THE INFLATIONARY PARADIGM

The hot Big Bang theory is nowadays a very robust edifice, with many independent observational checks: the expansion of the universe; the abundance of light elements; the cosmic microwave background; a predicted age of the universe compatible with the age of the oldest objects in it, and the formation of structure via gravitational collapse of initially small inhomogeneities. Today, these observations are confirmed to within a few percent accuracy, and have helped establish the hot Big Bang as the preferred model of the universe. All the physics involved in the above observations is routinely tested in the laboratory (atomic and nuclear physics experiments) or in the solar system (general relativity).

However, this theory leaves a range of crucial questions unanswered, most of which are initial conditions' problems. There is the reasonable assumption that these cosmological problems will be solved or explained by *new physical principles* at high energies, in the early universe. This assumption leads to the natural conclusion that accurate observations of the present state of the universe may shed light onto processes and physical laws at energies above those reachable by particle accelerators, present or future. We will see that this is a very optimistic approach indeed, and that there are many unresolved issues related to those problems. However, there might be in the near future reasons to be optimistic.

Shortcomings of Big Bang Cosmology

The Big Bang theory could not explain the origin of matter and structure in the universe; that is, the origin of the matter–antimatter asymmetry, without which the universe today would be filled by a uniform radiation continuosly expanding and cooling, with no traces of matter, and thus without the possibility to form gravitationally bound systems like galaxies, stars and planets that could sustain life. Moreover, the standard Big Bang theory assumes, but cannot explain, the origin of the extraordinary smoothness and flatness of the universe on the very large scales seen by the microwave background probes and the largest galaxy catalogs. It cannot explain the origin of the primordial density perturbations that gave rise to cosmic structures like galaxies, clusters and superclusters, via gravitational collapse; the quantity and nature of the dark matter that we believe holds the universe together; nor the origin of the Big Bang itself.

A summary of the problems that the Big Bang theory cannot explain is:

- The global structure of the universe.
  - Why is the universe so close to spatial flatness?
  - Why is matter so homogeneously distributed on large scales?
- The origin of structure in the universe.
  - How did the primordial spectrum of density perturbations originate?
- The origin of matter and radiation.
  - Where does all the energy in the universe come from?
  - What is the nature of the dark matter in the universe?
  - How did the matter-antimatter asymmetry arise?
- The initial singularity.
  - Did the universe have a beginning?
  - What is the global structure of the universe beyond our observable patch?

Let me discuss one by one the different issues:

The Flatness Problem

The Big Bang theory assumes but cannot explain the extraordinary spatial flatness of our local patch of the universe. In the general FRW metric the parameter K that characterizes spatial curvature is a free parameter. There is nothing in the theory that determines this parameter a priori. However, it is directly related, via the Friedmann equation, to the dynamics, and thus the matter content, of the universe,

$$K = \frac{8\pi G}{3}\rho a^2 - H^2 a^2 \;=\; \frac{8\pi G}{3}\rho a^2\Big(\frac{\Omega-1}{\Omega}\Big)\,. \tag{1}$$

We can therefore define a new variable,

$$x \equiv \frac{\Omega - 1}{\Omega} = \frac{\text{const.}}{\rho a^2}\,, \tag{2}$$

whose time evolution is given by

$$x' = \frac{dx}{dN} = (1+3\omega)\,x\,, \tag{3}$$

where $N = \ln(a/a_i)$ characterizes the *number of e-folds* of universe expansion ($dN = Hdt$) and where we have used the evolution of the total energy density, ρa^3, which only depends on the barotropic ratio ω. It is clear from Eq. (3) that the phase-space diagram (x, x') presents an unstable critical (saddle) point at $x = 0$ for $\omega > -1/3$, i.e. for the radiation ($\omega = 1/3$) and matter ($\omega = 0$) eras. A small perturbation from $x = 0$ will drive the system towards $x = \pm\infty$. Since we know the universe went through both the radiation era (because of primordial nucleosynthesis) and the matter era (because of structure formation), tiny deviations from $\Omega = 1$ would have grown since then, such that today

$$x_0 = \frac{\Omega_0 - 1}{\Omega_0} = x_{\rm in}\left(\frac{T_{\rm in}}{T_{\rm eq}}\right)^2(1+z_{\rm eq})\,. \tag{4}$$

In order that today's value be in the range

$$\Omega_0 = 0.9992 \pm 0.0040\,,$$

or $x_0 \approx 10^{-3}$, it is required that at, say, primordial nucleosynthesis ($T_{\rm NS} \simeq 10^6\, T_{\rm eq}$) its value be

$$\Omega(t_{\rm NS}) = 1 \pm 10^{-18}\,, \tag{5}$$

which represents a tremendous finetuning. Perhaps the universe indeed started with such a peculiar initial condition, but it is epistemologically more satisfying if we give a fundamental dynamical reason for the universe to have started so close to spatial flatness. These arguments were first used by Robert Dicke in the 1960s, much before inflation. He argued that the most natural initial condition for the spatial curvature should have been the Planck scale curvature, ${}^{(3)}\!R = 6K/l_{\rm P}^2$, where the Planck length is $l_{\rm P} = (\hbar G/c^3)^{1/2} = 1.62 \times 10^{-33}$ cm, that is, 60 orders of magnitude smaller than the present size of the universe, $a_0 = 1.38 \times 10^{28}$ cm. A universe with this immense curvature would have collapsed within a Planck time, $t_{\rm P} = (\hbar G/c^5)^{1/2} = 5.39 \times 10^{-44}$ s, again 60 orders of magnitude smaller than the present age of the universe, $t_0 = 4.1 \times 10^{17}$ s. Therefore, the flatness problem is also related to the Age Problem, why is it that the universe is so old and flat when, under ordinary circumstances (based on the fundamental scale of gravity) it should have lasted only a Planck time and reached a size of order the Planck length? As we will see, inflation gives a dynamical reason to such a peculiar initial condition.

The Homogeneity Problem

An expanding universe has *particle horizons*, that is, spatial regions beyond which causal communication cannot occur. The horizon distance can be defined as the maximum distance that light could have travelled since the origin of the universe,

$$d_{\rm H}(t) \equiv a(t) \int_0^t \frac{dt'}{a(t')} \sim H^{-1}(t)\,, \tag{6}$$

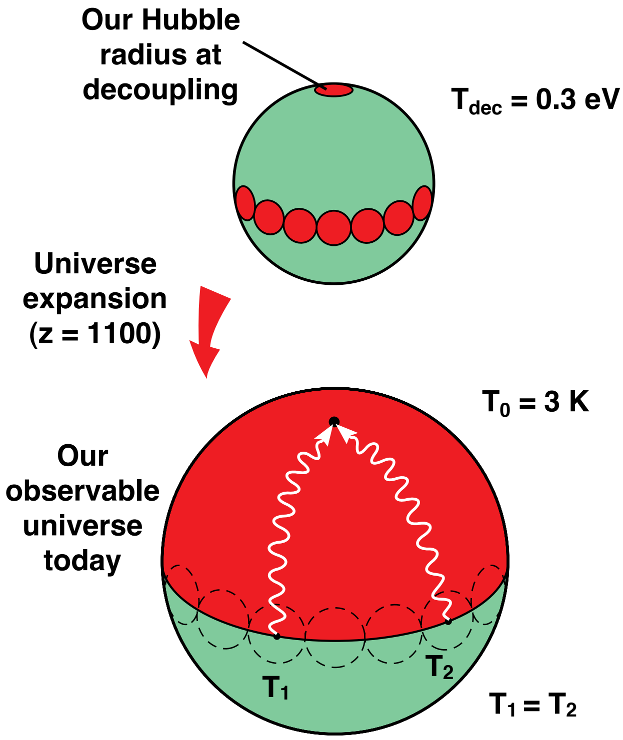

Figure 1: Perhaps the most acute problem of the Big Bang theory is explaining the extraordinary homogeneity and isotropy of the microwave background. At the time of decoupling, the volume that gave rise to our present universe contained many causally disconnected regions (top figure). Today we observe a blackbody spectrum of photons coming from those regions and they appear to have the same temperature, $T_1 = T_2$, to one part in 10^5. Why is the universe so homogeneous? This constitutes the so-called horizon problem, which is spectacularly solved by inflation.

which is proportional to the Hubble scale.[1] For instance, at the beginning of nucleosynthesis the horizon distance is a few light-seconds, but grows *linearly* with time and by the end of nucleosynthesis it is a few light-minutes, i.e. a factor 100 larger, while the scale factor has increased *only* a factor of 10. The fact that the causal horizon increases faster, $d_{\rm H} \sim t$, than the scale factor, $a \sim t^{1/2}$, implies that at any given time the universe contains regions within itself that, according to the Big Bang theory, were *never* in causal contact before. For instance, the

[1]For the radiation era, the horizon distance is equal to the Hubble scale. For the matter era it is twice the Hubble scale.

number of causally disconnected regions at a given redshift z present in our causal volume today, $d_{\rm H}(t_0) \equiv a_0$, is

$$N_{\rm CD}(z) \sim \left(\frac{a(t)}{d_{\rm H}(t)}\right)^3 \simeq (1+z)^{3/2}\,, \tag{7}$$

which, for the time of decoupling, is of order $N_{\rm CD}(z_{\rm dec}) \sim 10^5 \gg 1$.

This phenomenon is particularly acute in the case of the observed microwave background. Information cannot travel faster than the speed of light, so the causal region at the time of photon decoupling could not be larger than $d_{\rm H}(t_{\rm dec}) \sim 3 \times 10^5$ light years across, or about 1° projected in the sky today. So why should regions that are separated by more than 1° in the sky today have exactly the same temperature, to within 10 ppm, when the photons that come from those two distant regions could not have been in causal contact when they were emitted? This constitutes the so-called horizon problem, see Fig. 1, and was first discussed by Robert Dicke in the 1970s as a profound inconsistency of the Big Bang theory.

Cosmological Inflation

In the 1980s, a new paradigm, deeply rooted in fundamental physics, was put forward by Alan H. Guth, Andrei D. Linde and others, to address these fundamental questions. According to the inflationary paradigm, the early universe went through a period of exponential expansion, driven by the approximately constant energy density of a scalar field called the inflaton. In modern physics, elementary particles are represented by quantum fields, which resemble the familiar electric, magnetic and gravitational fields. A field is simply a function of space and time whose quantum oscillations are interpreted as particles. In our case, the inflaton field has, associated with it, a large potential energy density, which drives the exponential expansion during inflation. We know from general relativity that the density of matter determines the expansion of the universe, but a constant energy density acts in a very peculiar way: as a repulsive force that makes any two points in space separate at exponentially large speeds. (This does not violate the laws of causality because there is no information carried along in the expansion, it is simply the stretching of space-time.)

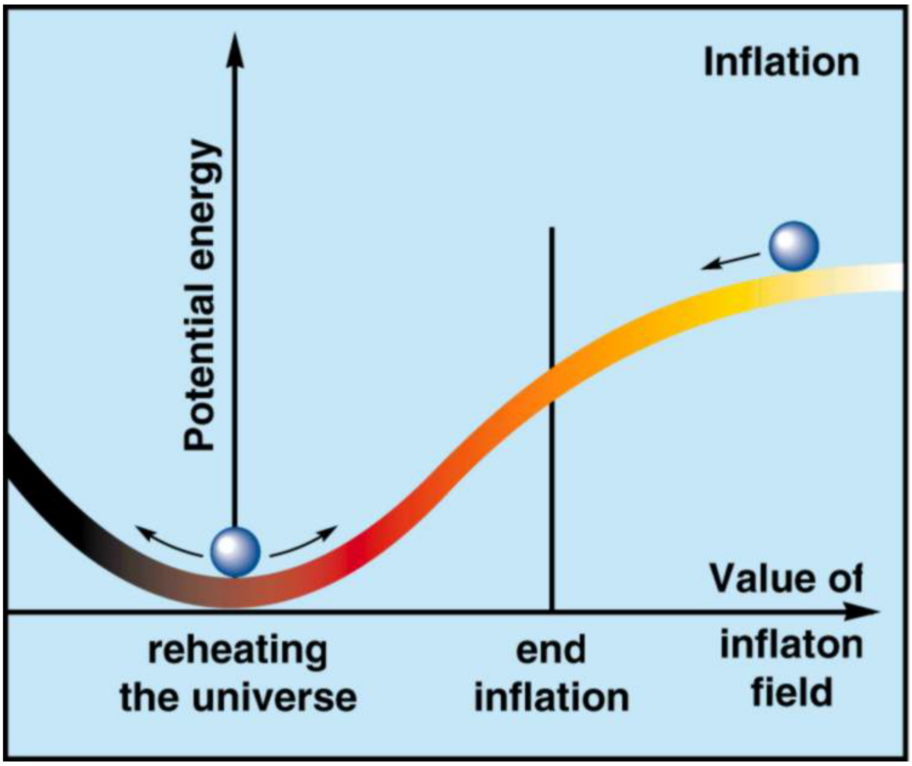

Figure 2: The inflaton field can be represented as a ball rolling down a hill. During inflation, the energy density is approximately constant, driving the tremendous expansion of the universe. When the ball starts to oscillate around the bottom of the hill, inflation ends and the inflaton energy decays into particles. In certain cases, the coherent oscillations of the inflaton could generate a resonant production of particles which soon thermalize, reheating the universe.

This superluminal expansion is capable of explaining the large scale homogeneity of our observable universe and, in particular, why the microwave background looks so isotropic: regions separated today by more than 1° in the sky were, in fact, in causal contact before inflation, but were stretched to cosmological distances by the expansion.

Any inhomogeneities present before the tremendous expansion would be washed out. This explains why photons from supposedly causally disconneted regions have actually the same spectral distribution with the same temperature, see Fig. 1.

Moreover, in the usual Big Bang scenario a flat universe, one in which the gravitational attraction of matter is exactly balanced by the cosmic expansion, is unstable under perturbations: a small deviation from flatness is amplified and soon produces either an empty universe or a collapsed one. As we discussed above, for the universe to be nearly flat today, it must have been extremely flat at nucleosynthesis, deviations not exceeding more than one part in 10^{15}. This extreme fine tuning of initial conditions was also solved by the inflationary paradigm, see Fig. 3. Thus inflation is an extremely elegant hypothesis that explains how a region much, much greater that our own observable universe could have become smooth and flat without recourse to *ad hoc* initial conditions. Furthermore, inflation dilutes away any "unwanted" relic species that could have remained from early universe phase transitions, like monopoles, cosmic strings, etc., which are predicted in grand unified theories and whose energy density could be so large that the universe would have become unstable, and collapsed, long ago. These relics are diluted by the superluminal expansion, which leaves at most one of these particles per causal horizon, making them harmless to the subsequent evolution of the universe.

The only thing we know about this peculiar scalar field, the *inflaton*, is that it has a mass and a self-interaction potential $V(\phi)$ but we ignore everything else, even the scale at which its dynamics determines the superluminal expansion. In particular, we still do not know the nature of the inflaton field itself, is it some new *fundamental* scalar field in the electroweak symmetry breaking sector, or is it just some *effective* description of a more fundamental high energy interaction? Hopefully,

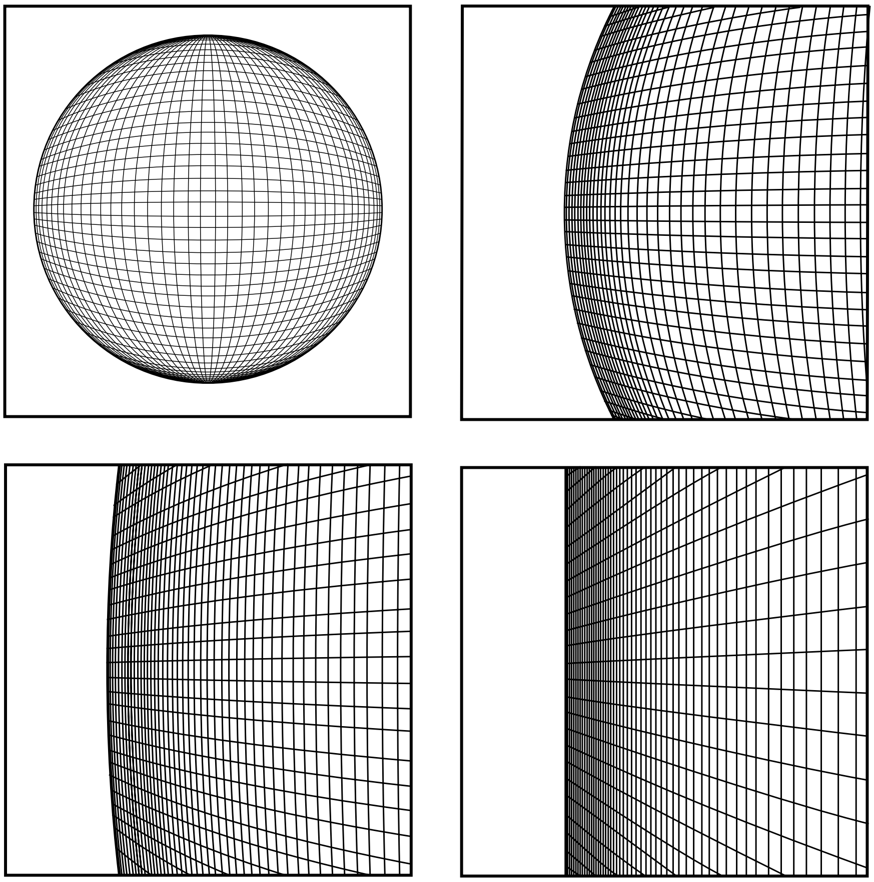

Figure 3: The exponential expansion during inflation made the radius of curvature of the universe so large that our observable patch of the universe today appears essentialy flat, analogous (in three dimensions) to how the surface of a balloon appears flatter and flatter as we inflate it to enormous sizes. This is a crucial prediction of cosmological inflation that will be tested to extraordinary accuracy in the next few years.

in the near future, experiments in particle physics might give us a clue to its nature. Inflation had its original inspiration in the Higgs field, the scalar field responsible for the masses of elementary particles (quarks and leptons) and the breaking of the electroweak symmetry. Its discovery at the large hadron collider at CERN opened the possibility of exploring its effects in the early universe. In particular, if the Higgs has some non-minimal coupling to gravity, it could actually be the inflaton. Such a possibility is for the moment perfectly in agreement with CMB observations by the Planck satellite.

