G. R. Dvali and S. H. Henry Tye, 1999, C. P. Burgess, P. Martineau, F. Quevedo, G. Rajesh, and R. J. Zhang, 2002, Eva Silverstein and Alexander Westphal, 2008, M. Cicoli, C. P. Burgess, and F. Quevedo, 2009.

1) Plateau-models

$$V(\phi) = \Lambda^4 \left[1 - \exp\left(-\gamma\kappa\phi\right)\right]^2$$



2) Slow roll and observables:

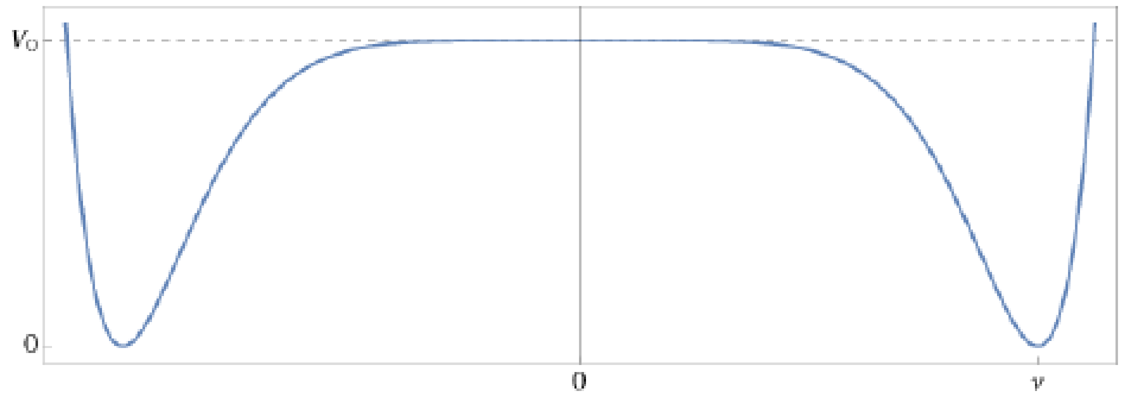
$$N \simeq \frac{\exp(\gamma\phi)}{2\gamma^2} \quad \Rightarrow \quad n_s = 1 - \frac{2}{N}\,, \qquad r = \frac{8}{\gamma^2 N^2}\,.$$

Specific models

Lotfi Boubekeur and David.H. Lyth 2005

1) Hilltop models

$$\Lambda^4 \left[1 - \left(\frac{\phi}{v}\right)^p\right]^2$$



2) Slow roll and observables

$$N \simeq \frac{\kappa^2 v^2}{2p(p-2)} \left(\frac{\phi}{v}\right)^{2-p}$$

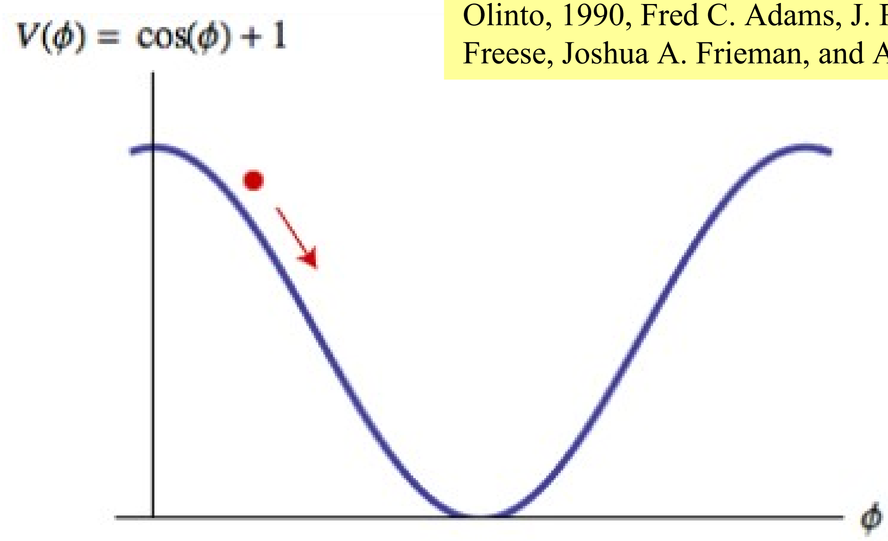
$$n_s \simeq 1 - \frac{2(p-1)}{(p-2)N}, \qquad r \simeq \frac{32p^2}{\kappa^2 v^2} \left[\frac{2p(p-2)}{\kappa^2 v^2} N\right]^{\frac{2p-2}{2-p}}$$

Specific models

Katherine Freese, Joshua A. Frieman, and Angela V. Olinto, 1990, Fred C. Adams, J. Richard Bond, Katherine Freese, Joshua A. Frieman, and Angela V. Olinto, 1993.