Homogeneous scalar field dynamics

In this subsection I will describe the theoretical basis for the phenomenon of inflation. Consider a scalar field ϕ, a singlet under any given interaction, with an effective potential $V(\phi)$. The Lagrangian for such a field in a curved background is

$$\mathcal{L}_{\rm inf} = \frac{1}{2}\, g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\,, \tag{8}$$

whose evolution equation in a Friedmann-Robertson-Walker metric and for a *homogeneous* field $\phi(t)$ is given by

$$\ddot\phi + 3H\dot\phi + V'(\phi) = 0\,, \tag{9}$$

where H is the rate of expansion, together with the Einstein equations,

$$H^2 \;=\; \frac{\kappa^2}{3}\Big(\frac{1}{2}\,\dot\phi^2 + V(\phi)\Big)\,, \tag{10}$$

$$\dot H \;=\; -\frac{\kappa^2}{2}\,\dot\phi^2\,, \tag{11}$$

where $\kappa^2 \equiv 8\pi G$. The dynamics of inflation can be described as a perfect fluid with a time dependent pressure and energy density given by

$$\rho \;=\; \frac{1}{2}\,\dot\phi^2 + V(\phi)\,, \tag{12}$$

$$p \;=\; \frac{1}{2}\,\dot\phi^2 - V(\phi)\,. \tag{13}$$

The field evolution equation (9) can then be written as the energy conservation equation,

$$\dot\rho + 3H(\rho + p) = 0\,. \tag{14}$$

If the potential energy density of the scalar field dominates the kinetic energy, $V(\phi) \gg \dot\phi^2$, then we see that

$$p \simeq -\rho \quad\Rightarrow\quad \rho \simeq {\rm const.} \quad\Rightarrow\quad H(\phi) \simeq {\rm const.}\,, \tag{15}$$

which leads to the solution

$$a(t) \sim \exp(Ht) \quad \Rightarrow \quad \frac{\ddot{a}}{a} > 0 \quad \text{accelerated expansion}\,. \tag{16}$$

Using the definition of the number of e-folds, $N = \ln(a/a_i)$, we see that the scale factor grows exponentially, $a(N) = a_i \exp(N)$. This solution of the Einstein equations solves immediately the flatness problem. Recall that the problem with the radiation and matter eras is that $\Omega = 1$ ($x = 0$) is an unstable critical point in phase-space. However, during inflation, with $p \simeq -\rho \;\Rightarrow\; \omega \simeq -1$, we have that $1 + 3\omega \geq 0$ and therefore $x = 0$ is a stable *attractor* of the equations of motion, see Eq. (3). As a consequence, what seemed an *ad hoc* initial condition, becomes a natural *prediction* of inflation. Suppose that during inflation the scale factor increased N e-folds, then

$$x_0 = x_{\rm in}\, e^{-2N} \left(\frac{T_{\rm rh}}{T_{\rm eq}}\right)^2 (1 + z_{\rm eq}) \;\simeq\; e^{-2N}\, 10^{56} \leq 1 \quad \Rightarrow \quad N \geq 65\,, \tag{17}$$

where we have assumed that inflation ended at the scale $\rho_{\rm inf}$, and the transfer of the inflaton energy density to thermal radiation at reheating occurred almost instantaneously[2] at the temperature $T_{\rm rh} \sim \rho_{\rm inf}^{1/4} \sim 10^{15}$ GeV. Note that we can now have initial conditions with a large uncertainty, $x_{\rm in} \simeq 1$, and still have today $x_0 \simeq 1$, thanks to the inflationary attractor towards $\Omega = 1$. This can be understood very easily by realizing that the three curvature evolves during inflation as

$${}^{(3)}\!R = \frac{6K}{a^2} = {}^{(3)}\!R_{\rm in}\, e^{-2N} \quad \longrightarrow \quad 0\,, \quad \text{for} \;\; N \gg 1\,. \tag{18}$$

Therefore, if cosmological inflation lasted over 65 e-folds, as most models predict, then today the universe (or at least our local patch) should be exactly flat, a prediction that can be tested with great accuracy in the near future and for which already seems to be some evidence from observations of the microwave background.

[2]There could be a small delay in thermalization, due to the intrinsic inefficiency of reheating, but this does not change significantly the required number of e-folds.

Furthermore, inflation also solves the homogeneity problem in a spectacular way. First of all, due to the superluminal expansion, any inhomogeneity existing prior to inflation will be washed out,

$$\delta_k \sim \left(\frac{k}{aH}\right)^2 \Phi_k \ \propto \ e^{-2N} \quad \longrightarrow \quad 0\,, \quad \text{for} \ \ N \gg 1\,. \tag{19}$$

Moreover, since the scale factor grows exponentially, while the horizon distance remains essentially constant, $d_H(t) \simeq H^{-1} = \text{const.}$, any scale within the horizon during inflation will be stretched by the superluminal expansion to enormous distances, in such a way that at photon decoupling all the causally disconnected regions that encompass our present horizon actually come from a single region during inflation, about 65 e-folds before the end. This is the reason why two points separated more than 1° in the sky have the same backbody temperature, as observed by the COBE satellite: they were actually in causal contact during inflation. There is at present no other proposal known that could solve the homogeneity problem without invoquing an acausal mechanism like inflation.

Finally, any relic particle species (relativistic or not) existing prior to inflation will be diluted by the expansion,

$$\rho_{\rm M} \ \propto \ a^{-3} \ \sim \ e^{-3N} \quad \longrightarrow \quad 0\,, \quad \text{for} \ \ N \gg 1\,, \tag{20}$$

$$\rho_{\rm R} \ \propto \ a^{-4} \ \sim \ e^{-4N} \quad \longrightarrow \quad 0\,, \quad \text{for} \ \ N \gg 1\,. \tag{21}$$

Note that the vacuum energy density ρ_v remains constant under the expansion, and therefore, very soon it is the only energy density remaining to drive the expansion of the universe.

The slow-roll approximation

In order to simplify the evolution equations during inflation, we will consider the slow-roll approximation (SRA). Suppose that, during inflation, the scalar field evolves very slowly down its effective potential, then we can define the slow-roll parameters,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\kappa^2}{2}\frac{\dot{\phi}^2}{H^2} \ll 1\,, \tag{22}$$

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} \ll 1\,, \tag{23}$$

$$\xi \equiv \frac{\dddot{\phi}}{H^2\dot{\phi}} - \delta^2 \ll 1\,. \tag{24}$$

It is easy to see that the condition

$$\epsilon < 1 \quad \Longleftrightarrow \quad \frac{\ddot{a}}{a} > 0 \tag{25}$$

characterizes inflation: it is all you need for superluminal expansion, i.e. for the horizon distance to grow more slowly than the scale factor, in order to solve the homogeneity problem, as well as for the spatial curvature to decay faster than usual, in order to solve the flatness problem.

The number of e-folds during inflation can be written with the help of Eq. (22) as

$$N = \ln\frac{a_{\rm end}}{a_i} = \int_{t_i}^{t_e} H dt = \int_{\phi_i}^{\phi_e} \frac{\kappa d\phi}{\sqrt{2\epsilon(\phi)}}\,, \tag{26}$$

which is an exact expression in terms of $\epsilon(\phi)$.

In the limit given by Eqs. (22), the evolution equations (9) and (10) become

$$H^2\Big(1-\frac{\epsilon}{3}\Big) \simeq H^2 = \frac{\kappa^2}{3}V(\phi)\,, \tag{27}$$

$$3H\dot{\phi}\left(1-\frac{\delta}{3}\right) \simeq 3H\dot{\phi} = -V'(\phi)\,. \tag{28}$$

Note that this corresponds to a reduction of the dimensionality of phase-space from two to one dimensions, $H(\phi,\dot{\phi}) \to H(\phi)$. In fact, it is possible to prove a theorem, for single-field inflation, which states that the slow-roll approximation is an attractor of the equations of motion, and thus we can always evaluate the inflationary trajectory in phase-space within the SRA, therefore reducing the number of initial conditions to just one, the initial value of the scalar field. If $H(\phi)$ only depends on ϕ, then $H'(\phi) = -\kappa^2\dot{\phi}/2$ and we can rewrite the slow-roll parameters as

$$\epsilon = \frac{2}{\kappa^2}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 \simeq \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \equiv \epsilon_V \ll 1\,, \tag{29}$$

$$\delta = \frac{2}{\kappa^2}\frac{H''(\phi)}{H(\phi)} \simeq \frac{1}{\kappa^2}\frac{V''(\phi)}{V(\phi)} - \frac{1}{2\kappa^2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \equiv \eta_V - \epsilon_V \ll 1\,, \tag{30}$$

$$\begin{aligned}\xi &= \frac{4}{\kappa^4}\frac{H'(\phi)H'''(\phi)}{H^2(\phi)} \simeq \frac{1}{\kappa^4}\frac{V'(\phi)V'''(\phi)}{V^2(\phi)} - \frac{3}{2\kappa^4}\frac{V''(\phi)}{V(\phi)}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \\ &+ \frac{3}{4\kappa^4}\left(\frac{V'(\phi)}{V(\phi)}\right)^4 \equiv \xi_V - 3\eta_V\epsilon_V + 3\epsilon_V^2 \ll 1\,.\end{aligned} \tag{31}$$

These expressions define the new slow-roll parameters ϵ_V, η_V and ξ_V. The number of e-folds can also be rewritten in this approximation as

$$N \simeq \int_{\phi_i}^{\phi_e} \frac{\kappa d\phi}{\sqrt{2\epsilon_V(\phi)}} = \kappa^2 \int_{\phi_i}^{\phi_e} \frac{V(\phi)\, d\phi}{V'(\phi)}\,, \tag{32}$$

a very useful expression for evaluating N for a given effective scalar potential $V(\phi)$.

The origin of density perturbations

If cosmological inflation made the universe so extremely flat and homogeneous, where did the galaxies and clusters of galaxies come from? One of the most astonishing predictions of inflation, one that was not even expected, is that quantum fluctuations of the inflaton field are stretched by the exponential expansion and generate large-scale perturbations in the metric. Inflaton fluctuations are small wave packets of energy that, according to general relativity, modify the space-time fabric, creating a whole spectrum of curvature perturbations. The use of the word spectrum here is closely related to the case of light waves propagating in a medium: a spectrum characterizes the amplitude of each given wavelength. In the case of inflation, the inflaton fluctuations induce waves in the space-time metric that can be decomposed into different wavelengths, all with approximately the same amplitude, that is, corresponding to a scale-invariant spectrum. These patterns of perturbations in the metric are like fingerprints that unequivocally characterize a period of inflation. When matter fell in the troughs of these waves, it created density perturbations that collapsed gravitationally to form galaxies, clusters and superclusters of galaxies, with a spectrum that is also scale invariant. Such a type of spectrum was proposed in the early 1970s (before inflation) by Harrison and Zel'dovich, to explain the distribution of galaxies and clusters of galaxies on very large scales in our observable universe. Perhaps the most interesting aspect of structure formation is the possibility that the detailed knowledge of what seeded galaxies and clusters of galaxies will allow us to test the idea of inflation.

Reparametrization invariant perturbation theory

Until now we have considered only the unperturbed FRW metric described by a scale factor $a(t)$ and a homogeneous scalar field $\phi(t)$,

$$ds^2 = a^2(\eta) \left[- d\eta^2 + \gamma_{ij} \, dx^i dx^j \right] , \tag{33}$$

$$\phi = \phi(\eta) \, , \tag{34}$$

where $\eta = \int dt/a(t)$ is the conformal time, under which the background equations of motion can be written as

$$\mathcal{H}^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\phi'^2 + a^2 V(\phi)\right], \tag{35}$$

$$\mathcal{H}' - \mathcal{H}^2 = -\frac{\kappa^2}{2}\phi'^2, \tag{36}$$

$$\phi'' + 2\mathcal{H}\phi' + a^2 V'(\phi) = 0, \tag{37}$$

where $\mathcal{H} = aH$ and $\phi' = a\dot{\phi}$.

During inflation, the quantum fluctuations of the scalar field will induce metric perturbations which will backreact on the scalar field. Let us consider, in linear perturbation theory, the most general line element with both scalar and tensor metric perturbations,[3] together with the scalar field perturbations

$$\begin{aligned} ds^2 &= a^2(\eta)\left[-(1+2A)\,d\eta^2 + 2B_{|i}\,dx^i d\eta \right.\\ &\qquad \left. + \Big\{(1+2\mathcal{R})\gamma_{ij} + 2E_{|ij} + 2h_{ij}\Big\}dx^i dx^j\right], \qquad (38)\\ \phi &= \phi(\eta) + \delta\phi(\eta, x^i). \qquad (39)\end{aligned}$$

The indices $\{i, j\}$ label the three-dimensional spatial coordinates with metric γ_{ij}, and the $|i$ denotes covariant derivative with respect to that metric. The gauge invariant tensor perturbation h_{ij} corresponds to a transverse traceless gravitational wave, $\nabla^i h_{ij} = h^i_i = 0$. The four scalar perturbations $(A, B, \mathcal{R}, E)$ are *gauge dependent* functions of (η, x^i). Under a general coordinate (gauge) transformation

$$\tilde{\eta} = \eta + \xi^0(\eta, x^i), \tag{40}$$

$$\tilde{x}^i = x^i + \gamma^{ij}\xi_{|j}(\eta, x^i), \tag{41}$$

with arbitrary functions (ξ^0, ξ), the scalar and tensor perturbations

[3] Note that inflation cannot generate, to linear order, a vector perturbation.

transform, to linear order, as

$$\tilde{A} = A - \xi^{0\prime} - \mathcal{H}\xi^0 \,, \qquad \tilde{B} = B + \xi^0 - \xi' \,, \tag{42}$$

$$\tilde{\mathcal{R}} = \mathcal{R} - \mathcal{H}\xi^0 \,, \qquad \tilde{E} = E - \xi \,, \tag{43}$$

$$\tilde{h}_{ij} = h_{ij} \,, \tag{44}$$

where a prime denotes derivative with respect to conformal time. It is possible to construct, however, two gauge-invariant gravitational potentials,

$$\Phi = A + (B - E')' + \mathcal{H}(B - E') \,, \tag{45}$$

$$\Psi = \mathcal{R} + \mathcal{H}(B - E') \,, \tag{46}$$

which are related through the perturbed Einstein equations,

$$\Phi \;=\; \Psi \,, \tag{47}$$

$$\frac{k^2 - 3K}{a^2}\Psi \;=\; \frac{\kappa^2}{2}\delta\rho \,, \tag{48}$$

where $\delta\rho$ is the gauge-invariant density perturbation, and the latter expression is nothing but the Poisson equation for the gravitational potential, written in relativistic form.

During inflation, the energy density is given in terms of a scalar field, and thus the gauge-invariant equations for the perturbations on comoving hypersurfaces (constant energy density hypersurfaces) are

$$\Phi'' + 3\mathcal{H}\Phi' + (\mathcal{H}' + 2\mathcal{H}^2)\Phi \;=\; \frac{\kappa^2}{2}[\phi'\delta\phi' - a^2 V'(\phi)\delta\phi] \,, \tag{49}$$

$$-\nabla^2\Phi + 3\mathcal{H}\Phi' + (\mathcal{H}' + 2\mathcal{H}^2)\Phi \;=\; -\frac{\kappa^2}{2}[\phi'\delta\phi' + a^2 V'(\phi)\delta\phi] \,, \tag{50}$$

$$\Phi' + \mathcal{H}\Phi \;=\; \frac{\kappa^2}{2}\phi'\delta\phi \,, \tag{51}$$

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi = 4\phi'\Phi' \;-\; 2a^2 V'(\phi)\Phi - a^2 V''(\phi)\delta\phi \,. \tag{52}$$

This system of equations seem too difficult to solve at first sight. However, there is a gauge invariant combination of variables that allows one to find exact solutions. Let us define

$$u \equiv a\delta\phi + z\Phi\,, \tag{53}$$

$$z \equiv a\frac{\phi'}{\mathcal{H}}\,. \tag{54}$$

Under this redefinition, the above equations simplify enormously to just three independent equations,

$$u'' - \nabla^2 u - \frac{z''}{z}u = 0\,, \tag{55}$$

$$\nabla^2\Phi = \frac{\kappa^2}{2}\frac{\mathcal{H}}{a^2}(zu' - z'u)\,, \tag{56}$$

$$\left(\frac{a^2\Phi}{\mathcal{H}}\right)' = \frac{\kappa^2}{2}zu\,. \tag{57}$$

From Equation (55) we can find a solution $u(z)$, which substituted into (57) can be integrated to give $\Phi(z)$, and together with $u(z)$ allow us to obtain $\delta\phi(z)$.

Quantum Mechanics in curved space-time

Until now we have treated the perturbations as classical, but we should in fact consider the perturbations Φ and $\delta\phi$ as quantum fields. Note that the perturbed action for the scalar mode u can be written as

$$\delta S = \frac{1}{2}\int d^3x\, d\eta\left[(u')^2 - (\nabla u)^2 + \frac{z''}{z}u^2\right]. \tag{58}$$

In order to quantize the field u in the curved background defined by the metric (33), we can write the operator

$$\hat{u}(\eta,\mathbf{x}) = \int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\left[u_k(\eta)\,\hat{a}_{\mathbf{k}}\,e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\eta)\,\hat{a}_{\mathbf{k}}^\dagger\,e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{59}$$

where the creation and annihilation operators satisfy the commutation relation of bosonic fields, and the scalar field's Fock space is defined through the vacuum condition,

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta^3(\mathbf{k}-\mathbf{k}')\,, \tag{60}$$

$$\hat{a}_{\mathbf{k}}|0\rangle = 0\,. \tag{61}$$

Note that we are not assuming that the inflaton is a fundamental scalar field, but that is can be written as a quantum field with its commutation relations (as much as a pion can be described as a quantum field).

The equations of motion for each mode $u_k(\eta)$ are decoupled in linear perturbation theory,

$$u_k'' + \left(k^2 - \frac{z''}{z}\right)u_k = 0\,. \tag{62}$$

The ratio z''/z acts like a time-dependent potential for this Schrödinger like equation. In order to find exact solutions to the mode equation, we will use the slow-roll parameters (22),

$$\epsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = \frac{\kappa^2}{2}\frac{z^2}{a^2}\,, \tag{63}$$

$$\delta = 1 - \frac{\phi''}{\mathcal{H}\phi'} = 1 + \epsilon - \frac{z'}{\mathcal{H}z}\,, \tag{64}$$

$$\xi = -\left(2 - \epsilon - 3\delta + \delta^2 - \frac{\phi'''}{\mathcal{H}^2\phi'}\right)\,. \tag{65}$$

In terms of these parameters, the conformal time and the effective potential for the u_k mode can be written as

$$\eta = \frac{-1}{\mathcal{H}} + \int \frac{\epsilon da}{a\mathcal{H}}\,, \tag{66}$$

$$\frac{z''}{z} = \mathcal{H}^2\left[(1+\epsilon-\delta)(2-\delta) + \mathcal{H}^{-1}(\epsilon'-\delta')\right]\,. \tag{67}$$

Note that the slow-roll parameters, (63) and (64), can be taken as *constant*,[4] to order ϵ^2,

$$\begin{aligned} \epsilon' &= 2\mathcal{H}\left(\epsilon^2 - \epsilon\delta\right) = \mathcal{O}(\epsilon^2)\,, \\ \delta' &= \mathcal{H}\left(\epsilon\delta - \xi\right) = \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{68}$$

In that case, for constant slow-roll parameters, we can write

$$\eta = \frac{-1}{\mathcal{H}}\frac{1}{1-\epsilon}\,, \tag{69}$$

$$\frac{z''}{z} = \frac{1}{\eta^2}\Big(\nu^2 - \frac{1}{4}\Big)\,, \qquad \text{where} \quad \nu = \frac{1+\epsilon-\delta}{1-\epsilon} + \frac{1}{2}\,. \tag{70}$$

We are now going to search for approximate solutions of the mode equation (62), where the effective potential (67) is of order $z''/z \simeq 2\mathcal{H}^2$ in the slow-roll approximation. In quasi-de Sitter there is a characteristic scale given by the (event) horizon size or Hubble scale during inflation, H^{-1}. There will be modes u_k with physical wavelengths much smaller than this scale, $k/a \gg H$, that are well within the de Sitter horizon and therefore do not feel the curvature of space-time. On the other hand, there will be modes with physical wavelengths much greater than the Hubble scale, $k/a \ll H$. In these two asymptotic regimes, the solutions can be written as

$$u_k = \frac{1}{\sqrt{2k}}\,e^{-ik\eta} \qquad k \gg aH\,, \tag{71}$$

$$u_k = C_1\, z \qquad\qquad k \ll aH\,. \tag{72}$$

In the limit $k \gg aH$ the modes behave like ordinary quantum modes in Minkowsky space-time, appropriately normalized, while in the opposite limit, u/z becomes constant on superhorizon scales. For approximately constant slow-roll parameters one can find exact solutions to (62), with the effective potential given by (70), that interpolate between the two

[4]For instance, there are models of inflation, like power-law inflation, $a(t) \sim t^p$, where $\epsilon = \delta = 1/p < 1$, that give constant slow-roll parameters.

asymptotic solutions,

$$u_k(\eta) = \frac{\sqrt{\pi}}{2}\, e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}\, (-\eta)^{1/2}\, H_\nu^{(1)}(-k\eta)\,, \tag{73}$$

where $H_\nu^{(1)}(z)$ is the Hankel function of the first kind, and ν is given by (70) in terms of the slow-roll parameters. In the limit $k\eta \to 0$, the solution becomes

$$|u_k| = \frac{2^{\nu-\frac{3}{2}}}{\sqrt{2k}}\,\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\,(-k\eta)^{\frac{1}{2}-\nu} \equiv \frac{C(\nu)}{\sqrt{2k}}\left(\frac{k}{aH}\right)^{\frac{1}{2}-\nu}, \tag{74}$$

$$C(\nu) = 2^{\nu-\frac{3}{2}}\,\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\,(1-\epsilon)^{\nu-\frac{1}{2}} \;\simeq 1 \qquad \text{for} \quad \epsilon, \delta \ll 1\,. \tag{75}$$

We can now compute Φ and $\delta\phi$ from the super-Hubble-scale mode solution (72), for $k \ll aH$. Substituting into Eq. (57), we find

$$\Phi = C_1\left(1 - \frac{\mathcal{H}}{a^2}\int a^2 d\eta\right) + C_2\,\frac{\mathcal{H}}{a^2}\,, \tag{76}$$

$$\delta\phi = \frac{C_1}{a^2}\int a^2 d\eta - \frac{C_2}{a^2}\,. \tag{77}$$

The term proportional to C_1 corresponds to the growing solution, while that proportional to C_2 corresponds to the decaying solution, which can soon be ignored. These quantities are gauge invariant but evolve with time outside the horizon, during inflation, and before entering again the horizon during the radiation or matter eras. We would like to write an expression for a gauge invariant quantity that is also *constant* for superhorizon modes. Fortunately, in the case of adiabatic perturbations, there is such a quantity:

$$\zeta \equiv \Phi + \frac{1}{\epsilon\mathcal{H}}\,(\Phi' + \mathcal{H}\Phi) = \frac{u}{z}\,, \tag{78}$$

which is constant, see Eq. (72), for $k \ll aH$. In fact, this quantity ζ is identical, for superhorizon modes, to the gauge invariant curvature metric perturbation $\mathcal{R}_c$ on comoving (constant energy density) hypersurfaces,

$$\zeta = \mathcal{R}_c + \frac{1}{\epsilon\mathcal{H}^2}\,\nabla^2\Phi\,. \tag{79}$$

Using Eq. (56) we can write the evolution equation for $\zeta = \frac{u}{z}$ as $\zeta' = \frac{1}{\epsilon\mathcal{H}}\,\nabla^2\Phi$, which confirms that ζ is constant for (adiabatic[5]) superhorizon modes, $k \ll aH$. Therefore, we can evaluate the Newtonian potential Φ_k when the perturbation reenters the horizon during radiation/matter eras in terms of the curvature perturbation $\mathcal{R}_k$ when it left the Hubble scale during inflation,

$$\Phi_k = \left(1 - \frac{\mathcal{H}}{a^2}\int a^2 d\eta\right)\mathcal{R}_k = \frac{3+3\omega}{5+3\omega}\,\mathcal{R}_k = \begin{cases} \frac{2}{3}\,\mathcal{R}_k & \text{radiation era}\,, \\ \frac{3}{5}\,\mathcal{R}_k & \text{matter era}\,. \end{cases} \tag{80}$$

Let us now compute the tensor or gravitational wave metric perturbations generated during inflation. The perturbed action for the tensor mode can be written as

$$\delta S = \frac{1}{2}\int d^3x\,d\eta\,\frac{a^2}{2\kappa^2}\Big[(h'_{ij})^2 - (\nabla h_{ij})^2\Big]\,, \tag{81}$$

with the tensor field h_{ij} considered as a quantum field,

$$\hat{h}_{ij}(\eta,\mathbf{x}) = \int\frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\sum_{\lambda=1,2}\Big[h_k(\eta)\,e_{ij}(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda}\,e^{i\mathbf{k}\cdot\mathbf{x}} + h.c.\Big]\,, \tag{82}$$

where $e_{ij}(\mathbf{k},\lambda)$ are the two polarization tensors, satisfying symmetric, transverse and traceless conditions

$$e_{ij} = e_{ji}\,,\quad k^i e_{ij} = 0\,,\quad e_{ii} = 0\,, \tag{83}$$

$$e_{ij}(-\mathbf{k},\lambda) = e^*_{ij}(\mathbf{k},\lambda)\,,\quad \sum_\lambda e^*_{ij}(\mathbf{k},\lambda)e^{ij}(\mathbf{k},\lambda) = 4\,, \tag{84}$$

while the creation and annihilation operators satisfy the usual commutation relation of bosonic fields, Eq. (60). We can now redefine our

[5] This conservation fails for entropy or isocurvature perturbations

gauge invariant tensor amplitude as

$$v_k(\eta) = \frac{a}{\sqrt{2}\kappa}\, h_k(\eta)\,, \tag{85}$$

which satisfies the following evolution equation, decoupled for each mode $v_k(\eta)$ in linear perturbation theory,

$$v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0\,. \tag{86}$$

The ratio a''/a acts like a time-dependent potential for this Schrödinger like equation, analogous to the term z''/z for the scalar metric perturbation. For constant slow-roll parameters, the potential becomes

$$\frac{a''}{a} = 2\mathcal{H}^2\left(1 - \frac{\epsilon}{2}\right) = \frac{1}{\eta^2}\left(\mu^2 - \frac{1}{4}\right)\,, \tag{87}$$

$$\mu = \frac{1}{1-\epsilon} + \frac{1}{2}\,. \tag{88}$$

We can solve equation (86) in the two asymptotic regimes,

$$v_k = \frac{1}{\sqrt{2k}}\, e^{-ik\eta} \qquad k \gg aH\,, \tag{89}$$

$$v_k = C\, a \qquad\qquad k \ll aH\,. \tag{90}$$

In the limit $k \gg aH$ the modes behave like ordinary quantum modes in Minkowsky space-time, appropriately normalized, while in the opposite limit, the metric perturbation h_k becomes *constant* on superhorizon scales. For constant slow-roll parameters one can find exact solutions to (86), with effective potential given by (87), that interpolate between the two asymptotic solutions. These are identical to Eq. (73) except for the substitution $\nu \to \mu$. In the limit $k\eta \to 0$, the solution becomes

$$|v_k| = \frac{C(\mu)}{\sqrt{2k}} \left(\frac{k}{aH}\right)^{\frac{1}{2}-\mu}\,. \tag{91}$$

Since the mode h_k becomes constant on superhorizon scales, we can evaluate the tensor metric perturbation when it reentered during the radiation or matter era directly in terms of its value during inflation.

Power spectrum of scalar and tensor metric perturbations

Not only do we expect to measure the amplitude of the metric perturbations generated during inflation and responsible for the anisotropies in the CMB and density fluctuations in LSS, but we should also be able to measure its power spectrum, or two-point correlation function in Fourier space. Let us consider first the scalar metric perturbations $\mathcal{R}_k$, which enter the horizon at $a = k/H$. Its correlator is given by

$$\langle 0|\mathcal{R}_k^*\mathcal{R}_{k'}|0\rangle = \frac{|u_k|^2}{z^2}\,\delta^3(\mathbf{k}-\mathbf{k}') \equiv \frac{\mathcal{P}_\mathcal{R}(k)}{4\pi k^3}\,(2\pi)^3\,\delta^3(\mathbf{k}-\mathbf{k}')\,, \quad (92)$$

$$\mathcal{P}_\mathcal{R}(k) = \frac{k^3}{2\pi^2}\,\frac{|u_k|^2}{z^2} = \frac{\kappa^2}{2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \equiv A_S^2\left(\frac{k}{aH}\right)^{n_s-1}, \quad (93)$$

where we have used $\mathcal{R}_k = \zeta_k = \frac{u_k}{z}$ and Eq. (74). This last equation determines the power spectrum in terms of its amplitude at horizon-crossing, A_S, and a tilt,

$$n_s - 1 \equiv \frac{d\ln\mathcal{P}_\mathcal{R}(k)}{d\ln k} = 3 - 2\nu = 2\Big(\frac{\delta-2\epsilon}{1-\epsilon}\Big) \simeq 2\eta_V - 6\epsilon_V\,, \quad (94)$$

see Eqs. (29), (30). Note from this equation that it is possible, in principle, to obtain from inflation a scalar tilt which is either positive ($n > 1$) or negative ($n < 1$). Furthermore, depending on the particular inflationary model, we can have significant departures from scale invariance.

Note that at horizon entry $k\eta = -1$, and thus we can alternatively evaluate the tilt as

$$\begin{aligned} n_s - 1 \equiv -\frac{d\ln\mathcal{P}_\mathcal{R}}{d\ln\eta} &= -2\eta\mathcal{H}\Big[(1-\epsilon)-(\epsilon-\delta)-1\Big] \\ &= 2\Big(\frac{\delta-2\epsilon}{1-\epsilon}\Big) \simeq 2\eta_V - 6\epsilon_V\,, \end{aligned} \quad (95)$$

and the running of the tilt

$$\frac{dn_s}{d\ln k} = -\frac{dn_s}{d\ln\eta} = -\eta\mathcal{H}\left(2\xi + 8\epsilon^2 - 10\epsilon\delta\right) \simeq 2\xi_V + 24\epsilon_V^2 - 16\eta_V\epsilon_V\,, \tag{96}$$

where we have used Eqs. (68).

Let us consider now the tensor (gravitational wave) metric perturbation, which enter the horizon at $a = k/H$,

$$\sum_\lambda \langle 0|h^*_{k,\lambda} h_{k',\lambda}|0\rangle = \frac{8\kappa^2}{a^2}\,|v_k|^2\delta^3(\mathbf{k}-\mathbf{k}') \equiv \frac{\mathcal{P}_g(k)}{4\pi k^3}\,(2\pi)^3\,\delta^3(\mathbf{k}-\mathbf{k}')\,,$$

$$\mathcal{P}_g(k) = 8\kappa^2\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\mu} \equiv A_T^2\left(\frac{k}{aH}\right)^{n_T}, \tag{97}$$

where we have used Eqs. (85) and (91). Therefore, the power spectrum can be approximated by a power-law expression, with amplitude A_T and tilt

$$n_T \equiv \frac{d\ln\mathcal{P}_g(k)}{d\ln k} = 3 - 2\mu = \frac{-2\epsilon}{1-\epsilon} \simeq -2\epsilon_V < 0\,, \tag{98}$$

which is always negative. In the slow-roll approximation, $\epsilon \ll 1$, the tensor power spectrum is scale invariant.

Alternatively, we can evaluate the tensor tilt by

$$n_T \equiv -\frac{d\ln\mathcal{P}_g}{d\ln\eta} = -2\eta\mathcal{H}\Big[(1-\epsilon) - 1\Big] = \frac{-2\epsilon}{1-\epsilon} \simeq -2\epsilon_V\,, \tag{99}$$

and its running by

$$\frac{dn_T}{d\ln k} = -\frac{dn_T}{d\ln\eta} = -\eta\mathcal{H}\left(4\epsilon^2 - 4\epsilon\delta\right) \simeq 8\epsilon_V^2 - 4\eta_V\epsilon_V\,, \tag{100}$$

where we have used Eqs. (68).

The anisotropies of the microwave background

The metric fluctuations generated during inflation are not only responsible for the density perturbations that gave rise to galaxies via gravitational collapse, but one should also expect to see such ripples in the metric as temperature anisotropies in the cosmic microwave background, that is, minute deviations in the temperature of the blackbody spectrum when we look at different directions in the sky. Such anisotropies had been looked for ever since Penzias and Wilson's discovery of the CMB, but had eluded all detection, until COBE satellite discovered them in 1992. The reason why they took so long to be discovered was that they appear as perturbations in temperature of only one part in 10^5. Soon after COBE, other groups quickly confirmed the detection of temperature anisotropies at around $30\,\mu$K, at higher multipole numbers or smaller angular scales.

The Sachs-Wolfe effect

The anisotropies corresponding to large angular scales are only generated via gravitational red-shift and density perturbations through the Einstein equations, $\delta\rho/\rho = -2\Phi$ for adiabatic perturbations; we can ignore the Doppler contribution, since the perturbation is non-causal. In that case, the temperature anisotropy in the sky today is given by

$$\frac{\delta T}{T}(\theta,\phi) = \frac{1}{3}\Phi(\eta_{\rm LS})\,Q(\eta_0,\theta,\phi) + 2\int_{\eta_{\rm LS}}^{\eta_0} dr\,\Phi'(\eta_0 - r)\,Q(r,\theta,\phi)\,, \tag{101}$$

where η_0 is the *coordinate distance* to the last scattering surface, i.e. the present conformal time, while $\eta_{\rm LS} \simeq 0$ determines that comoving hypersurface. The above expression is known as the Sachs-Wolfe effect, and contains two parts, the intrinsic and the Integrated Sachs-Wolfe (ISW) effect, due to integration along the line of sight of time variations in the gravitational potential.

In linear perturbation theory, the scalar metric perturbations can be separated into $\Phi(\eta,\mathbf{x}) \equiv \Phi(\eta)\,Q(\mathbf{x})$, where $Q(\mathbf{x})$ are the scalar harmonics, eigenfunctions of the Laplacian in three dimensions,

$$\nabla^2 Q_{klm}(r,\theta,\phi) = -k^2\, Q_{klm}(r,\theta,\phi)\,.$$

These functions have the general form

$$Q_{klm}(r,\theta,\phi) = \Pi_{kl}(r)\, Y_{lm}(\theta,\phi)\,, \tag{102}$$

where $Y_{lm}(\theta,\phi)$ are the usual spherical harmonics.

In order to compute the temperature anisotropy associated with the Sachs-Wolfe effect, we have to know the evolution of the metric perturbation during the matter era,

$$\Phi'' + 3\mathcal{H}\,\Phi' + a^2\Lambda\,\Phi - 2K\,\Phi = 0\,. \tag{103}$$

In the case of a flat universe without cosmological constant, the Newtonian potential remains constant during the matter era and only the intrinsic SW effect contributes to $\delta T/T$. In case of a non-vanishing Λ, since its contribution is negligible in the past, most of the photon's trajectory towards us is unperturbed, and the only difference with respect to the $\Lambda = 0$ case is an overall factor. We will consider here the approximation $\Phi =$ constant during the matter era and ignore that factor.

In a flat universe, the radial part of the eigenfunctions (102) can be written as

$$\Pi_{kl}(r) = \sqrt{\frac{2}{\pi}}\, k\, j_l(kr)\,, \tag{104}$$

where $j_l(z)$ are the spherical Bessel functions. The growing mode solution of the metric perturbation that left the Hubble scale during inflation contributes to the temperature anisotropies on large scales (101) as

$$\frac{\delta T}{T}(\theta,\phi) \;=\; \frac{1}{3}\Phi(\eta_{\rm LS})\,Q \;=\; \frac{1}{5}\,\mathcal{R}\,Q(\eta_0,\theta,\phi) \;\equiv\; \sum_{l=2}^{\infty}\sum_{m=-l}^{l} a_{lm}\,Y_{lm}(\theta,\phi)\,, \tag{105}$$

where we have used the fact that at reentry (at the surface of last scattering) the gauge invariant Newtonian potential Φ is related to the curvature perturbation $\mathcal{R}$ at Hubble-crossing during inflation, see Eq. (80); and we have expanded $\delta T/T$ in spherical harmonics.