1) Natural inflation

$$V(\phi) = \Lambda^4 \left[1 + \cos\left(\frac{\phi}{v}\right)\right]$$



2) Slow roll and observables:

$$N \simeq -2\kappa^2 v^2 \ln\left[\sin\left(\frac{\phi}{2v}\right)\right] \Rightarrow$$

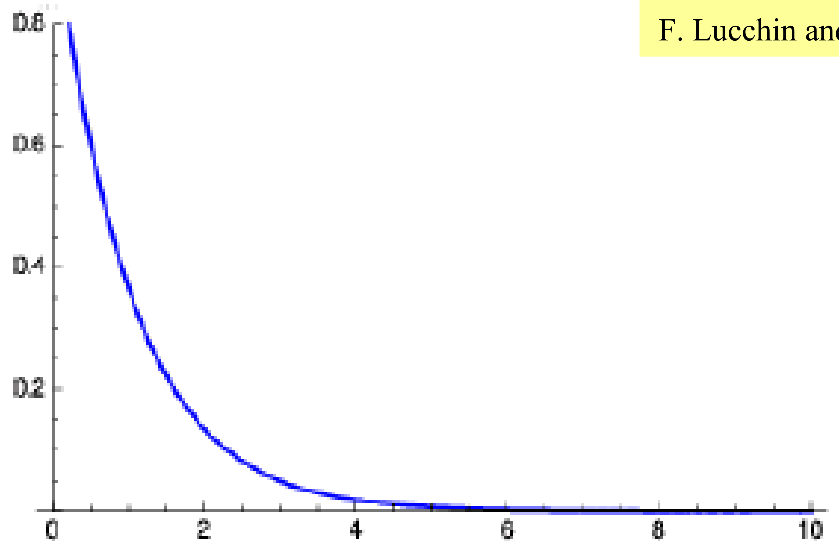
$$n_s \simeq 1 - \frac{1}{\kappa^2 v^2} \frac{\exp\left(\frac{N}{\kappa^2 v^2}\right) + 1}{\exp\left(\frac{N}{\kappa^2 v^2}\right) - 1}, \qquad r \simeq \frac{8}{\kappa^2 v^2}\left[\exp\left(\frac{N}{\kappa^2 v^2}\right) - 1\right]^{-1}$$

Specific models

1) Power-law inflation

F. Lucchin and S. Matarrese, 1985.

$$V(\phi) = \Lambda^4 \exp\left(-\lambda\kappa\phi\right)$$



2) Observables (independent from N):

$$n_s = 1 - \lambda^2 \qquad r = 8\lambda^2$$

K-essence inflation

1) K-essence (most general action for minimally coupled scalar field)

Dark Energy,
L.A. and S.T.

$$X \equiv -(1/2)(\nabla\phi)^2 \quad \Rightarrow \quad S = \int d^4x \sqrt{-g}\, p(\phi, X)$$

$$\Rightarrow \quad S_E = \int d^4x \sqrt{-g} \left[\frac{1}{2} R + K(\phi) X + L(\phi) X^2 + \cdots \right],$$

2) Equation of state:

$$T^{(\phi)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}P)}{\delta g^{\mu\nu}} = P_{,X} \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} P \quad \Rightarrow \quad P_\phi = P, \quad \rho_\phi = 2X P_{,X} - P \quad \Rightarrow$$

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{P}{2X P_{,X} - P}$$

Can cross w=-1!

Modified gravity

1) Simplest thing we can add to GR Lagrangian (on LHS!) is R→ f(R)

- i) Just a scalar degree of freedom
- ii) Has been used in Dark Energy!
- iii) Dynamics well understood, but rich phenomenology
- iv) High energy physics inspired

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\, R + S_m \quad \Longrightarrow \quad S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\, f(R) + S_m$$

2) Simplest example of f(R) is ΛCDM!

$$f(R) \simeq f(R_0) + f'(R_0)R + \cdots \quad \Longrightarrow$$

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\, f(R) + S_m \quad \Longrightarrow \quad S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R - 2\Lambda) + S_m$$

GR is just a special case, not unique theory!

Modified gravity

3) High energy physics inspired. New terms appear when trying to renormalize GR at one-loop order:

$$R \Rightarrow R + \alpha \left[\frac{1}{180} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} - \frac{1}{6}\left(\frac{1}{5} - \xi\right) \Box R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2 + \cdots \right]$$

Birrell & Davis 1986, Sec 6.2, pg 159

=?

f(R)!

4) Most general (pure) modified gravity theory is of the form:

$$R \Rightarrow f(R, P, Q, \Box^n, G)$$

$$\begin{aligned} R &= g_{\mu\nu} R^{\mu\nu} \\ P &= R_{\mu\nu} R^{\mu\nu} \\ Q &= R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \\ \Box &= g^{\mu\nu} \nabla_\mu \nabla_\nu \\ G &= Q - 4P + R^2 \end{aligned}$$