We can now compute the two-point correlation function or angular power spectrum, $C(\theta)$, of the CMB anisotropies on large scales, defined as an expansion in multipole number,

$$C(\theta) = \left\langle \frac{\delta T^*}{T}(\mathbf{n}) \frac{\delta T}{T}(\mathbf{n}') \right\rangle_{\mathbf{n}\cdot\mathbf{n}'=\cos\theta} = \frac{1}{4\pi}\sum_{l=2}^{\infty}(2l+1)\, C_l\, P_l(\cos\theta)\,, \tag{106}$$

where $P_l(z)$ are the Legendre polynomials, and we have averaged over different universe realizations. Since the coefficients a_{lm} are isotropic (to first order), we can compute the $C_l = \langle |a_{lm}|^2\rangle$ as

$$C_l^{(S)} = \frac{4\pi}{25}\int_0^\infty \frac{dk}{k}\,\mathcal{P}_{\mathcal{R}}(k)\, j_l^2(k\eta_0)\,, \tag{107}$$

where we have used Eqs. (105) and (92). In the case of scalar metric perturbation produced during inflation, the scalar power spectrum at reentry is given by $\mathcal{P}_{\mathcal{R}}(k) = A_S^2 (k\eta_0)^{n-1}$, in the power-law approximation, see Eq. (93). In that case, one can integrate (107) to give

$$C_l^{(S)} = \frac{2\pi}{25}\, A_S^2\, \frac{\Gamma[\frac{3}{2}]\,\Gamma[1-\frac{n-1}{2}]\,\Gamma[l+\frac{n-1}{2}]}{\Gamma[\frac{3}{2}-\frac{n-1}{2}]\,\Gamma[l+2-\frac{n-1}{2}]}\,, \tag{108}$$

$$\frac{l(l+1)\, C_l^{(S)}}{2\pi} = \frac{A_S^2}{25} = \text{constant}\,, \qquad \text{for } n=1\,. \tag{109}$$

This last expression corresponds to what is known as the Sachs-Wolfe plateau, and is the reason why the coefficients C_l are always plotted multiplied by $l(l+1)$.

Tensor metric perturbations also contribute with an approximately constant angular power spectrum, $l(l+1)C_l$. The Sachs-Wolfe effect for a gauge invariant tensor perturbation is given by

$$\frac{\delta T}{T}(\theta,\phi) = \int_{\eta_{\rm LS}}^{\eta_0} dr\, h'(\eta_0 - r)\, Q_{rr}(r,\theta,\phi)\,, \tag{110}$$

where Q_{rr} is the rr-component of the tensor harmonic along the line of sight. The tensor perturbation h during the matter era satisfies the following evolution equation

$$h_k'' + 3\mathcal{H}\, h_k' + (k^2 + 2K)\, h_k = 0\,, \tag{111}$$

which depends on the wavenumber k, contrary to what happens with the scalar modes, see Eq. (103). For a flat ($K = 0$) universe, the solution to this equation is $h_k(\eta) = h\, G_k(\eta)$, where h is the constant tensor metric perturbation at horizon crossing and $G_k(\eta) = 3\, j_1(k\eta)/k\eta$, normalized so that $G_k(0) = 1$ at the surface of last scattering. The radial part of the tensor harmonic Q_{rr} in a flat universe can be written as

$$Q_{kl}^{rr}(r) = \left[\frac{(l-1)l(l+1)(l+2)}{\pi k^2}\right]^{1/2} \frac{j_l(kr)}{r^2}\,. \tag{112}$$

The tensor angular power spectrum can finally be expressed as

$$C_l^{(T)} = \frac{9\pi}{4}\,(l-1)l(l+1)(l+2)\int_0^\infty \frac{dk}{k}\,\mathcal{P}_g(k)\, I_{kl}^2\,, \tag{113}$$

$$I_{kl} = \int_0^{x_0} dx\, \frac{j_2(x_0 - x) j_l(x)}{(x_0 - x)x^2}\,, \tag{114}$$

where $x \equiv k\eta$, and $\mathcal{P}_g(k)$ is the primordial tensor spectrum (97). For a scale invariant spectrum, $n_T = 0$, we can integrate (113) to give

$$l(l+1)\, C_l^{(T)} = \frac{\pi}{36}\Big(1 + \frac{48\pi^2}{385}\Big)\, A_T^2\, B_l\,, \tag{115}$$

with $B_l = (1.1184, 0.8789, \ldots, 1.00)$ for $l = 2, 3, \ldots, 30$. Therefore, $l(l+1)\, C_l^{(T)}$ also becomes constant for large l. Beyond $l \sim 30$, the Sachs-Wolfe expression is not a good approximation and the tensor angular power spectrum decays very quickly at large l.

The consistency relation

In spite of the success of inflation in predicting a homogeneous and isotropic background on which to imprint a scale-invariant spectrum of inhomogeneities, it is difficult to test the idea of inflation. A CMB cosmologist before the 1980s would have argued that *ad hoc* initial conditions could have been at the origin of the homogeneity and flatness of the universe on large scales, while a LSS cosmologist would have agreed with Harrison and Zel'dovich that the most natural spectrum needed to explain the formation of structure was a scale-invariant spectrum. The surprise was that inflation incorporated an understanding of *both* the globally homogeneous and spatially flat background, and the approximately scale-invariant spectrum of perturbations in the same formalism. But that could have been just a coincidence.

What is *unique* to inflation is the fact that inflation determines not just one but *two* primordial spectra, corresponding to the scalar (density) and tensor (gravitational waves) metric perturbations, from a single continuous function, the inflaton potential $V(\phi)$. In the slow-roll approximation, one determines, from $V(\phi)$, two continuous functions, $\mathcal{P}_{\mathcal{R}}(k)$ and $\mathcal{P}_g(k)$, that in the power-law approximation reduces to two amplitudes, A_S and A_T, and two tilts, n and n_T. It is clear that there must be a relation between the four parameters. Indeed, one can see from Eqs. (115) and (109) that the ratio of the tensor to scalar contribution to the angular power spectrum is proportional to the tensor tilt,

$$r \equiv \frac{A_T^2}{A_S^2} = 16\epsilon \simeq -8\, n_T\,. \tag{116}$$

This is a unique prediction of inflation, which could not have been postulated a priori by any cosmologist. If we finally observe a tensor spectrum of anisotropies in the CMB, or a stochastic gravitational wave background in laser interferometers like LIGO or LISA, with sufficient accuracy to determine their spectral tilt, one might have some chance to test the idea of inflation, via the consistency relation (116). For the moment, observations of the microwave background anisotropies suggest that the Sachs-Wolfe plateau exists, see but it is still premature to determine the tensor contribution. Perhaps in the near future, from the analysis of CMB polarization as well as temperature anisotropies, with the COrE+ or Litebird satellites, we might have a chance of determining the validity of the consistency relation.

Assuming that the scalar contribution dominates over the tensor on large scales, i.e. $r \ll 1$, one can actually give a measure of the amplitude of the scalar metric perturbation from the observations of the Sachs-Wolfe plateau in the angular power spectrum,

$$\left[\frac{l(l+1)\,C_l^{(S)}}{2\pi}\right]^{1/2} = \frac{A_S}{5} = (0.926 \pm 0.0106) \times 10^{-5}\,, \quad (117)$$

$$n = 0.9667 \pm 0.0040\,, \quad (118)$$

$$\frac{dn}{d\ln k} = -0.002 \pm 0.013\,. \quad (119)$$

These measurements can be used to normalize the primordial spectrum and determine the parameters of the model of inflation. In the near future these parameters will be determined with much better accuracy.

The acoustic peaks

The Sachs-Wolfe plateau is a distinctive feature of Fig. 24. These observations confirm the existence of a primordial spectrum of scalar (density) perturbations on all scales, otherwise the power spectrum would have started from zero at $l = 2$. However, we see that the spectrum starts to rise around $l = 20$ towards the first acoustic peak, where the SW approximation breaks down and the above formulae are no longer valid.

As mentioned above, the first peak in the photon distribution corresponds to overdensities that have undergone half an oscillation, that is, a compression, and appear at a scale associated with the size of the horizon at last scattering, about 1° projected in the sky today. Since photons scatter off baryons, they will also feel the acoustic wave and create a peak in the correlation function. The height of the peak is proportional to the amount of baryons: the larger the baryon content of the universe, the higher the peak. The position of the peak in the power spectrum depends on the geometrical size of the particle horizon at last scattering. Since photons travel along geodesics, the projected size of the causal horizon at decoupling depends on whether the universe is flat, open or closed. In a flat universe the geodesics are straight lines and, by looking at the angular scale of the first acoustic peak, we would be measuring the actual size of the horizon at last scattering. In an open universe, the geodesics are inward-curved trajectories, and therefore the projected size on the sky appears smaller. In this case, the first acoustic peak should occur at higher multipoles or smaller angular scales. On the other hand, for a closed universe, the first peak occurs at smaller multipoles or larger angular scales. The dependence of the position of the first acoustic peak on the spatial curvature can be approximately given by $l_{\rm peak} \simeq 220\,\Omega_0^{-1/2}$, where $\Omega_0 = \Omega_{\rm M}+\Omega_\Lambda = 1-\Omega_K$. Past observations from the balloon experiment BOOMERANG, suggested clearly a few years ago that the first peak was between $l = 180$ and 250 at 95% c.l., with an amplitude $\delta T = 80 \pm 10\ \mu$K, and therefore the universe was most probably flat. However, with the high precision Planck data we can now pinpoint the spatial curvature to less than a tenth of a percent,

$$\Omega_0 = 0.9992 \pm 0.0040 \quad (95\%\ \text{c.l.}) \tag{120}$$

Therefore, the universe is spatially flat (i.e. Euclidean), within 0.1%

uncertainty, which is much better than we could ever do before, and is one the most robust predictions of inflation.

With WMAP and specially with Planck, we have now evidence of at least nine distinct acoustic peaks. These peaks should occur at harmonics of the first one, but are typically much lower because of Silk damping. Since the amplitude and position of the primary and secondary peaks are directly determined by the sound speed (and, hence, the equation of state) and by the geometry and expansion of the universe, they can be used as a powerful test of the density of baryons and dark matter, and other cosmological parameters.

By looking at these patterns in the anisotropies of the microwave background, cosmologists can determine not only the cosmological parameters, but also the primordial spectrum of density perturbations produced during inflation. It turns out that the observed temperature anisotropies are compatible with a scale-invariant spectrum, see Eq. (118), as predicted by inflation. This is remarkable, and gives very strong support to the idea that inflation may indeed be responsible for both the CMB anisotropies and the large-scale structure of the universe. Different models of inflation have different specific predictions for the fine details associated with the spectrum generated during inflation. It is these minute differences that will allow cosmologists to differentiate between alternative models of inflation and discard those that do not agree with observations. However, most importantly, perhaps, the pattern of anisotropies predicted by inflation is completely different from those predicted by alternative models of structure formation, like cosmic defects: strings, vortices, textures, etc. These are complicated networks of energy density concentrations left over from an early universe phase transition, analogous to the defects formed in the laboratory in certain kinds of liquid crystals when they go through a phase transition. The cosmological defects have spectral properties very different from those generated by inflation. That is why it is so important to launch more sensitive instruments, and with better angular resolution, to determine the properties of the CMB anisotropies.

ARNOWITT-DESER-MISNER FORMALISM

So far we have given a broad brush account of inflationary models and issues like initial conditions for inflation and the large scale structure of the Universe cannot be described without a complete mathematical formalism. In particular, the character of the slow-roll attractor that I will discuss in the next section cannot be understood without the ADM formalism.

The level of this part of the lectures goes beyond that of the main course, and is intended here only as a guide for those students more mathematically inclined that feel that the interplay between gravitation and inflation in the early universe described in the previous sections was too naive. This section will therefore not be compulsory for passing.

Those students that feel they have enough material already may want to go on with the notes until the section on "Inflationary Model Building".

The Arnowitt–Deser–Misner formalism gives a (3+1)-splitting of space-time, a foliation in which the four dimensional metric $g_{\mu\nu}$ is parametrized by the three-metric h_{ij} and the lapse and shift functions, N and N^i, which describe the evolution of time-like hypersurfaces, with proper interval ds, between $x^\alpha = (t, x^i)$ and $x^\alpha + dx^\alpha = (t+dt, x^i+dx^i)$, given by $ds^2 = -(Ndt)^2 + h_{ij}(dx^i+N^i dt)(dx^j+N^j dt)$. The components of the metric thus become

$$g_{00} = -N^2 + h^{ij}N_iN_j, \quad g_{0i} = g_{i0} = N_i, \quad g_{ij} = h_{ij}, \tag{121}$$

and inverse metric

$$g^{00} = -N^{-2}, \quad g^{0i} = g^{i0} = N^{-2}N^i, \quad g^{ij} = h^{ij} - N^{-2}N^iN^j, \tag{122}$$

where the 3-metric is used to raise and lower spatial indices, $N^i = h^{ij}N_j$, with $h^{ik}h_{kj} = \delta^i{}_j$. This splitting corresponds to a 3-hypersurface Σ and a timelike unit vector normal to it, with components

$$n_\alpha = (-N, 0)\,, \quad n^\alpha = (N^{-1}, -N^{-1}N^i)\,,$$

satisfying $n_\alpha n^\alpha = -1$. We can then define an intrinsic curvature to the 3-surface, ${}^{(3)}R_{ij}$, written in terms of the 3-metric h_{ij}, as well as an extrinsic curvature, related to the normal vector,

$$K_{ij} = -n_{i|j} = -N\Gamma^0_{ij} = \frac{1}{2N}\Big(2N_{(i|j)} - \partial_0 h_{ij}\Big)\,, \tag{123}$$

where bars denote 3-space covariant derivatives with connections derived from h_{ij}, and subindices in parenthesis denote symmetrization, $2A_{(ij)} = A_{ij} + A_{ji}$, while brakets denote antisymmetrization, $2A_{[ij]} = A_{ij} - A_{ji}$. The traceless part of a tensor is denoted by an overbar. In particular, the trace and traceless parts of the extrinsic curvature are

$$\bar{K}_{ij} = K_{ij} - \frac{1}{3}Kh_{ij}, \qquad K = K_i{}^i = \frac{1}{N}\Big[N_{i|i} - \partial_0 \ln\sqrt{h}\Big]\,. \tag{124}$$

The trace K is a generalization of the Hubble parameter, as will be shown below.

Instead of the coordinate basis $(\mathbf{e}_0 = \partial_0\,, \mathbf{e}_i = \partial_i)$, with 1-forms $(\mathbf{d}t\,, \mathbf{d}x^i)$, we will use a basis with the normal 3-vector $\mathbf{n}$ instead of the time vector,

$$
\begin{array}{lll}
\underline{\text{vectors}} & \underline{1-\text{forms}} & \\
\mathbf{e}_n = \dfrac{1}{N}(\partial_0 - N^i\partial_i) & \mathbf{w}^n = (\mathbf{n}\cdot\mathbf{n})\mathbf{n} = N\mathbf{d}t & (125)\\
\mathbf{e}_i = \partial_i & \mathbf{w}^i = \mathbf{d}x^i + N^i\mathbf{d}t\,. & (126)
\end{array}
$$

In this case, for instance, the kinetic term of a scalar field is written as

$$
-\partial_\mu\phi\partial^\mu\phi = (\Pi^\phi)^2 - (\partial_i\phi)^2\,, \tag{127}
$$

where Π^ϕ is the scalar-field's conjugate momentum

$$
\Pi^\phi = \frac{1}{N}(\dot\phi - N^i\phi_{|i})\,. \tag{128}
$$

The gravitational Lagrangian can be written as

$$
\mathcal{L}_G = \sqrt{-g}R = N\sqrt{h}\,\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right) \tag{129}
$$

from which we can obtain the conjugate momentum of the metric

$$
\Pi^{ij} = \frac{\partial\mathcal{L}_G}{\partial\dot h_{ij}} = -\sqrt{h}(K^{ij} - Kh^{ij}), \tag{130}
$$

with trace and traceless parts given by

$$
\Pi = 2\sqrt{h}K\,, \qquad \bar\Pi^{ij} = -\sqrt{h}\,\bar K^{ij}\,. \tag{131}
$$

After some algebra it can be shown that the gravitational Lagrangian (129) can be written as

$$
\begin{aligned}
\mathcal{L}_G &= N\sqrt{h}{}^{(3)}R + \frac{N}{\sqrt{h}}\Big(\Pi_{ij}\Pi^{ij} - \frac{1}{2}\Pi^2\Big)\\
&= \Pi^{ij}\dot h_{ij} - N\mathcal{H} - N_i\mathcal{H}^i - 2\nabla_i(\Pi^{ij}N_j)\,,
\end{aligned} \tag{132}
$$

where the lapse and shift functions appear as Lagrange multipliers, and

$$
\mathcal{H}(h_{ij},\,\Pi^{ij}) = -\sqrt{h}{}^{(3)}R + \frac{1}{\sqrt{h}}\Big(\Pi_{ij}\Pi^{ij} - \frac{1}{2}\Pi^2\Big)\,, \tag{133}
$$

$$
\mathcal{H}^i(h_{ij},\,\Pi^{ij}) = -2\Pi^{ij}{}_{|j}\,, \tag{134}
$$

The gravitational Hamiltonian then becomes

$$\mathcal{H}_G = \Pi^{ij}\dot{h}_{ij} - \mathcal{L}_G = -N\sqrt{h}{}^{(3)}R + \frac{N}{\sqrt{h}}\Big(\Pi_{ij}\Pi^{ij} - \frac{1}{2}\Pi^2\Big) + 2\Pi^{ij}N_{(i|j)}\,. \tag{135}$$

and the Hamiltonian and momentum constraints,

$$\frac{\delta\mathcal{H}_G}{\delta N} = \mathcal{H} = 0\,, \tag{136}$$

$$\frac{\delta\mathcal{H}_G}{\delta N_i} = \mathcal{H}^i = 0\,. \tag{137}$$

While the Hamiltonian evolution equations for the independent variables h_{ij} and Π^{ij} can be written as

$$\dot{h}_{ij} = \frac{\delta\mathcal{H}_G}{\delta\Pi^{ij}} = -2N\,K_{ij} + 2N_{(i|j)}\,, \tag{138}$$

$$\begin{aligned}\dot{\Pi}^{ij} = -\frac{\delta\mathcal{H}_G}{\delta h_{ij}} &= -N\sqrt{h}\Big({}^{(3)}R^{ij} - \frac{1}{2}{}^{(3)}R\,h^{ij}\Big) \\ &\quad + \frac{N}{2\sqrt{h}}h^{ij}\Big(\Pi_{kl}\Pi^{kl} - \frac{1}{2}\Pi^2\Big) \\ &\quad - \frac{2N}{\sqrt{h}}\Big(\Pi^{ik}\Pi_k^{\ j} - \frac{1}{2}\Pi\,\Pi^{ij}\Big) + \sqrt{h}\Big(N^{|ij} - h^{ij}\,N^{|k}_{\ \ |k}\Big) \\ &\quad + \Big(N^k\Pi^{ij}\Big)_{|k} - 2\Pi^{k(i}N^{j)}_{\ \ |k}\,.\end{aligned} \tag{139}$$

With these equations we can evaluate the derivative of the trace Π,

$$\begin{aligned}\dot{\Pi} = \dot{\Pi}^{ij}\,h_{ij} + \Pi^{ij}\,\dot{h}_{ij} \;&=\; \frac{1}{2}N\sqrt{h}\,{}^{(3)}R + \frac{3}{2}N\sqrt{h}(K_{ij}K^{ij} - K^2) \\ &\quad -2\sqrt{h}N^{|i}_{\ \ |i} + 2\sqrt{h}(K\,N^i)_{|i}\end{aligned} \tag{140}$$

while from (131) we have

$$\dot{\Pi} = 2\sqrt{h}\dot{K} + \sqrt{h}K\,h^{ij}\dot{h}_{ij} = 2\sqrt{h}\dot{K} - 2N\sqrt{h}K^2 + 2\sqrt{h}K\,N^i_{\ |i}\,,$$

and therefore the derivate of the trace of the extrinsic curvature is

$$\dot{K} - N^i\, K_{|i} = -N^{|i}_{\ \ |i} + N\Big(\frac{1}{4}{}^{(3)}\!R + \frac{3}{4}\bar{K}_{ij}\bar{K}^{ij} + \frac{1}{2}K^2\Big)\,. \tag{141}$$

Now we can evaluate the derivative of the traceless part $\bar{\Pi}^{ij}$. Using the identity

$$\frac{2N}{\sqrt{h}}\Big(\Pi^{ik}\Pi_k^{\ j} - \frac{1}{2}\Pi\,\Pi^{ij}\Big) = 2N\sqrt{h}\Big(\bar{K}^{ik}\bar{K}_k^{\ j} - \frac{1}{3}K\,\bar{K}^{ij} - \frac{2}{9}K^2h^{ij}\Big)\,,$$

and

$$2N\bar{K}^{ik}\bar{K}_k^{\ j} = \bar{K}^{ik}\Big(N^j_{\ |k} + N_k^{\ |j}\Big) - \Big(\frac{2}{3}NK\,\bar{K}^{ij} + \bar{K}^{ik}h_{kl}\dot{h}^{lj}\Big)\,,$$

we have, after some algebra,

$$\begin{aligned}\dot{\bar{\Pi}}^{ij} \;&=\; \dot{\Pi}^{ij} - \frac{1}{3}\dot{\Pi}\,h^{ij} - \frac{1}{3}\Pi\,\dot{h}^{ij} = -N\sqrt{h}\,{}^{(3)}\!\bar{R}^{ij} - 2N\sqrt{h}\bar{K}^{ik}\bar{K}_k^{\ j} \\ &\quad -\frac{2}{3}NK\,\bar{K}^{ij} + \sqrt{h}\Big(N^{|ij} - \frac{1}{3}N^{|k}_{\ \ |k}\,h^{ij}\Big) \\ &\quad -\sqrt{h}N^k\,\bar{K}^{ij}_{\ \ |k} - \sqrt{h}\bar{K}^{ij}N^k_{\ |k} + 2\sqrt{h}\bar{K}^{k(i}N^{j)}_{\ \ |k}\,. \end{aligned} \tag{142}$$

Also, from $\bar{\Pi}^{ij} = -\sqrt{h}\,\bar{K}^{ij}$, we obtain

$$\dot{\bar{\Pi}}^{ij} = -\sqrt{h}\,\dot{\bar{K}}^{ij} + N\sqrt{h}K\,\bar{K}^{ij} - \sqrt{h}\bar{K}^{ij}\,N^k_{\ |k} \tag{143}$$

and therefore, comparing the two expressions, we deduce

$$\begin{aligned}&\dot{\bar{K}}^i_{\ j} - N^k\bar{K}^i_{\ j|k} + N^i_{\ |k}\bar{K}^k_{\ j} - N^k_{\ |j}\bar{K}^i_{\ k} = \\ &\qquad = -N^{|i}_{\ \ |j} + \frac{1}{3}N^{|k}_{\ \ |k}\,\delta^i_{\ j} + N\Big({}^{(3)}\!\bar{R}^i_{\ j} + K\,\bar{K}^i_{\ j}\Big)\,. \end{aligned} \tag{144}$$

Let us consider now the matter content and write the gravitational action for a scalar field with potential $V(\phi)$ in the ADM formalism as

$$\begin{aligned}\mathcal{S} &= \int d^4x\sqrt{-g}\Big[\frac{1}{2\kappa^2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\Big] \\ &= \int d^4x N\sqrt{h}\Big[\frac{1}{2\kappa^2}\Big({}^{(3)}\!R + \bar{K}_{ij}\bar{K}^{ij} - \frac{2}{3}K^2\Big) + \\ &\qquad + \frac{1}{2}\Big[(\Pi^\phi)^2 - \phi_{|i}\phi^{|i}\Big] - V(\phi)\Big]\,, \end{aligned} \tag{145}$$

Variation of the action with respect to N and N^i yields the energy and momentum constraint equations respectively

$$-{}^{(3)}R + \bar{K}_{ij}\bar{K}^{ij} - \frac{2}{3}K^2 + 2\kappa^2\, T_{00} = 0\,, \tag{146}$$

$$\bar{K}^{j}{}_{i|j} - \frac{2}{3}K_{|i} + \kappa^2\, T^0_{\ i} = 0\,. \tag{147}$$

Variation with respect to h_{ij} gives the dynamical gravitational field equations, which can be separated into the trace and traceless parts

$$\dot{K} - N^i K_{|i} = -\, N^{|i}_{\ |i} + N\left(\frac{1}{4}{}^{(3)}R + \frac{3}{4}\bar{K}_{ij}\bar{K}^{ij} + \frac{1}{2}K^2 + \frac{\kappa^2}{2}\,T\right)\,, \tag{148}$$

$$\begin{aligned}\dot{\bar{K}}^{i}{}_{j} - N^k\bar{K}^{i}{}_{j|k} + N^{i}{}_{|k}\bar{K}^{k}{}_{j} - N^{k}{}_{|j}\bar{K}^{i}{}_{k} &= \\ = -N^{|i}{}_{|j} + \frac{1}{3}N^{|k}{}_{|k}\,\delta^{i}{}_{j} + N\Big({}^{(3)}\bar{R}^{i}{}_{j} + K\,\bar{K}^{i}{}_{j} - \kappa^2\bar{T}^{i}{}_{j}\Big)\,.\end{aligned} \tag{149}$$

The matter energy-momentum tensor is

$$T_{00} = \frac{1}{2}(\Pi^\phi)^2 + \frac{1}{2}\phi_{|k}\phi^{|k} + V(\phi)\,, \tag{150}$$

$$T^0_{\ i} = \Pi^\phi\phi_{|i}\,, \tag{151}$$

$$T^{i}{}_{j} = \phi^{|i}\phi_{|j} + \delta^{i}{}_{j}\left[\frac{1}{2}(\Pi^\phi)^2 - \frac{1}{2}\phi_{|k}\phi^{|k} - V(\phi)\right]\,, \tag{152}$$

$$\bar{T}^{i}{}_{j} = \phi^{|i}\phi_{|j} - \frac{1}{3}\phi_{|k}\phi^{|k}\delta^{i}{}_{j}\,, \tag{153}$$

$$T = \frac{3}{2}(\Pi^\phi)^2 - \frac{1}{2}\phi_{|k}\phi^{|k} - 3V(\phi) \tag{154}$$

Variation with respect to ϕ gives the scalar-field's equation of motion

$$\frac{1}{N}(\dot{\Pi}^\phi - N^i\Pi^\phi_{|i}) - \ K\Pi^\phi - \frac{1}{N}N_{|i}\phi^{|i} - \phi_{|i}^{\ |i} + \frac{\partial V}{\partial\phi} = 0\,. \tag{155}$$

It is extremely difficult to solve these highly nonlinear coupled equations in a cosmological scenario without making some approximations.

The usual approach is to assume homogeneity of the fields to give a background solution and then linearize the equations to study deviations from spatial uniformity. The smallness of cosmic microwave background anisotropies gives some justification for this perturbative approach at least in our local part of the Universe. However, there is no reason to believe it will be valid on much larger scales. In fact, the stochastic approach to inflation suggests that the Universe is extremely inhomogeneous on very large scales. Fortunately, in this framework one can coarse-grain over a horizon distance and separate the short- from the long-distance behavior of the fields, where the former communicates with the latter through stochastic forces. The equations for the long-wavelength background fields are obtained by neglecting large-scale gradients, leading to a self-consistent set of equations, as we will discuss in the next section.

Spatial gradient expansion

It is reasonable to expand in spatial gradients whenever the forces arising from time variations of the fields are much larger than forces from spatial gradients. In linear perturbation theory one solves the perturbation equations for evolution outside of the horizon: a typical time scale is the Hubble time H^{-1}, which is assumed to exceed the gradient scale a/k, where k is the comoving wave number of the perturbation. Since we are interested in structures on scales larger than the horizon, it is reasonable to expand in $k/(aH)$. In particular, for inflation this is an appropriate parameter of expansion since spatial gradients become exponentially negligible after a few e-folds of expansion beyond horizon crossing, $k = aH$.

It is therefore useful to split the field ϕ into coarse-grained long-wavelength background fields $\phi(t, x^j)$ and residual short-wavelength fluctuating fields $\delta\phi(t, x^j)$. There is a preferred timelike hypersurface within the stochastic inflation approach in which the splitting can be made consistently, but the definition of the background field will depend on the choice of hypersurface, *i.e.* the smoothing is not gauge invariant. For stochastic inflation the natural smoothing scale is the comoving Hubble length $(aH)^{-1}$ and the natural hypersurfaces are those on which aH is constant. In that case a fundamental difference between between ϕ and $\delta\phi$ is that the short-wavelength components are essentially uncorrelated at different times, while long-wavelength components are deterministically correlated through the equations of motion.

In order to solve the equations for the background fields, we will have to make suitable approximations. The idea is to expand in the spatial gradients of ϕ and to treat the terms that depend on the fluctuating fields as stochastic forces describing the connection between short- and long-wavelength components. In this Section we will neglect the stochastic forces due to quantum fluctuations of the scalar fields and will derive the approximate equation of motion for the background fields. We retain only those terms that are at most first order in spatial gradients, neglecting such terms as $\phi_{|i}^{\ |i}$, $\phi_{|i}\phi^{|i}$, ${}^{(3)}R$, ${}^{(3)}\bar{R}^i_{\ j}$, and $\bar{T}^i_{\ j}$.

We will also choose the simplifying gauge $N^i = 0$ [Note that for the evolution during inflation this is a consequence of the rapid expansion, more than a gauge choice]. The evolution equation (149) for the traceless part of the extrinsic curvature is then $\dot{\bar{K}}^i_{\ j} = NK\bar{K}^i_{\ j}$. Using $NK = -\partial_t \ln\sqrt{h}$ from (124), we find the solution $\bar{K}^i_{\ j} \propto h^{-1/2}$, where h is the determinant of h_{ij}. During inflation $h^{-1/2} \equiv a^{-3}$, with a the overall expansion factor, therefore $\bar{K}^i_{\ j}$ decays extremely rapidly and can be set to zero in the approximate equations. The most general form of the three-metric with vanishing $\bar{K}^i_{\ j}$ is

$$h_{ij} = a^2(t,x^k)\,\gamma_{ij}(x^k), \quad a(t,x^k) \equiv e^{\alpha(t,x^k)}, \tag{156}$$

where the time-dependent conformal factor is interpreted as a space-dependent expansion factor. The time-independent three-metric γ_{ij}, of unit determinant, describes the three-geometry of the conformally transformed space. Since $a(t,x^k)$ is interpreted as a scale factor, we can substitute the trace K of the extrinsic curvature for the local space-dependent Hubble parameter

$$H(t,x^i) \equiv \frac{1}{N(t,x^i)}\,\dot{\alpha}(t,x^i) = -\frac{1}{3}K(t,x^i). \tag{157}$$

The energy and momentum constraint equations, (146) and (147), can now be written as

Hamiltonian and momentum constraint equations

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}(\Pi^\phi)^2 + V(\phi)\right], \tag{158}$$

$$H_{|i} = -\frac{\kappa^2}{2}\Pi^\phi \phi_{|i}, \tag{159}$$

together with the evolution equation (148)

$$-\frac{1}{N}\dot{H} = \frac{3}{2}H^2 + \frac{\kappa^2}{6}T = \frac{\kappa^2}{2}(\Pi^\phi)^2, \tag{160}$$

Evolution equations

$$\dot{H} = -\frac{\kappa^2}{2}(\Pi^\phi)^2, \tag{161}$$

where $T = \frac{3}{2}(\Pi^\phi)^2 - 3V(\phi)$.

In general, H is a function of the scalar field and time, $H(t, x^i) \equiv H(\phi(t, x^i), t)$. From the momentum constraint (159) we find that the scalar-field's momentum must obey

$$\Pi^\phi = -\frac{2}{\kappa^2}\left(\frac{\partial H}{\partial \phi}\right)_t. \tag{162}$$

Scalar Field Dynamics

Comparing Eq. (161) with the time derivative of $H(\phi, t)$,

$$\frac{1}{N}\left(\frac{\partial H}{\partial t}\right)_x = \Pi^\phi\left(\frac{\partial H}{\partial \phi}\right)_t + \frac{1}{N}\left(\frac{\partial H}{\partial t}\right)_\phi = -\frac{\kappa^2}{2}(\Pi^\phi)^2 + \frac{1}{N}\left(\frac{\partial H}{\partial t}\right)_\phi, \tag{163}$$

we find $\left(\frac{\partial H}{\partial t}\right)_\phi = 0$. In fact, we should not be surprised since this is actually a consequence of the general covariance of the theory.

On the other hand, the scalar field's equation (155) can be written to first order in spatial gradients as

$$\frac{1}{N}\dot{\Pi}^\phi + 3H\Pi^\phi + \frac{\partial V}{\partial \phi} = 0\,. \tag{164}$$

$$\dot{\Pi}^\phi + 3H\Pi^\phi + \frac{\partial V}{\partial \phi} = 0\,. \tag{165}$$

We can also show that the conjugate momentum Π^ϕ does not depend explicitly on time, its only dependence comes through ϕ. For this, differentiate Eq. (158) w.r.t. ϕ to obtain

$$\Pi^\phi \left(\frac{\partial \Pi^\phi}{\partial \phi}\right)_t + 3H\,\Pi^\phi + \frac{\partial V}{\partial \phi} = 0$$

and compare with (165), where

$$\frac{1}{N}\dot{\Pi}^\phi = \Pi^\phi \left(\frac{\partial \Pi^\phi}{\partial \phi}\right)_t + \left(\frac{\partial \Pi^\phi}{\partial t}\right)_\phi\,, \tag{166}$$

which implies $\left(\frac{\partial \Pi^\phi}{\partial t}\right)_\phi = 0$.

Hamilton-Jacobi formalism

We can now summarise what we have learned. The evolution of a general foliation of space-time in the presence of a scalar field fluid can be described solely in terms of the rate of expansion, which is a function of the scalar field only, $H \equiv H(\phi(t, x^i))$, satisfying the Hamiltonian constraint equation:

$$3H^2(\phi) = \frac{2}{\kappa^2}\left(\frac{\partial H}{\partial \phi}\right)^2 + \kappa^2 V(\phi)\,, \tag{167}$$

together with the momentum constraint and the evolution of the scale factor,

$$\dot{\phi} = -\frac{2}{\kappa^2}\left(\frac{\partial H}{\partial \phi}\right) = \Pi^\phi\,, \qquad \frac{\dot{a}}{a} = H(\phi)\,, \tag{168}$$

$$\frac{1}{N}\dot{\phi} \;=\; -\frac{2}{\kappa^2}\left(\frac{\partial H}{\partial \phi}\right) = \Pi^\phi \tag{169}$$

$$\frac{1}{N}\dot{\alpha} \;=\; H(\phi)\,, \tag{170}$$

as well as the dynamical gravitational and scalar field evolution equations

$$\dot{H} \;=\; -\frac{2}{\kappa^2}\left(\frac{\partial H}{\partial \phi}\right)^2 = -\frac{\kappa^2}{2}(\Pi^\phi)^2\,, \tag{171}$$

$$\dot{\Pi}^\phi \;=\; -3H\,\Pi^\phi - V'(\phi)\,. \tag{172}$$

Therefore, $H(\phi)$ is all you need to specify (to second order in field gradients) the evolution of the scale factor and the scalar field during inflation.

These equations are still too complicated to solve for arbitrary potentials $V(\phi)$. In the next section we will find solutions to them in the slow-roll approximation.

Slow-roll approximation and attractor

Given the complete set of constraints (158)-(159) and evolution equations (161)-(165), we can construct the following parameters,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{2}{\kappa^2}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 = -\frac{\partial \ln H}{\partial \ln a}, \tag{173}$$

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{2}{\kappa^2}\left(\frac{H''(\phi)}{H(\phi)}\right) = -\frac{\partial \ln H'}{\partial \ln a}, \tag{174}$$

in terms of which we can define the number of e-folds N_e as

$$N_e \equiv \ln\frac{a_{\rm end}}{a(t)} = \int_t^{t_{\rm end}} H dt = -\frac{\kappa^2}{2}\int_\phi^{\phi_{\rm end}} \frac{H(\phi)d\phi}{H'(\phi)}\,. \tag{175}$$

In order for inflation to be predictive, you need to ensure that inflation is independent of initial conditions. That is, one should ensure that there is an attractor solution to the dynamics, such that differences between solutions corresponding to different initial conditions rapidly vanish.

Let $H_0(\phi)$ be an exact, particular, solution of the constraint equation (167), either inflationary or not. Add to it a *homogeneous* linear perturbation $\delta H(\phi)$,

$$H(\phi) = H_0(\phi) + \delta H(\phi)$$

and substitute into (167).