D'Alambertian in curved space

Gauss-Bonnet term (topological invariant in 4D)

f(R) models

1) Get f(R) equations of motion by varying action with respect to metric (F=f '(R)):

$$S = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g}\, f(R) + S_m$$

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu} \leftrightarrow \tilde{g}^{\mu\nu} = g^{\mu\nu} - \delta g^{\mu\nu}$$

$$\begin{aligned}
\delta\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2} g^{\lambda a} \left(\nabla_\mu \delta g_{a\nu} + \nabla_\nu \delta g_{a\mu} - \nabla_a \delta g_{\mu\nu}\right) \\
\delta R^{\nu}_{k\lambda a} &= \nabla_\lambda \delta\Gamma^{\nu}_{ka} - \nabla_a \delta\Gamma^{\nu}_{k\lambda} \\
\delta R_{\mu\nu} &= \frac{1}{2}\left(-\Box \delta g_{\mu\nu} + \nabla_a \nabla_\mu \delta g^a_\nu + \nabla_a \nabla_\nu \delta g^a_\mu - \nabla_\mu \nabla_\nu \delta g^a_a\right) \\
\delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}\, g_{ab}\, \delta g^{ab} \\
\delta R &= \delta(g^{\mu\nu} R_{\mu\nu}) = \delta g^{\mu\nu} R_{\mu\nu} + g_{\mu\nu} \Box \delta g^{\mu\nu} - \nabla_\mu \nabla_\nu \delta g^{\mu\nu}
\end{aligned}$$

$\Rightarrow$

$$F G_{\mu\nu} - \frac{1}{2}(f(R) - R\, F) g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu)\, F = \kappa T^{(m)}_{\mu\nu}$$

2) Conservation equation:

$$\begin{aligned}
S &= \int d^4x \sqrt{-g}\,\mathcal{L} \Rightarrow \\
\delta S &= \int d^4x \sqrt{-g} \left[\frac{\sqrt{-g}\mathcal{L}}{\delta g^{\mu\nu}} \frac{1}{\sqrt{-g}}\right] \delta g^{\mu\nu} \\
&= \int d^4x \sqrt{-g}\, S_{\mu\nu}\, \delta g^{\mu\nu}
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\delta g_{\mu\nu} &= \pounds_V g_{\mu\nu} = V^\sigma \nabla_\sigma g_{\mu\nu} + (\nabla_\mu V^\lambda) g_{\lambda\nu} + (\nabla_\nu V^\lambda) g_{\mu\lambda} \\
&= \nabla_\mu V_\nu + \nabla_\nu V_\mu
\end{aligned}$$

$$\begin{aligned}
\delta S &= \int d^4x \sqrt{-g}\left[(V_\mu \nabla_\nu) S^{\mu\nu} + (V_\nu \nabla_\mu) S^{\mu\nu}\right] \\
&= -\int d^4x \sqrt{-g}\, V_\nu \left[\nabla_\mu S^{\mu\nu} \nabla_\mu S^{\mu\nu}\right] \Rightarrow
\end{aligned}$$

$$\nabla_\mu S^{\mu\nu} = 0$$

f(R) models

3) f(R) Friedman equations for FRW and acceleration!

$$ds^2 = c^2dt^2 - \alpha(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin(\theta)^2 d\phi^2) \right) \Longrightarrow$$

$$\begin{aligned} 3FH^2 &= \rho_{\rm m} + \rho_{\rm rad} + \frac{1}{2}(FR - f) - 3H\dot{F} \\ -2F\dot{H} &= \rho_{\rm m} + \frac{4}{3}\rho_{\rm rad} + \ddot{F} - H\dot{F} \end{aligned}$$

Properly chosen,
can give acceleration!

4) Perturbations and Geff: f(R) modifies Newton's constant!

$$f(R) \simeq f(R_0) + f'(R_0)R + \cdots \Longrightarrow$$

$$S = \frac{1}{8\pi G_N} \int d^4x\sqrt{-g} f(R) \simeq \frac{1}{8\pi G_N} \int d^4x\sqrt{-g}\,[f(R_0) + f'(R_0)R] \simeq \frac{1}{8\pi G_{eff}} \int d^4x\sqrt{-g}\,[R - 2\Lambda]$$

$$G_{eff} \sim G_N/f'(R_0)$$

f(R) models

5) Conformal transformation (Jordan→Einstein frame): f(R) is just a scalar field!

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}\,, \quad \Longrightarrow \quad R = \Omega^2(\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega) \quad \Longrightarrow$$

$$\omega \equiv \ln\,\Omega$$

$$S = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa^2}F\Omega^{-2}(\tilde{R} + 6\tilde{\Box}\omega - 6\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega) - \Omega^{-4}U\right] + \int \mathrm{d}^4x\mathcal{L}_M(\Omega^{-2}\,\tilde{g}_{\mu\nu}, \Psi_M)$$

$$\Omega^2 = F \qquad U = \frac{FR - f}{2\kappa^2}$$

Redefine "field":

$$\kappa\phi \equiv \sqrt{3/2}\;\ln\;F \quad \Longrightarrow$$

$$S_E = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[\frac{1}{2\kappa^2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + \int \mathrm{d}^4x\mathcal{L}_M(F^{-1}(\phi)\tilde{g}_{\mu\nu}, \Psi_M)$$

Quintessence!!!