The linear perturbation equation reads

$$H_0'(\phi)\,\delta H'(\phi) = (3\kappa^2/2)\,H_0\delta H\,,$$

whose general solution is

$$\delta H(\phi) = \delta H(\phi_i)\,\exp\left(\frac{3\kappa^2}{2}\int_{\phi_i}^{\phi}\frac{H_0(\phi)d\phi}{H_0'(\phi)}\right) = \delta H(\phi_i)\,\exp(-3\Delta N)\,,$$

where $\Delta N = N_i - N > 0$, and we have used (175) with the particular solution $H_0(\phi)$. This means that any deviation from the attractor dies away exponentially fast. This ensures that we can effectively reduce our two-dimensional space (ϕ, Π^ϕ) to just a single trajectory in phase space.

As a consequence, regardless of the initial condition, the attractor behaviour implies that late-time solutions are the same *up to a constant time shift*, which cannot be measured.

An example: Power-Law Inflation

An exponential potential is a particular case where the attractor can be found explicitly and one can study the approach to it, for an arbitrary initial condition. Consider the inflationary potential

$$V(\phi) = V_0 \, e^{-\beta\kappa\phi} \,, \tag{176}$$

with $\beta \ll 1$ for inflation to proceed. A particular solution to the Hamiltonian constraint equation (167) is

$$H_{\rm att}(\phi) \;=\; H_0 \, e^{-\frac{1}{2}\beta\kappa\phi} \,, \tag{177}$$

$$H_0^2 \;=\; \frac{\kappa^2}{3} V_0 \left(1 - \frac{\beta^2}{6}\right)^{-1} . \tag{178}$$

This model corresponds to an inflationary universe with a scale factor that grows like

$$a(t) \sim t^p \,, \qquad p = \frac{2}{\beta^2} \gg 1 \,. \tag{179}$$

The slow-roll parameters are both constant,

$$\epsilon = \frac{2}{\kappa^2}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 = \frac{\beta^2}{2} = \frac{1}{p} \ll 1\,, \tag{180}$$

$$\delta = \frac{2}{\kappa^2}\left(\frac{H''(\phi)}{H(\phi)}\right) = \frac{\beta^2}{2} = \frac{1}{p} \ll 1\,, \tag{181}$$

$$\xi = \frac{4}{\kappa^4}\left(\frac{H'\,H'''(\phi)}{H^2(\phi)}\right) = \frac{\beta^4}{4} = \frac{1}{p^2} \ll 1\,. \tag{182}$$

All trajectories tend to the attractor (177), while we can also write down the solution corresponding to the slow-roll approximation, $\epsilon = \delta = 0$,

$$H^2_{\rm SR}(\phi) = \frac{\kappa^2}{3}\,V_0\,e^{-\beta\kappa\phi}\,, \tag{183}$$

which differs from the actual attractor by a tiny constant factor, $3p/(3p-1) \simeq 1$, responsible for a constant time-shift which cannot be measured.

INFLATIONARY MODEL BUILDING

For the moment, observations of the microwave background anisotropies suggest that the Sachs-Wolfe plateau exists, but it is still premature to determine the tensor contribution. Perhaps in the near future, from the analysis of polarization as well as temperature anisotropies, with the CMB satellites WMAP and Planck, we might have a chance of determining the validity of the consistency relation. Assuming that the scalar contribution dominates over the tensor on large scales, i.e. $r \ll 1$, one can actually give a measure of the amplitude of the scalar metric perturbation from the observations of the Sachs-Wolfe plateau in the angular power spectrum,

$$\left[\frac{l(l+1)\,C_l^{(S)}}{2\pi}\right]^{1/2} = \frac{A_S}{5} = (0.926 \pm 0.0106) \times 10^{-5}\,, \quad (184)$$

$$r < 0.11\,, \qquad n = 0.9667 \pm 0.0040\,, \quad (185)$$

$$\frac{dn}{d\ln k} = -0.002 \pm 0.013\,. \quad (186)$$

These measurements can be used to normalize the primordial spectrum and determine the parameters of a particular model of inflation. In the near future these parameters will be determined with much better accuracy, to less than a percent.

In the next sections we will consider specific models of inflation. The formulae we will be using are

$$\epsilon = \frac{1}{2\kappa^2}\left(\frac{V'}{V}\right)^2, \qquad \eta = \frac{1}{\kappa^2}\left(\frac{V''}{V}\right), \quad (187)$$

$$\xi = \frac{1}{\kappa^4}\left(\frac{V'V'''}{V^2}\right), \qquad N = \int_{\phi_{\rm end}}^{\phi} \frac{\kappa d\phi}{\sqrt{2\epsilon}}, \quad (188)$$

together with the formula for the amplitude and tilt of scalar and tensor anisotropies

$$A_S = \frac{\kappa}{\sqrt{2\,\epsilon}}\,\frac{H}{2\pi}, \qquad n_s = 1 + 2\eta - 6\epsilon\,, \qquad \frac{d\,n_s}{d\,\ln k} = 2\xi + 24\epsilon^2 - 16\epsilon\eta\,,$$

$$A_T = 2\sqrt{2}\,\kappa\,\frac{H}{2\pi}, \qquad n_T = -2\epsilon\,, \qquad r = 16\epsilon = -8\,n_T\,.$$

Power-law inflation

$$V(\phi) = V_0 e^{-\beta\kappa\phi} \qquad \beta \ll 1 \quad \text{for inflation}$$

$$3H^2(\phi) = \frac{2}{\kappa^2}\left(\frac{\partial H}{\partial\phi}\right)^2 + \kappa^2 V(\phi)\,, \qquad \beta H = \kappa\dot{\phi}$$

$$H(\phi) = H_0 e^{-\frac{1}{2}\beta\kappa\phi} \qquad \Rightarrow \qquad \frac{1}{H}\frac{\partial H}{\partial\phi} = -\frac{1}{2}\beta\kappa = \text{const}$$

$$H_0^2 = \frac{\kappa^2}{3}V_0\left(1 - \frac{\beta^2}{6}\right)^{-1} \qquad \text{where} \qquad V_0 \equiv M^4 \tag{189}$$

$$\epsilon = \frac{2}{\kappa^2}\left(\frac{H'}{H}\right)^2 = \frac{1}{2}\beta^2 < 1 \tag{190}$$

$$\delta = \frac{2}{\kappa^2}\left(\frac{H''}{H}\right)^2 = \frac{1}{2}\beta^2 < 1 \tag{191}$$

$$\xi = \frac{1}{\kappa^4}\left(\frac{H'H'''}{H^2}\right) = \frac{1}{4}\beta^4 < 1 \tag{192}$$

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2}\beta^2 \qquad \Rightarrow \qquad a \propto t^p \;\xrightarrow{\epsilon=1/p}\; p = \frac{2}{\beta^2} \tag{193}$$

$$\epsilon = \delta = \frac{1}{p} = \text{const} \tag{194}$$

$$N = \int_{\phi}^{\phi_{\rm end}} \frac{\kappa d\phi}{\sqrt{2\epsilon}} = \frac{\kappa}{\beta}(\phi_{\rm end} - \phi) = 65 \tag{195}$$

$$A_S = \frac{\kappa}{\sqrt{2\epsilon}}\frac{H}{2\pi} = 5\times 10^{-5} \qquad \Rightarrow \qquad M \simeq 10^{-3}M_P \simeq M_{\rm GUT} \tag{196}$$

$$n_s - 1 = 2\left(\frac{\delta - 2\epsilon}{1-\epsilon}\right) = -\,\frac{2}{p-1} \qquad \Rightarrow \qquad \frac{d\,n_s}{d\ln k} = 0 \tag{197}$$

$$n_s = 0.96 \qquad \Rightarrow \qquad p = 51 \tag{198}$$

$$n_T = -\,\frac{2\epsilon}{1-\epsilon} = -\frac{2}{p-1} = n_s - 1\,, \qquad\qquad r = -8\,n_T = 0.32 > 0.11 \tag{199}$$

Chaotic inflation $(m^2\phi^2)$

$$V(\phi) = \frac{1}{2}m^2\phi^2 \quad \Rightarrow \quad H^2 \simeq \frac{\kappa^2}{6}m^2\phi^2 \tag{200}$$

$$\epsilon = \frac{1}{2\kappa^2}\left(\frac{V'}{V}\right)^2 = \frac{2}{\kappa^2\phi^2} = 1 \quad \Rightarrow \quad \phi_{\rm end} = \frac{M_P}{2\sqrt{\pi}} \simeq \frac{M_P}{3.5} \tag{201}$$

$$\eta = \frac{1}{\kappa^2}\left(\frac{V''}{V}\right) = \frac{2}{\kappa^2\phi^2} = \epsilon = \frac{1}{2N}\,, \qquad \xi = 0 \tag{202}$$

$$N = \int_{\phi_{\rm end}}^{\phi} \frac{\kappa d\phi}{\sqrt{2\epsilon}} = \left(\frac{\kappa\phi}{2}\right)^2\Bigg|_{\phi_{\rm end}}^{\phi} \approx \frac{\kappa^2\phi^2}{4} \quad \Rightarrow \quad \phi_{60} = 3M_P \tag{203}$$

$$A_S = \frac{\kappa m}{\sqrt{6}}\frac{\kappa^2\phi^2}{4\pi} = N\sqrt{\frac{4}{3\pi}}\frac{m}{M_P} = 5\times 10^{-5} \quad \Rightarrow \tag{204}$$

$$m = 1.2\times 10^{-6}M_P = 1.4\times 10^{13}\,\mathrm{GeV} \tag{205}$$

$$n_s = 1 - \frac{2}{N} \approx 0.967\,, \qquad \frac{dn_s}{d\ln k} = \frac{2}{N^2} = 6\times 10^{-4}$$

$$A_T = \frac{4}{\sqrt{\pi}}\frac{H}{M_P} < 10^{-5}\,, \qquad n_T = -2\epsilon = -\frac{1}{N} \simeq -0.016 \tag{206}$$

$$r = \frac{8}{N} \simeq 0.13 \tag{207}$$

Chaotic inflation ($\lambda\phi^4$)

$$V(\phi) = \frac{1}{4}\lambda\phi^4 \quad \Rightarrow \quad H^2 \simeq \frac{\kappa^2}{12}\,\lambda\phi^4 \tag{208}$$

$$\epsilon = \frac{1}{2\kappa^2}\left(\frac{V'}{V}\right)^2 = \frac{8}{\kappa^2\phi^2} = 1 \quad \Rightarrow \quad \phi_{\rm end} = \frac{M_P}{\sqrt{\pi}} \simeq \frac{M_P}{1.8} \tag{209}$$

$$\eta = \frac{1}{\kappa^2}\left(\frac{V''}{V}\right) = \frac{12}{\kappa^2\phi^2} = \frac{3\epsilon}{2} = \frac{3}{2N}\,, \qquad \xi = \frac{3}{2N^2}\,, \tag{210}$$

$$N = \int_{\phi_{\rm end}}^{\phi} \frac{\kappa d\phi}{\sqrt{2\epsilon}} = \left(\frac{\kappa\phi}{8}\right)^2\Bigg|_{\phi_{\rm end}}^{\phi} \approx \frac{\kappa^2\phi^2}{8} \Rightarrow \phi_{60} = 4.3 M_P \tag{211}$$

$$A_S(\phi_*) = \sqrt{\frac{\lambda}{3}}\,\frac{\kappa^3\phi_*^3}{16\pi} = \sqrt{\frac{\lambda}{3}}\,\frac{(8N)^{3/2}}{16\pi} = 5\times 10^{-5} \tag{212}$$

$$\Rightarrow \quad \lambda = 1.3\times 10^{-13} \tag{213}$$

$$n_s = 1 + 2\eta - 6\epsilon = 1 - \frac{3}{N} \approx 0.95\,, \tag{214}$$

$$\frac{dn_s}{d\ln k} = \frac{3}{N^2} = 1.7\times 10^{-3} \tag{215}$$

$$A_T = \frac{4}{\sqrt{\pi}}\,\frac{H}{M_P} < 10^{-5}\,, \tag{216}$$

$$n_T = -2\epsilon = -\frac{2}{N} \simeq -0.03 \tag{217}$$

$$r = -8\,n_T = \frac{16}{N} \simeq 0.27 \tag{218}$$

New inflation $\quad (v \sim M_P)$

$$V(\phi) = \frac{\lambda}{4}\Big(\phi^2 - v^2\Big)^2 = M^4\left(1 - \frac{\phi^2}{v^2}\right)^2 \quad \Rightarrow \quad H^2 \simeq \frac{\kappa^2}{12}\,\lambda\left(\phi^2 - v^2\right)^2 \tag{219}$$

$$\epsilon \;=\; \frac{8\phi^2}{\kappa^2(\phi^2 - v^2)^2} = 1 \;\;\Rightarrow\;\; \phi_{\rm end} = v \pm \frac{M_P}{2\sqrt{\pi}}\,. \tag{220}$$

$$\eta \;=\; \frac{4(3\phi^2 - v^2)}{\kappa^2(\phi^2 - v^2)^2}\,, \qquad \xi = \frac{96\phi^2}{\kappa^4(\phi^2 - v^2)^3}\,, \tag{221}$$

$$N \;=\; \int_{\phi_{\rm end}}^{\phi} \frac{\kappa d\phi}{\sqrt{2\epsilon}} = \frac{\kappa^2}{8}\Big(\phi^2 - \phi_{\rm end}^2\Big) - \frac{\kappa^2 v^2}{4}\ln\frac{\phi}{\phi_{end}} \tag{222}$$

$$\Rightarrow\;\; \phi_{60} = \phi_{\rm end}\exp\Big(-\frac{30M_P^2}{\pi v^2}\Big) \tag{223}$$

$$A_S(\phi_*) = \sqrt{\frac{\lambda}{3}}\,\frac{\kappa^3}{16\pi\phi_*}\Big(\phi_*^2 - v^2\Big)^2 = \frac{\kappa^2 M^2}{2\pi\sqrt{6}}\frac{\phi_{\rm end}}{\phi_*} = 5\times 10^{-5} \tag{224}$$

$$\Rightarrow\;\; M \sim 10^{13}\ \text{GeV} \tag{225}$$

$$n_s = 1 + 2\eta - 6\epsilon \simeq 1 - \frac{8}{\kappa^2 v^2} \approx 0.96 \tag{226}$$

$$\Rightarrow \quad v \simeq \frac{5M_p}{\sqrt{\pi}}\,, \qquad \frac{dn_s}{d\ln k} \approx 0 \tag{227}$$

$$A_T = \frac{4}{\sqrt{\pi}}\,\frac{H}{M_P} < 10^{-5}\,, \qquad n_T = -2\epsilon \approx 0 \tag{228}$$

$$r \;=\; -8\,n_T \approx 0 \tag{229}$$

Natural inflation

$$V(\phi) = M^2 f^2 \left(1-\cos\frac{\phi}{f}\right) = 2M^2 f^2 \sin^2\frac{\phi}{2f} \tag{230}$$

$$= \frac{1}{2} M^2\phi^2 - \frac{1}{4!}\frac{M^2}{f^2}\phi^4 + \mathcal{O}(\phi^6) \tag{231}$$

$$\epsilon = \frac{1}{2\kappa^2 f^2}\left(\frac{\sin\frac{\phi}{f}}{1-\cos\frac{\phi}{f}}\right)^2 = \frac{\cot^2\frac{\phi}{2f}}{2\kappa^2 f^2} \ll 1 \tag{232}$$

$$\eta = \frac{1}{\kappa^2 f^2}\frac{\cos\frac{\phi}{f}}{1-\cos\frac{\phi}{f}} = \epsilon - \frac{1}{2\kappa^2 f^2} \ll 1 \tag{233}$$

$$\xi = \frac{-1}{\kappa^4 f^4}\left(\frac{\sin\frac{\phi}{f}}{1-\cos\frac{\phi}{f}}\right)^2 = \frac{-2\epsilon}{\kappa^2 f^2} \ll 1 \tag{234}$$

$$N = 2\kappa^2 f^2 \int_{x_{\rm end}}^{x} dx\ \tan x = -2\kappa^2 f^2 \ln\cos\frac{\phi}{2f}\bigg|_{\phi_{\rm end}}^{\phi} \tag{235}$$

$$\epsilon = 1 \qquad \Rightarrow \quad \cos\frac{\phi_{\rm end}}{2f} = \left(\frac{2\kappa^2 f^2}{1+2\kappa^2 f^2}\right)^{1/2} < 1 \tag{236}$$

$$\cos\frac{\phi}{2f} = \left(\frac{2\kappa^2 f^2}{1+2\kappa^2 f^2}\right)^{1/2} e^{-\frac{N}{2\kappa^2 f^2}} \tag{237}$$

$$\epsilon_{65} = \frac{1}{2\kappa^2 f^2}\left(e^{\frac{N}{\kappa^2 f^2}} - 1\right)^{-1} \ll \frac{1}{2\kappa^2 f^2} \tag{238}$$

$$\eta_{65} = \epsilon_{65} - \frac{1}{2\kappa^2 f^2} \quad \Rightarrow \quad n_s \simeq 1 - \frac{1}{\kappa^2 f^2} \tag{239}$$

$$A_S = \sqrt{\frac{2}{3}}\frac{\kappa M}{2\pi}\,(2\kappa^2 f^2)\,\sinh\frac{N}{2\kappa^2 f^2} \tag{240}$$

$$\text{If} \quad f = M_P \;\Rightarrow\; M = 9\times 10^{-7} M_P = 10^{13}\,\text{GeV} \tag{241}$$

$$n_s = 1 - \frac{1}{8\pi} = 0.96\,, \qquad \frac{d\,n_s}{d\ln k} = \frac{1.3\epsilon}{2\pi} = 4.2\times 10^{-4} \tag{242}$$

$$r = -8\,n_T = \frac{1}{\pi}\left(e^{\frac{N}{8\pi}} - 1\right)^{-1} \simeq 0.032 \tag{243}$$

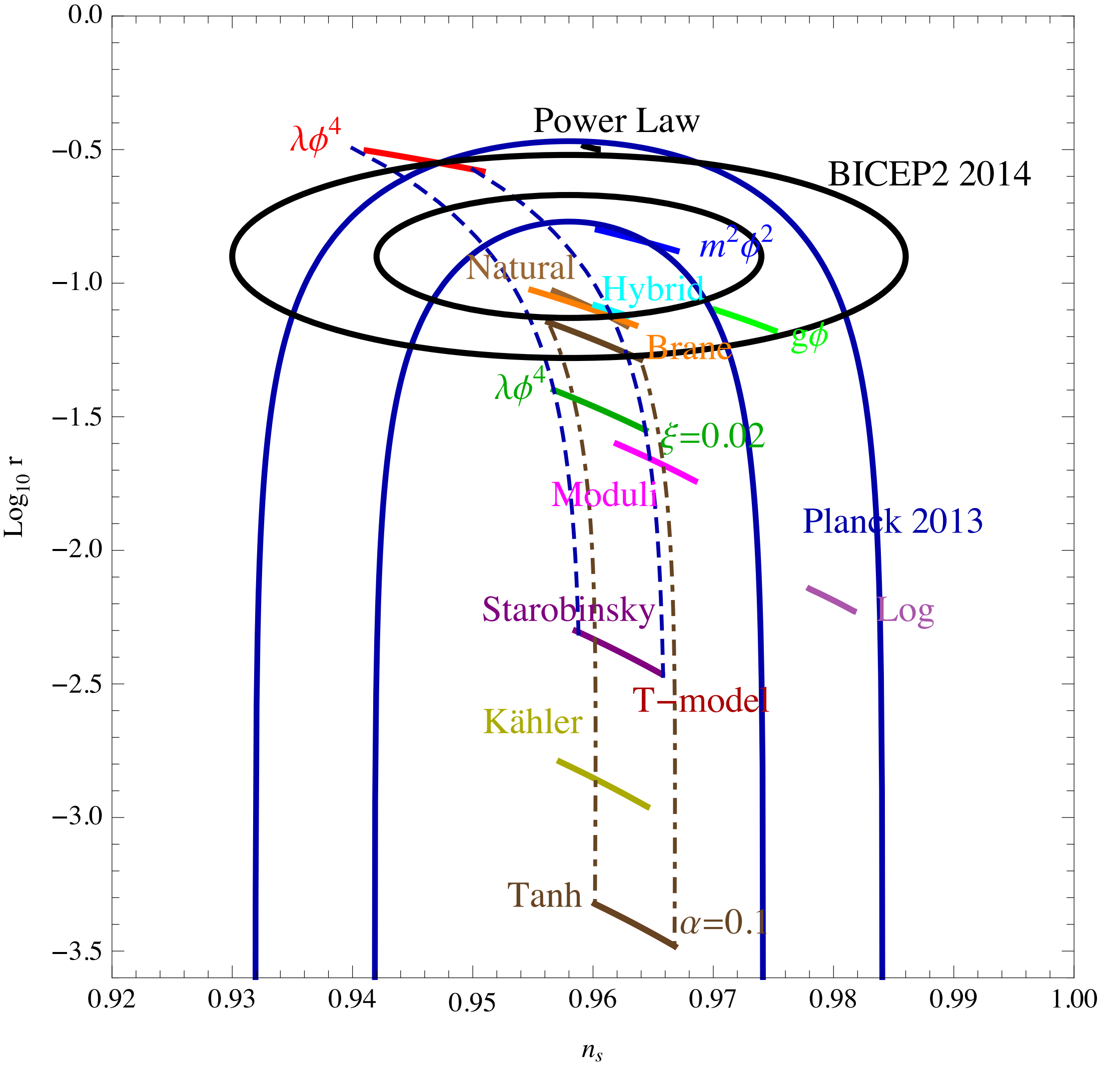

Figure 4: The plane (n_s, r) allowed by Planck, with all the models discussed in the text. The ranges of values correspond to the interval $N \in [50, 60]$. Also shown is the motion along the plane due to a increasing non-minimal coupling, as discussed below.

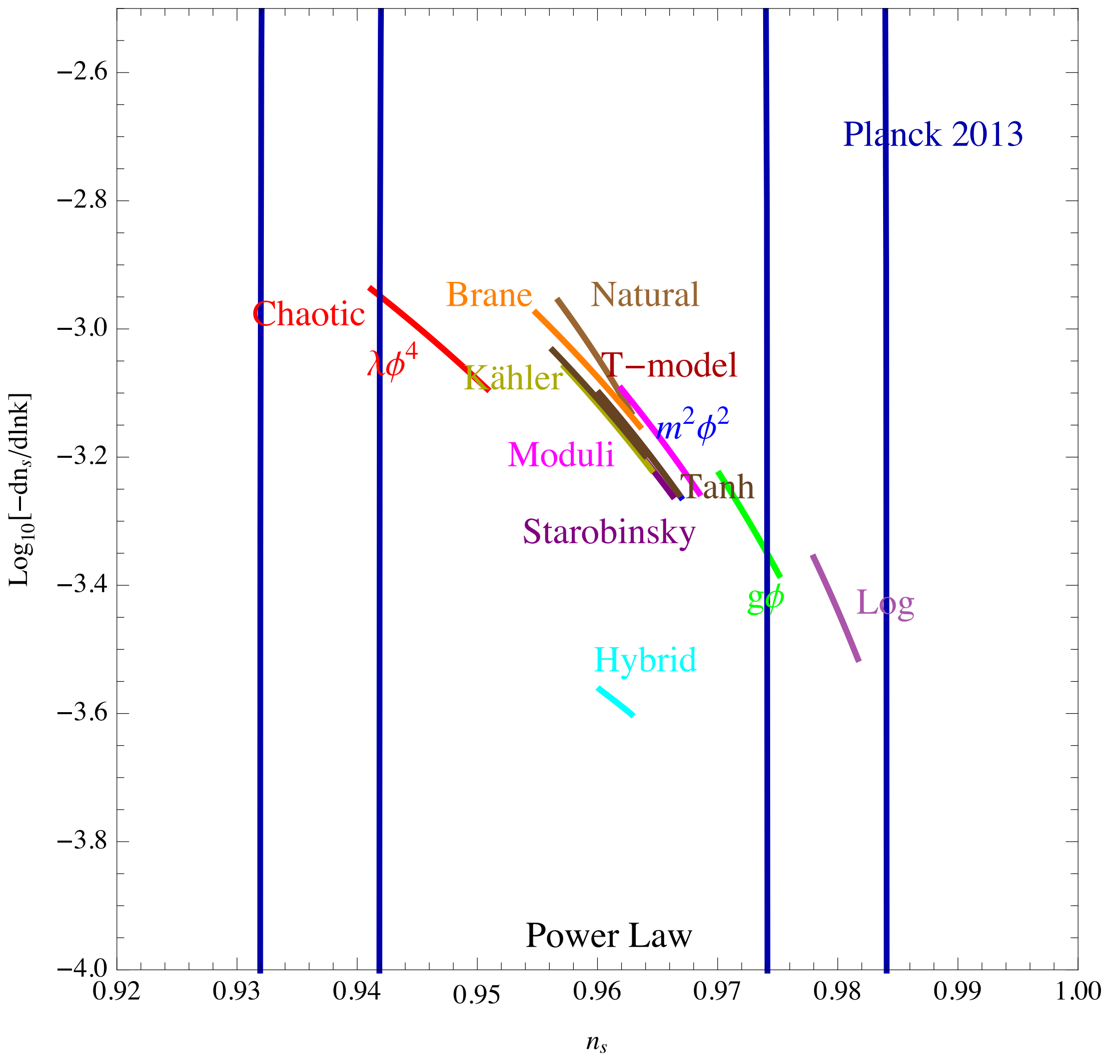

Figure 5: The plane $(n_s, dn_s/d\ln k)$ with all the models discussed in the text. The ranges of values correspond to the interval $N \in [50, 60]$. Note that they all lie in a small region around $dn_s/d\ln k \simeq 10^{-3}$, where PRISM is possibly capable of detecting it, but where unfortunately Planck has no resolution.

Starobinski inflation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,R = \kappa^2\,\langle T_{\mu\nu}\rangle_{\rm ren} = \frac{1}{6M^2}\,{}^{(1)}H_{\mu\nu} + \frac{1}{H_0^2}\,{}^{(3)}H_{\mu\nu}\,, \tag{244}$$

$$ {}^{(1)}H_{\mu\nu} \;=\; 2(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2)R + 2R\,R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^2 \tag{245}$$

$$ {}^{(3)}H_{\mu\nu} \;=\; R_\mu^{\ \lambda}R_{\lambda\nu} - \frac{2}{3}R\,R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\rho\sigma}R_{\rho\sigma} + \frac{1}{4}g_{\mu\nu}R^2 \tag{246}$$

Substituting FRW metric and using the Slow Roll Approximation,

$$\dot{H} = -\frac{M^2}{6}\left(1 - \frac{H^2}{H_0^2}\right) . \tag{247}$$

At first stage: $H_0^2 \gg M^2 \;\Rightarrow\; -\dot{H} < M^2/6 \ll H_0^2 \;\Rightarrow\; H \approx H_0 = \text{const}$. However, H grows and becomes unstable. When $H \sim M$ inflation ends. Alternatively, one can study the evolution in the effective action formalism, including higher derivatives,

$$\mathcal{S}_g = \int d^4x\,\sqrt{-g}\,\frac{1}{2\kappa^2}\left(R - \frac{R^2}{6M^2}\right) \equiv \int d^4x\,\sqrt{-g}\,\frac{1}{2\kappa^2}\,f(R) \tag{248}$$

which gives rise to Eq. (245). One can then write this action as the usual Einstein-Hilbert action plus a scalar field, making use of the conformal transformation

$$\begin{aligned}
\tilde{g}_{\mu\nu} &= F(R)\,g_{\mu\nu} \equiv e^{\alpha\kappa\phi} g_{\mu\nu} \quad \Rightarrow \quad \sqrt{-\tilde{g}} = e^{2\alpha\kappa\phi}\sqrt{-g} && (249)\\
\tilde{R}_{\mu\nu} &= R_{\mu\nu} - \frac{\alpha\kappa}{2}\left(g_{\mu\nu}\nabla^2\phi + 2\nabla_\mu\nabla_\nu\phi\right) && (250)\\
\tilde{R} &= e^{-\alpha\kappa\phi}\left[R - 3\alpha\kappa\,\nabla^2\phi - \frac{3}{2}\alpha^2\kappa^2(\partial\phi)^2\right] && (251)
\end{aligned}$$

The scalar field ϕ will have canonical kinetic term for $\alpha^2 = 2/3$. From the equations of motion one finds the relationship $F(R) = f'(R)$, and therefore the effective scalar potential becomes

$$V(\phi) = \frac{1}{2\kappa^2}\,\frac{f(R) - R\,f'(R)}{(f'(R))^2} = \frac{R^2}{12\kappa^2 M^2}\left(1 - \frac{R}{3M^2}\right)^{-2} \tag{252}$$

$$V(\phi) = \frac{3M^2}{4\kappa^2}\left(1 - e^{-\alpha\kappa\phi}\right)^2 = \frac{1}{2}M^2\phi^2\,(1 + \alpha\kappa\phi + \dots) \tag{253}$$

$$\epsilon = \frac{2\alpha^2}{(e^{\alpha\kappa\phi}-1)^2} = 1 \quad \Rightarrow \quad \phi_{\rm end} = \frac{\sqrt{3}\,M_P}{4\sqrt{\pi}} \ln\left(1+\frac{2}{\sqrt{3}}\right) \simeq \frac{M_P}{5.33}$$
$$\Rightarrow \quad H_{\rm end} = \frac{\sqrt{3}\,M}{2\sqrt{2+\sqrt{3}}} \tag{254}$$

$$\eta = \frac{2\alpha^2(2-e^{\alpha\kappa\phi})}{(e^{\alpha\kappa\phi}-1)^2} = 0 \quad \Rightarrow \quad \phi_* = \frac{\sqrt{3}M_P}{4\sqrt{\pi}} \ln 2 \simeq \frac{M_P}{5.90} \tag{255}$$

$$\xi = \frac{4\alpha^4(e^{\alpha\kappa\phi}-4)}{(e^{\alpha\kappa\phi}-1)^3} \tag{256}$$

$$N = \frac{e^{\alpha\kappa\phi}-\alpha\kappa\phi}{2\alpha^2}\Bigg|_{\phi_{\rm end}}^{\phi} \simeq \frac{3}{4}\,e^{\alpha\kappa\phi} \quad \Rightarrow \quad \phi_{60} = 1.09M_P \tag{257}$$

$$\epsilon_{60} \simeq \frac{1}{2\alpha^2N^2}\,, \qquad \eta_{60} \simeq -\frac{1}{N}\,, \qquad \xi_{60} \simeq \frac{1}{N^2}\,, \tag{258}$$
$$\alpha\kappa\phi_{60} = 4.38 \gg 1 \quad \Rightarrow \quad V(\phi_{60}) \simeq \frac{M^2}{2\alpha^2\kappa^2} \quad \Rightarrow \quad H_{60} \simeq \frac{M}{2} \tag{259}$$

$$A_S = \frac{\alpha N}{2\pi}\,\kappa H = 5\times 10^{-5} \quad \Rightarrow \quad M \simeq 2.4\times 10^{-6}M_P \tag{260}$$
$$n_s = 1-\frac{2}{N} \simeq 0.97\,, \qquad \frac{d\,n_s}{d\ln k} \simeq \frac{2}{N^2} = 5.6\times 10^{-4} \tag{261}$$

$$A_T = \frac{\sqrt{2}}{\pi}\frac{H}{M_P} = \frac{2}{\sqrt{\pi}}\frac{M}{M_P} = 2.7\times 10^{-6} \tag{262}$$

$$n_T = -2\epsilon \simeq -\frac{3}{2N^2} = -2.8\times 10^{-4} \tag{263}$$

$$r = -8\,n_T \simeq 2\times 10^{-3} \tag{264}$$

(Inverted) Hybrid inflation

$$V(\phi,\chi) = \frac{\lambda}{4}(\chi^2 - v^2)^2 + \frac{1}{2}g^2\phi^2\chi^2 - \frac{1}{2}m^2\phi^2 \tag{265}$$

The effective Higgs mass in the false vacuum ($\chi = 0$):

$$m_\chi^2 \equiv \frac{\partial^2 V}{\partial\chi^2} = g^2\phi^2 - \lambda v^2 = 0 \quad \Rightarrow \quad \phi_c \equiv \frac{M}{g} = \frac{\sqrt{\lambda}v}{g} \tag{266}$$

For large values of the inflaton, the Higgs has a large mass and sits at its minimum, and therefore the ffective potential during inflation is

$$V(\phi) \;=\; V_0 - \frac{1}{2}m^2\phi^2 \simeq V_0 = \text{const.} \tag{267}$$

$$H_0 \;\simeq\; \sqrt{\frac{2\pi}{3}}\,\frac{Mv}{M_P} \tag{268}$$

$$\epsilon \;=\; \frac{m^2}{\kappa^2 V_0}\,\frac{m^2\phi^2}{2V_0} \;\ll\; \eta = -\frac{m^2}{\kappa^2 V_0} \quad \Rightarrow \quad n = 1 - \frac{2m^2}{\kappa^2 V_0} \tag{269}$$

$$N \;=\; \frac{\kappa^2 V_0}{m^2}\,\ln\frac{\phi}{\phi_c} \quad \Rightarrow \quad \phi = \phi_c\, e^{\eta N} \tag{270}$$

Inflation ends not because of the end of slow-roll ($\epsilon = 1$) but because of symmetry breaking by the Higgs

$$A_S \;=\; \frac{H^2}{2\pi\dot\phi} = \frac{gH}{2\pi\eta\, M}\, e^{-\eta N} = 5\times 10^{-5} \quad \Rightarrow \tag{271}$$

$$g \;=\; \sqrt{\frac{3\pi}{8}}\,(n-1)\,10^{-4}\,\frac{M_P}{v}\,e^{(n-1)\frac{N}{2}} \tag{272}$$

Negligible gravitational waves:

$$r = -8\,n_T = 16\epsilon \;\ll\; 8\,(n-1) \tag{273}$$

Many possibilities of scales of inflation: e.g. GUT scale,

$$v \;=\; 10^{-3}M_P\,,\ \lambda = 0.1\,,\ g = 0.01\,, \quad \Rightarrow \quad n = 0.965\,, \tag{274}$$

$$M \;=\; 4\times 10^{15}\,\text{GeV}\,,\ m = 1.3\times 10^{12}\,\text{GeV}\,,\ r = 5\times 10^{-4} \tag{275}$$

Radiative corrections on SUSY hybrid inflation

Coleman-Weinberg potential

$$V_{1-\text{loop}} = \frac{1}{64\pi^2}\sum_i (-1)^{F_i}\, m_i^4\, \ln\frac{m_i^2}{\Lambda^2} \tag{276}$$

Supergravity hybrid model (units $\kappa = 1$)

$$\begin{aligned} W &= \sqrt{\lambda}\,\Phi(\bar\Sigma\Sigma - v^2) & (277)\\ V &= \lambda\,\left|\bar\sigma\sigma - v^2\right|^2 + \frac{\lambda}{2}\phi^2(|\sigma|^2 + |\bar\sigma|^2) + \mathrm{D-term} & (278)\end{aligned}$$

where $\phi = \sqrt{2}\,\Phi$ is the canonically normalized field. The absolute minimum appears at $\phi = 0,\ \sigma = \bar\sigma = v$. For $\phi > \phi_c = \sqrt{2}\,v$, the fields $\sigma, \bar\sigma$ have a positive mass squared and stay at the origin. Inflation takes place along that "flat" direction, which is lifted by radiative corrections. The masses of bosons are $m_B^2 = \frac{1}{2}\lambda(\phi^2 \pm 2v^2)$, while that of the fermion is $m_F^2 = \frac{1}{2}\lambda\phi^2$. The loop corrected potential along the flat direction is

$$\begin{aligned} V_{1-\text{loop}}(\phi) &= \frac{\lambda^2}{128\pi^2}\Bigg[(\phi^2 - 2v^2)^2\,\ln\left(\frac{\phi^2 - 2v^2}{\Lambda^2}\right)\\ &+ (\phi^2 + 2v^2)^2\,\ln\left(\frac{\phi^2 + 2v^2}{\Lambda^2}\right) - 2\phi^4\,\ln\left(\frac{\phi^2}{\Lambda^2}\right)\Bigg] & (279)\\ \Rightarrow\ V(\phi) &\simeq \lambda\, v^4\left(1 + \frac{\lambda}{8\pi^2}\ln\frac{\phi}{\phi_c}\right)\,, \qquad \phi \gg \phi_c & (280)\end{aligned}$$

$$\epsilon = \frac{\lambda^2}{128\pi^4\phi^2} = \frac{\lambda}{32\pi^2 N}\,, \qquad \eta = -\frac{\lambda}{8\pi^2\phi^2} = -\frac{1}{2N} \tag{281}$$

$$N = \int\frac{d\phi}{\sqrt{2\epsilon}} = \frac{4\pi^2\phi^2}{\lambda} \tag{282}$$

$$A_S = \sqrt{\frac{N}{3}}\,16\pi\,\frac{v^2}{M_P^2} = 5\times 10^{-5} \quad\Rightarrow\quad v = 5.6\times 10^{15}\,\mathrm{GeV} \tag{283}$$

$$n = 1-\frac{1}{N} = 0.98\,, \qquad r = -2\pi\, n_T = 4\pi\epsilon = \frac{\lambda}{8\pi N} \ll 1 \tag{284}$$