Potential

$$V(\phi) = \frac{U}{F^2} = \frac{FR - f}{2\kappa^2F^2}$$

Non-minimal coupling

Starobinsky inflation

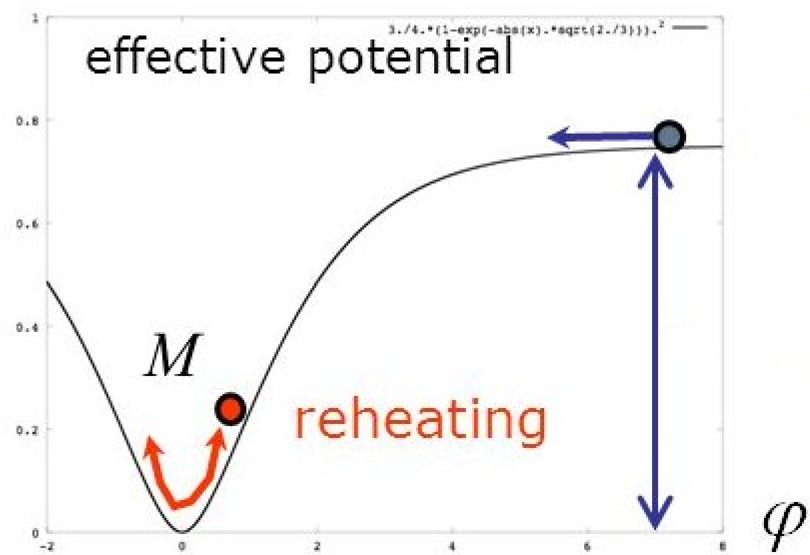
Alexei A. Starobinsky 1982

6) Simple f(R) model

$$f(R) = R + R^2/(6M^2) \Longrightarrow V(\phi) = \Lambda^4\left(1 - e^{-\sqrt{2/3}\phi/M_{pl}}\right)^2$$

7) Slow roll and observables:

$$n_s \simeq 1 - \frac{2}{N}, \qquad r \simeq \frac{12}{N^2}.$$



Modified gravity and ghosts

Ghosts+propagators in MoG:

ArXiv:0911.3094

$$S = \int d^4x\sqrt{-g}f(R,P,Q)$$

$$P \equiv R_{ab}R^{ab}$$
$$Q \equiv R_{abcd}R^{abcd}$$

$$F \equiv \frac{\partial f}{\partial R}, \quad f_P \equiv \frac{\partial f}{\partial P}, \quad f_Q \equiv \frac{\partial f}{\partial Q}$$

$$\begin{aligned} FG_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}(f - R\,F) - (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)F \\ &-2\left(f_P R^a_\mu R_{a\nu} + f_Q\, R_{abc\mu}R^{abc}{}_\nu\right) \\ &-g_{\mu\nu}\nabla_a\nabla_b(f_P R^{ab}) - \Box(f_P R_{\mu\nu}) \\ &+2\nabla_a\nabla_b\left(f_P\, R^a{}_{(\mu}\delta^b{}_{\nu)} + 2f_Q\, R^a{}_{(\mu\nu)}{}^b\right) \end{aligned}$$

Fourth order derivatives… Problem!!!

Linearize and find propagator G(k):

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{\bar{h}}{2}\,\eta_{\mu\nu} + \eta_{\mu\nu}h_f$$

$$\left(k^2 + \frac{k^4}{m^2_{spin2}}\right)\bar{h}_{\mu\nu} = 0$$

$$\Box h_f = m_s^2 h_f$$

$$m^2_{spin2} \equiv -\frac{F_0}{f_{P0}+4f_{Q0}}$$

$$m_s^2 \equiv \frac{1}{3}\frac{F_0}{F_{,R0}+\frac{2}{3}(f_{P0}+f_{Q0})}$$

$$G(k) \propto \frac{1}{k^2} - \frac{1}{k^2 + m^2_{spin2}}$$

Negative sign…

Other MoG models

1) Models with extra dimensions: Kaluza-Klein

i) Assume extra dimension y, which is compactified with cylindrical boundary conditions. Then 5D metric g_{MN} satisfies

$$f(x,y) = f(x, y + 2\pi r) \Longrightarrow \frac{\partial g_{MN}}{\partial y} = 0$$

Similar to U(1) symmetry!

ii) Expand 5D metric in Fourier modes:

$$g_{MN}(x,y) = \sum_n g^{(n)}_{MN}(x) e^{iny/r} \Longrightarrow g^{(0)}_{MN} = \phi^{-1/3} \begin{pmatrix} g_{\mu\nu} + \phi A_\mu A\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix}$$

Very general decomposition)

Other MoG models

iii) GR in 5D:

4D GR+Maxwell+scalar field!

$$S = \frac{1}{16\pi G_N^5}\int d^4x dy \sqrt{-g^{(5)}} R^{(5)}$$
$$= \frac{1}{16\pi G_N^4}\int d^4x \sqrt{-g^{(4)}}\left(R + \frac{1}{4}\phi F_{\mu\nu}F^{\mu\nu} + \frac{1}{6\phi^2}\partial^\mu\phi\partial_\mu\phi\right)$$

$$G_N^{(4)} = \frac{G_N^{(5)}}{2\pi r}$$

iv) Add extra scalar field:

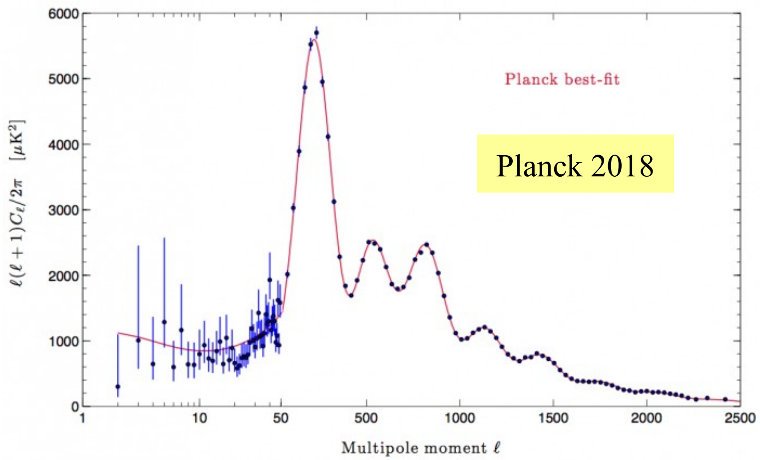
$$S_\Phi = \int d^4x dy \sqrt{-g^{(5)}}\left(g_{MN}^{(0)}\partial_M\Phi\partial_N\Phi\right)$$
$$= (2\pi r)\sum_n \int d^4x\sqrt{-g^{(4)}}\left[g^{\mu\nu}\left(\partial_\mu + \frac{in}{r}A_\mu\right)\Phi_n\left(\partial_\nu + \frac{in}{r}A_\nu\right)\Phi_n - \frac{n^2}{\phi r^2}\Phi_n^2\right]$$

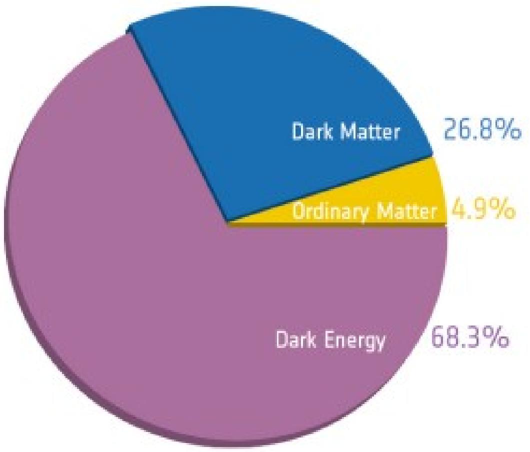
$$\Rightarrow \quad Q_n = \frac{8\pi G_N^{(4)} n}{r}\sqrt{\frac{2}{\phi}} \qquad M_n = \frac{|n|}{r\sqrt{\phi}}$$

$Q_n \sim M_n$… Problem!!!

Comparison with CMB

Inflation after Planck

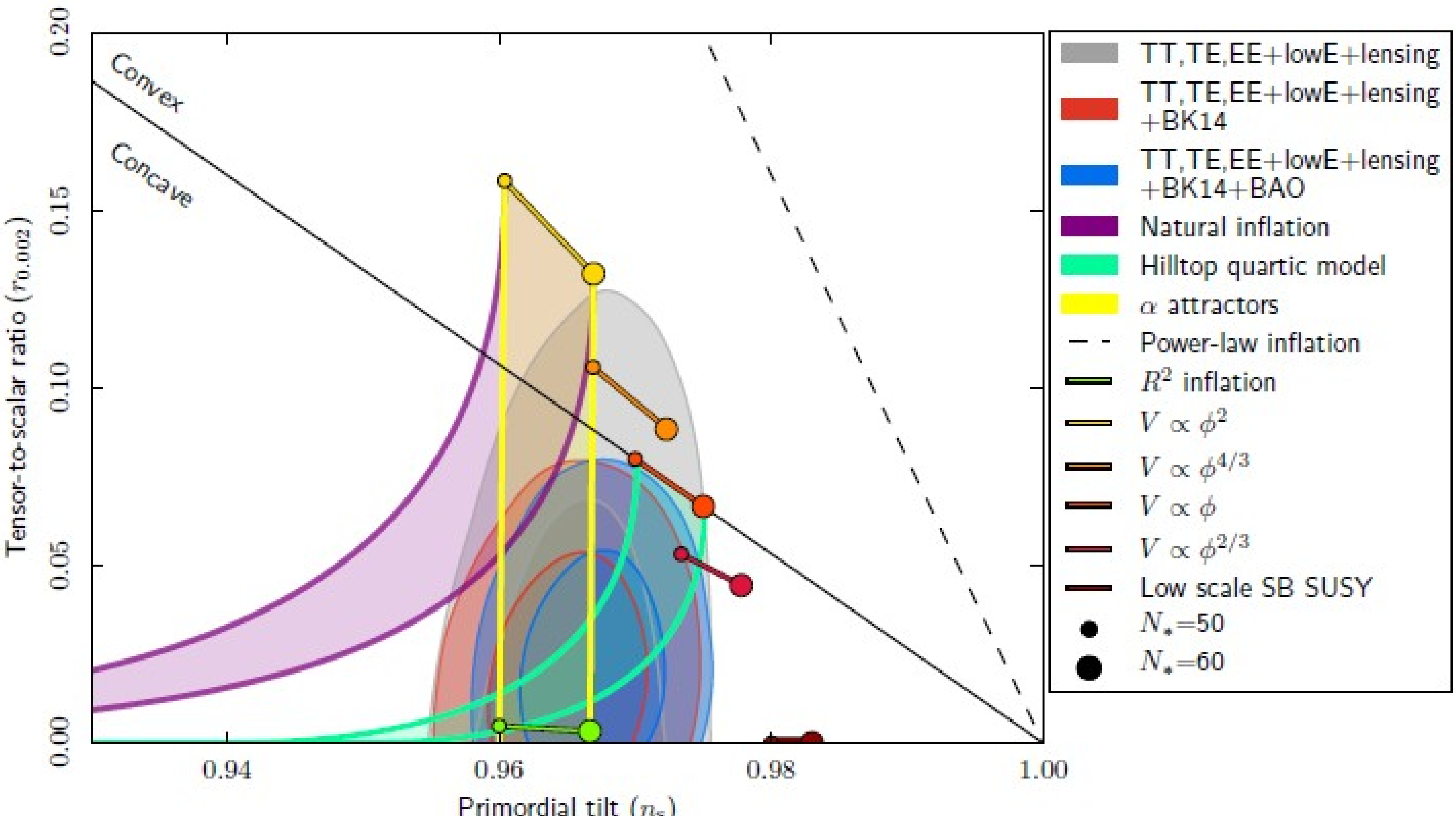




Parameter	TT+lowE 68% limits	TE+lowE 68% limits	EE+lowE 68% limits	TT,TE,EE+lowE 68% limits	TT,TE,EE+lowE+lensing 68% limits	TT,TE,EE+lowE+lensing+BAO 68% limits
$\Omega_b h^2$	0.02212 ± 0.00022	0.02249 ± 0.00025	0.0240 ± 0.0012	0.02236 ± 0.00015	0.02237 ± 0.00015	0.02242 ± 0.00014
$\Omega_c h^2$	0.1206 ± 0.0021	0.1177 ± 0.0020	0.1158 ± 0.0046	0.1202 ± 0.0014	0.1200 ± 0.0012	0.11933 ± 0.00091
$100\theta_{MC}$	1.04077 ± 0.00047	1.04139 ± 0.00049	1.03999 ± 0.00089	1.04090 ± 0.00031	1.04092 ± 0.00031	1.04101 ± 0.00029
τ	0.0522 ± 0.0080	0.0496 ± 0.0085	0.0527 ± 0.0090	$0.0544^{+0.0070}_{-0.0081}$	0.0544 ± 0.0073	0.0561 ± 0.0071
$\ln(10^{10}A_s)$	3.040 ± 0.016	$3.018^{+0.020}_{-0.018}$	3.052 ± 0.022	3.045 ± 0.016	3.044 ± 0.014	3.047 ± 0.014
n_s	0.9626 ± 0.0057	0.967 ± 0.011	0.980 ± 0.015	0.9649 ± 0.0044	0.9649 ± 0.0042	0.9665 ± 0.0038
H_0 [km s^{-1} Mpc^{-1}] . .	66.88 ± 0.92	68.44 ± 0.91	69.9 ± 2.7	67.27 ± 0.60	67.36 ± 0.54	67.66 ± 0.42
Ω_Λ	0.679 ± 0.013	0.699 ± 0.012	$0.711^{+0.033}_{-0.026}$	0.6834 ± 0.0084	0.6847 ± 0.0073	0.6889 ± 0.0056
Ω_m	0.321 ± 0.013	0.301 ± 0.012	$0.289^{+0.026}_{-0.033}$	0.3166 ± 0.0084	0.3153 ± 0.0073	0.3111 ± 0.0056
$\Omega_m h^2$	0.1434 ± 0.0020	0.1408 ± 0.0019	$0.1404^{+0.0034}_{-0.0039}$	0.1432 ± 0.0013	0.1430 ± 0.0011	0.14240 ± 0.00087
$\Omega_m h^3$	0.09589 ± 0.00046	0.09635 ± 0.00051	$0.0981^{+0.0016}_{-0.0018}$	0.09633 ± 0.00029	0.09633 ± 0.00030	0.09635 ± 0.00030
σ_8	0.8118 ± 0.0089	0.793 ± 0.011	0.796 ± 0.018	0.8120 ± 0.0073	0.8111 ± 0.0060	0.8102 ± 0.0060
$S_8 \equiv \sigma_8(\Omega_m/0.3)^{0.5}$.	0.840 ± 0.024	0.794 ± 0.024	$0.781^{+0.052}_{-0.060}$	0.834 ± 0.016	0.832 ± 0.013	0.825 ± 0.011
$\sigma_8\Omega_m^{0.25}$	0.611 ± 0.012	0.587 ± 0.012	0.583 ± 0.027	0.6090 ± 0.0081	0.6078 ± 0.0064	0.6051 ± 0.0058
z_{re}	7.50 ± 0.82	$7.11^{+0.91}_{-0.75}$	$7.10^{+0.87}_{-0.73}$	7.68 ± 0.79	7.67 ± 0.73	7.82 ± 0.71
$10^9 A_s$	2.092 ± 0.034	2.045 ± 0.041	2.116 ± 0.047	$2.101^{+0.031}_{-0.034}$	2.100 ± 0.030	2.105 ± 0.030
$10^9 A_s e^{-2\tau}$	1.884 ± 0.014	1.851 ± 0.018	1.904 ± 0.024	1.884 ± 0.012	1.883 ± 0.011	1.881 ± 0.010
Age [Gyr]	13.830 ± 0.037	13.761 ± 0.038	$13.64^{+0.16}_{-0.14}$	13.800 ± 0.024	13.797 ± 0.023	13.787 ± 0.020

Comparison with CMB

Planck 2018, arXiv: 1807.06211

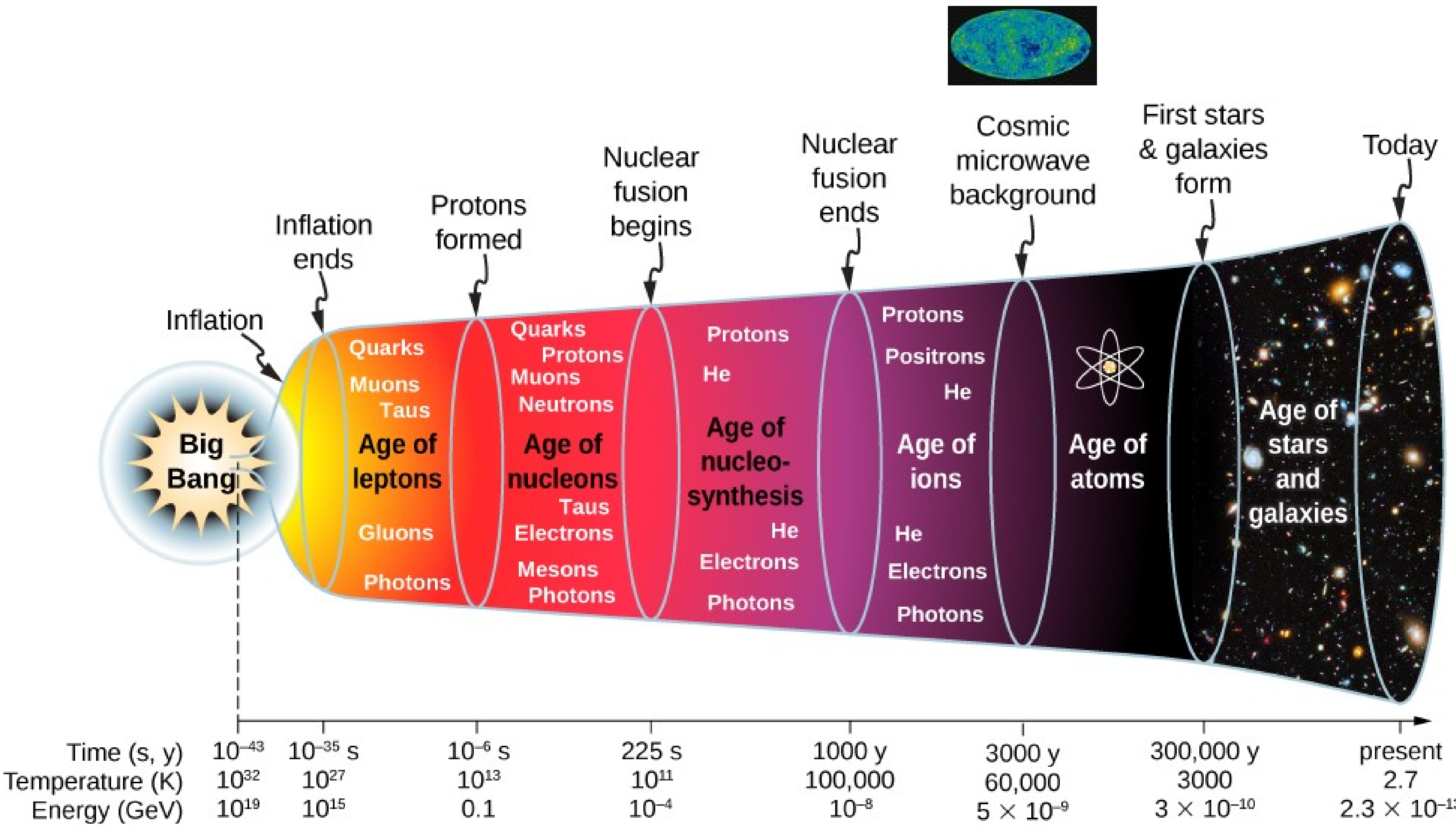


Comparison with CMB

Planck 2018, arXiv: 1807.06211

Inflationary model	Potential $V(\phi)$	Parameter range	$\Delta\chi^2$	$\ln B$
$R + R^2/(6M^2)$	$\Lambda^4\left(1 - e^{-\sqrt{2/3}\phi/M_{\rm Pl}}\right)^2$	...	...	...
Power-law potential	$\lambda M_{\rm Pl}^{10/3}\phi^{2/3}$	...	2.8	−2.6
Power-law potential	$\lambda M_{\rm Pl}^{3}\phi$	...	2.5	−1.9
Power-law potential	$\lambda M_{\rm Pl}^{8/3}\phi^{4/3}$	...	10.4	−4.5
Power-law potential	$\lambda M_{\rm Pl}^{2}\phi^{2}$	...	22.3	−7.1
Power-law potential	$\lambda M_{\rm Pl}\phi^{3}$	...	40.9	−19.2
Power-law potential	$\lambda\phi^4$	...	89.1	−33.3
Non-minimal coupling	$\lambda^4\phi^4 + \xi\phi^2 R/2$	$-4 < \log_{10}\xi < 4$	3.1	−1.6
Natural inflation	$\Lambda^4\left[1 + \cos(\phi/f)\right]$	$0.3 < \log_{10}(f/M_{\rm Pl}) < 2.5$	9.4	−4.2
Hilltop quadratic model	$\Lambda^4\left(1 - \phi^2/\mu_2^2 + \ldots\right)$	$0.3 < \log_{10}(\mu_2/M_{\rm Pl}) < 4.85$	1.7	−2.0
Hilltop quartic model	$\Lambda^4\left(1 - \phi^4/\mu_4^4 + \ldots\right)$	$-2 < \log_{10}(\mu_4/M_{\rm Pl}) < 2$	−0.3	−1.4
D-brane inflation ($p = 2$)	$\Lambda^4\left(1 - \mu_{\rm D2}^2/\phi^p + \ldots\right)$	$-6 < \log_{10}(\mu_{\rm D2}/M_{\rm Pl}) < 0.3$	−2.3	1.6
D-brane inflation ($p = 4$)	$\Lambda^4\left(1 - \mu_{\rm D4}^4/\phi^p + \ldots\right)$	$-6 < \log_{10}(\mu_{\rm D4}/M_{\rm Pl}) < 0.3$	−2.2	0.8
Potential with exponential tails	$\Lambda^4\left[1 - \exp(-q\phi/M_{\rm Pl}) + \ldots\right]$	$-3 < \log_{10} q < 3$	−0.5	−1.0
Spontaneously broken SUSY	$\Lambda^4\left[1 + \alpha_h \log(\phi/M_{\rm Pl}) + \ldots\right]$	$-2.5 < \log_{10}\alpha_h < 1$	9.0	−5.0
E-model ($n = 1$)	$\Lambda^4\left\{1 - \exp\left[-\sqrt{2}\phi\left(\sqrt{3\alpha_1^{\rm E}}M_{\rm Pl}\right)^{-1}\right]\right\}^{2n}$	$-2 < \log_{10}\alpha_1^{\rm E} < 4$	0.2	−1.0
E-model ($n = 2$)	$\Lambda^4\left\{1 - \exp\left[-\sqrt{2}\phi\left(\sqrt{3\alpha_2^{\rm E}}M_{\rm Pl}\right)^{-1}\right]\right\}^{2n}$	$-2 < \log_{10}\alpha_2^{\rm E} < 4$	−0.1	0.7
T-model ($m = 1$)	$\Lambda^4 \tanh^{2m}\left[\phi\left(\sqrt{6\alpha_1^{\rm T}}M_{\rm Pl}\right)^{-1}\right]$	$-2 < \log_{10}\alpha_1^{\rm T} < 4$	−0.1	0.1
T-model ($m = 2$)	$\Lambda^4 \tanh^{2m}\left[\phi\left(\sqrt{6\alpha_2^{\rm T}}M_{\rm Pl}\right)^{-1}\right]$	$-2 < \log_{10}\alpha_2^{\rm T} < 4$	−0.4	0.1

Big Bang theory timeline



Side-effects and prospects

1) Production of primordial Gravitational Waves (GWs), see next lecture.

2) Production of Primordial Black Holes (PBHs), see next lecture.

3) Inflation probes high-energy physics (GUT+), not in reach of experiments.

4) Can be used to test for BSM physics.

5) B-modes of CMB probe inflation (see CMB lecture).

Summary

1) Hot Big Bang theory has problems (horizon, flatness, monopole, LSS), that a phase of accelerate expansion, aka Inflation, can solve.

2) Scalar field Inflation is nice toy model that exhibits desired properties.

3) Zoo of plausible inflationary models, both scalar and MoG.

4) Inflation is strongly constrained by CMB: $n_s=0.96$, $r<0.1$.

5) Inflation is now established as a pillar of the Cosmological Model!

References

1) Scott Dodelson: “Modern Cosmology”.

2) Daniel Baumann: Arxiv 0907.5424.

3) John Peacock: “Cosmological Physics”.

4) Juan Garcia-Bellido’s notes at Cosmo website.

5) Andrew Liddle: “Cosmological Inflation & Large-Scale Structure”.