Higgs Inflation

Standard Model Higgs with non-minimal coupling ξ. For the moment I will consider only the most straightforward model

$$\mathcal{S} = \int d^4x\sqrt{g}\left[\frac{1}{2}(1+\xi\phi^2)R - \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4\right]. \tag{285}$$

The usual procedure is to do a conformal transformation to the Einstein frame,

$$\Omega^2 = 1+\xi\phi^2 \equiv e^{\alpha\varphi}\,, \quad \frac{d\varphi}{d\phi} = \frac{\sqrt{1+\xi(1+6\xi)\phi^2}}{1+\xi\phi^2}\,, \quad V(\varphi) = \frac{\lambda\phi^4}{4(1+\xi\phi^2)^2}\,. \tag{286}$$

This allows one to write the slow-roll parameters

$$\epsilon_V = \frac{1}{2V^2}\left(\frac{d\phi}{d\varphi}\frac{dV}{d\phi}\right)^2, \tag{287}$$

$$\eta_V = \frac{1}{V}\frac{d\phi}{d\varphi}\frac{d}{d\phi}\left(\frac{d\phi}{d\varphi}\frac{dV}{d\phi}\right), \tag{288}$$

$$\xi_V = \frac{1}{V^2}\frac{d\phi}{d\varphi}\frac{dV}{d\phi}\cdot\frac{d\phi}{d\varphi}\frac{d}{d\phi}\left(\frac{d\phi}{d\varphi}\frac{d}{d\phi}\left(\frac{d\phi}{d\varphi}\frac{dV}{d\phi}\right)\right), \tag{289}$$

while integrating the number of e-folds,

$$N = \frac{\phi^2}{8}(1+6\xi) - \frac{3}{4}\ln(1+\xi\phi^2)\,, \tag{290}$$

we can substitute into the expression for the slow-roll parameters

$$
\begin{aligned}
\epsilon_V &= \frac{1+6\xi}{N(1+8\xi N)}\,, && (291)\\
\eta_V &= \frac{3+2\xi(4N+9)-32\xi^2 N(4N-3)}{2N(1+8\xi N)^2}\,, && (292)\\
\xi_V &= \frac{3-4\xi(8N-9)-12\xi^2(80N^2+12N-9)}{2N^2(1+8\xi N)^4} - && (293)\\
&- \frac{96\xi^3 N(32N^2+60N-3)+2048\xi^4 N^3(4N-9)}{2N^2(1+8\xi N)^4}\,,
\end{aligned}
$$

with which we can construct the observables

$$
\begin{aligned}
n_s &= 1 - \frac{6+36\xi}{N(1+8\xi N)} && (294)\\
&+ \frac{3+2\xi(4N+9)-32\xi^2 N(4N-3)}{N(1+8\xi N)^2}\,,\\
\frac{d\ln n_s}{d\ln k} &= \frac{3+12\xi(8N+3)+4\xi^2(272N^2+252N+27)}{N^2(1+8\xi N)^4} && (295)\\
&+ \frac{32\xi^3 N(160N^2+300N+81)}{N^2(1+8\xi N)^4}\\
&+ \frac{2048\xi^4 N^2(4N^2+15N+9)}{N^2(1+8\xi N)^4}\,,\\
r &= \frac{16+96\xi}{N(1+8\xi N)}\,. && (296)
\end{aligned}
$$

We show in Fig. 4 the attractor behavior, as we increase the non-minimal coupling, from the chaotic inflation $\lambda\phi^4$ model towards the Starobinsky model.

Reheating after inflation

One of the fundamental quests of cosmology is to understand the origin of all the matter and radiation present in the universe today. We have seen how inflation produces a homogeneous and flat background space-time, and imprints on top of it a set of scalar and tensor quantum fluctuations that become classical Gaussian random fields outside the horizon, with an approximately scale invariant spectrum.

Inflation also dilutes any relic species left from a hypothetical earlier period of the universe, such that at the end of inflation there remains only a homogeneous zero mode of the inflaton field with tiny fluctuations on the homogeneous metric. That is, the universe is empty and very cold: the entropy of the universe is exponentially small and the temperature can be taken to be zero, $S = T = 0$.

Therefore we are left with the puzzle: How does the large entropy and energy of our present horizon, $S \sim 10^{89}$ and $M \sim 10^{23} M_\odot$, arise from such a cold and empty universe? The answer seems to lie in the process by which the large potential energy density present during inflation gets converted into radiation at the end of inflation, a process known as *reheating of the universe.*

This process was studied soon after the first models of inflation were proposed and considered the *perturbative* decay of the inflaton field into quanta of other fields to which it coupled, e.g. fermions, gauge fields, and other scalars. Such couplings exist during inflation but play no role (except for inducing radiative corrections, as we will discuss later), because even if those particles were produced during inflation the exponential expansion would dilute them almost instantaneously, and nothing would be left at the end of inflation.

Let us write down the most general Lagrangian with couplings of the inflaton to other fields and among themselves,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) + \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{2}\xi\chi^2 R \\ &+ \bar\psi(i\gamma^\mu\partial_\mu - m_\psi)\psi - h\phi\bar\psi\psi - \frac{1}{2}g^2\phi^2\chi^2 - g^2\sigma\phi\chi^2\,, \end{aligned} \tag{297}$$

where g, h, ξ, etc. are small couplings (to avoid large radiative corrections during inflation); σ is the possibly finite vev of the inflaton, and we have shifted the inflaton potential by $\phi - \sigma \to \phi$, such that the minimum is at $\phi = 0$ and the potential can be expanded around the minimum as

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \mathcal{O}(\phi^4)\,, \tag{298}$$

where m is the mass of the inflaton at the minimum. In chaotic inflation of the type $m^2\phi^2$ or $\lambda\phi^4$, this mass and self-coupling are bounded by observations of the CMB to be

$$m \sim 10^{13}\ \mathrm{GeV}\,, \qquad \lambda \lesssim 10^{-13}\,. \tag{299}$$

We will consider this mass to be much larger than that of the other fields to which it couples: $m^2 \gg m_\chi^2, m_\psi^2 \gg g^2\sigma\phi, h\phi$. Also, the end of inflation occurs in these models when $H \sim m$, and subsequently, the rate of expansion decays as $H \sim 1/t < m$.

Let us compute the evolution of the inflaton after inflation, whose amplitude satisfies the equation (we are neglecting here the couplings to other fields, but we will consider them later)

$$\ddot\phi + 3H(t)\dot\phi + m^2\phi = 0\,, \tag{300}$$

whose solution is oscillatory,

$$\phi(t) = \Phi(t)\,\sin mt\,, \tag{301}$$

with the amplitude of oscillations decaying like $\Phi \sim a^{-3/2}$, as we will prove now. Consider the average energy density and pressure of the homogeneous inflaton field over one period of oscillations,

$$\langle\rho\rangle = \frac{1}{2}\langle\dot{\phi}^2\rangle + \frac{1}{2}m^2\langle\phi^2\rangle = \frac{1}{2}m^2\Phi^2(t)\Big(\langle\cos^2 mt\rangle + \langle\sin^2 mt\rangle\Big)\,, \tag{302}$$

$$\langle p\rangle = \frac{1}{2}\langle\dot{\phi}^2\rangle - \frac{1}{2}m^2\langle\phi^2\rangle = \frac{1}{2}m^2\Phi^2(t)\Big(\langle\cos^2 mt\rangle - \langle\sin^2 mt\rangle\Big) \simeq 0 \tag{303}$$

where we have neglected the change in $\Phi(t)$ due to the condition $m \gg H$ during reheating. The fact that an oscillating homogeneous scalar field behaves like a pressureless fluid means that the universe during that period expands like a matter dominated universe,

$$\dot{\rho} + 3H(\rho + p) = 0 \qquad \Rightarrow \qquad \rho = \frac{1}{2}m^2\Phi^2(t) \sim a^{-3}\,, \tag{304}$$

and therefore $\Phi \sim a^{-3/2} \sim t^{-1}$. That is, a homogeneous scalar field oscillating with frequency equal to its mass can be considered as a coherent wave of ϕ particles with zero momenta and particle density

$$n_\phi = \rho_\phi/m = \frac{1}{2}m\Phi^2 \sim a^{-3}\,, \tag{305}$$

oscillating coherently with the same phase.

Until now we have considered only the effects of expansion, and ignored the effects due to the production of particles from the inflaton. This can be accounted for by including, in the equation of motion, the denominator of the QFT propagator,

$$\ddot{\phi} + 3H(t)\dot{\phi} + \Big(m^2 + \Pi(\omega)\Big)\phi = 0\,, \tag{306}$$

where $\Pi(\omega)$ is the Minkowski space polarization operator for ϕ with four-momentum $k^\mu = (\omega, \mathbf{0})$, with $\omega = m$. The real part of the polarization operator can be neglected (due to the small couplings), $\mathrm{Re}\,\Pi(\omega) \ll m^2$. However, due to phase space, if the frequency of oscillations satisfies $\omega \gg \min(2m_\chi, 2m_\psi)$, then the polarization operator acquires an imaginary part,

$$\mathrm{Im}\,\Pi(m) = m\,\Gamma_\phi\,, \tag{307}$$

where Γ_ϕ is the total decay rate of the inflaton, and we have used the optical theorem (i.e. unitarity) to relate both quantities at the physical pole, $\omega = m$.

The total decay rate can be written as a sum over partial decays,

$$\Gamma_\phi = \sum_i \Gamma(\phi \to \chi_i \chi_i) + \sum_i \Gamma(\phi \to \bar{\psi}_i \psi_i)\,, \tag{308}$$

$$\Gamma(\phi \to \chi_i \chi_i) = \frac{g_i^4 v^2}{8\pi m}\,, \qquad \Gamma(\phi \to \bar{\psi}_i \psi_i) = \frac{h_i^2 m}{8\pi}\,, \tag{309}$$

$$\Gamma_\phi \equiv \frac{h_{\rm eff}^2 m}{8\pi} \ll m\,, \qquad h_{\rm eff}^2 = \sum_i \left(h_i^2 + \frac{g_i^4 v^2}{m^2}\right) \tag{310}$$

The evolution of the inflaton during the period of oscillations after inflation can be described through the phenomenological equation

$$\ddot{\phi} + 3H(t)\dot{\phi} + \Gamma_\phi \dot{\phi} + m^2 \phi = 0\,, \tag{311}$$

which includes the decay rate Γ_ϕ as a friction term giving rise to the damping of the oscillations due to inflaton particle decay. It assumes the inflaton condensate (the homogeneous zero mode) is composed of very many inflaton particles, each of these decaying into other particles to which it couples. The solution to this equation is given by (301) with

$$\Phi(t) = \Phi_0\, e^{-\frac{1}{2}\int 3H dt}\, e^{-\frac{1}{2}\Gamma_\phi t} = \frac{\Phi_0}{t}\, e^{-\frac{1}{2}\Gamma_\phi t}\,, \tag{312}$$

where we have used $H = 2/3t$.

We can now compute the evolution of the energy and number density of the inflaton field under the effect of particle production,

$$\frac{d}{dt}\Big(\rho_\phi a^3\Big) = -\Gamma_\phi\, \rho_\phi a^3\,, \tag{313}$$

$$\frac{d}{dt}\Big(n_\phi a^3\Big) = -\Gamma_\phi\, n_\phi a^3\,, \tag{314}$$

which simply states the usual exponential decay law for particles with decay rate Γ. Initially, the total decay rate is much smaller than the rate of expansion, $\Gamma_\phi \ll 3H = 2/t \ll m$, and the total comoving energy and total number of inflaton particles is conserved, their energy and number densities decaying like a matter fluid, $\rho_\phi \simeq m\, n_\phi \sim a^{-3}$.

Eventually, the universe expands sufficiently (this may take many many inflaton oscillations) that the decay rate becomes larger than the rate of expansion, or alternatively, the inflaton life-time, $\eta_\phi = \Gamma_\phi^{-1}$, becomes smaller than the age of the universe, $\eta_\phi < t_U = H^{-1}$, and the inflaton decays suddenly (in less than one Hubble time), releasing *all* its energy density ρ_ϕ into relativistic particles χ and ψ, in an exponential burst of energy. Subsequently, the produced particles interact among themselves and soon thermalize to a common temperature. This process is responsible for the present abundance of matter and radiation energy, and could be associated with the Big Bang of the "old" cosmology.

At first sight it may seem paradoxical that the universe may have to "wait" until it is old enough for the inflaton to decay, because we are accustomed to very rapid decays in our particle physics detectors, where life-times of order 10^{-17}s are possible, while our universe is 10^{+17}s old! However, if inflation took place at energy densities of order the GUT scale, the Hubble time of a causal domain at the end of inflation would be of order 10^{-35}s, which is many orders of magnitude smaller than even the fastest decays of the inflaton, $\sim 10^{-25}$s. So the probability that the inflaton decays in such a short Hubble time is negligible, and the universe has to wait until it is old enough that there is any probability of decay of a single inflaton particle. Eventually, of course, once the universe is older than the inflaton life-time, it (the inflaton) will decay exponentially fast due to its *constant* decay rate Γ_ϕ.

Let us now compute the reheating temperature of the universe that arises from the thermalization of the products of decay of the inflaton. Note that the process of reheating, once possible, is essentially instantaneous and therefore the energy density at reheating can be estimated as that corresponding to a rate of expansion $H = \Gamma_\phi$. Since all that energy density will be quickly converted into a plasma of relativistic particles, we can estimate

$$\rho(t_{\rm rh}) = \frac{3\Gamma_\phi^2 M_{\rm P}^2}{8\pi} = \frac{\pi^2}{30}\, g(T_{\rm rh})\, T_{\rm rh}^4\,, \qquad (315)$$

$$\Rightarrow \qquad T_{\rm rh} \simeq 0.1\,\sqrt{\Gamma_\phi M_{\rm P}} \qquad , \qquad (316)$$

where we have assumed $g(T_{\rm rh}) \sim 10^2 - 10^3$. Let us estimate this temperature. If we substitute $\Gamma_\phi = h_{\rm eff}^2 m/8\pi$ with $m \sim 10^{13}$ GeV, we find

$$T_{\rm rh} \simeq 2 \times 10^{14}\, h_{\rm eff}\ {\rm GeV} \lesssim 10^{11}\ {\rm GeV}\,, \qquad (317)$$

where we have imposed the constraint $h_{\rm eff} \lesssim 10^{-3}$ from radiative corrections in chaotic type models. Let us estimate it: if we consider the quantum loop corrections to the inflaton potential due to its coupling to other fields like in the Lagrangian described above, we find

$$V(\phi) = \frac{1}{2}m^2\phi^2\left(1+\frac{3g^4}{16\pi^4\lambda}\right)+\frac{\lambda}{4}\phi^4\left(1+\frac{3g^4}{16\pi^4\lambda}-\frac{h^4}{16\pi^4\lambda}\right)+\ldots \qquad (318)$$

Therefore, the couplings of the inflaton to other fields cannot be very large otherwise they would modify the amplitude of CMB anisotropies. If we impose that the mass and self-coupling of the inflaton satisfy (299), then the other couplings are bound to

$$3g^4\,, h^4 < 16\pi^2\lambda \qquad \Rightarrow \qquad g\,, h \lesssim 10^{-3}\,. \qquad (319)$$

For completeness, let us mention that in theories with only gravitational interactions, like e.g. in Starobinsky model, the decay of the inflaton is induced via gravity only and

$$\Gamma_{\rm grav} \sim \frac{m^3}{M_{\rm P}^2} \sim 10^{-18}\, M_{\rm P} \qquad \Rightarrow \qquad T_{\rm rh} \sim 10^9\ {\rm GeV}\,. \qquad (320)$$

All this indicates that, although the energy density at the end of inflation may be large, the *final* reheating temperature $T_{\rm rh}$ may not be higher than 10^{12} GeV, and thus the usually assumed thermal phase transition at Grand Unification, which was the basis for most of the early universe phenomenology, like production of topological defects, GUT baryogenesis, etc., could not have taken place.

We will see shortly that such phenomenology may be resucitated in the context of preheating and non-thermal phase transitions, but for the moment let us focus our attention onto two well differentiated and concrete cases of ordinary reheating:

Reheating in chaotic inflation models

Consider a $m^2\phi^2$ model of inflation, for which the value of the inflaton at the end of inflation is $\phi_{\rm end} = M_{\rm P}/2\sqrt{\pi}$, and the corresponding energy density

$$\rho_{\rm end} = \frac{3}{2}V(\phi_{\rm end}) = \frac{3m^2M_{\rm P}^2}{16\pi} = \Big(6.5\times 10^{15}\ {\rm GeV}\Big)^4 . \tag{321}$$

On the other hand, CMB anisotropies require

$$A_S = N\sqrt{\frac{4}{3\pi}\frac{m}{M_{\rm P}}} = 5\times10^{-5} \qquad \Rightarrow \qquad m \simeq 1.4\times10^{13}\,{\rm GeV}\,, \tag{322}$$

while radiative corrections impose the constraint $h_{\rm eff} \lesssim 10^{-3}$.

We are thus left with three time scales:

$$\left.\begin{array}{r} t_{\rm osc} \sim m^{-1} \sim 10^{-36}\,{\rm s} \\ t_{\rm exp} \gtrsim H_{\rm end}^{-1} \sim 10^{-35}\,{\rm s} \\ t_{\rm dec} \sim \Gamma_\phi^{-1} \sim 10^{-25}\,{\rm s} \end{array}\right\} \Rightarrow \qquad t_{\rm osc} \ll t_{\rm exp} \ll t_{\rm dec}\,, \tag{323}$$

so there are several oscillations per Hubble time, and we also expect many oscillations of the inflaton field before it decays. This result is typical of most high-scale models of inflation.

Reheating in low-scale hybrid inflation models

In this case, reheating occurs in very different circumstances. Most models of inflation occur at scales of order the GUT scale, because their parameters are fixed by the amplitude of CMB anisotropies, $\delta T/T \sim m/M_{\rm P} \sim 10^{-5}$. However, in models of hybrid inflation, which end due to the symmetry breaking of a field coupled to the inflaton, and not because of the end of slow-roll, it is possible to decouple the amplitude of CMB fluctuations from the scale of inflation. For instance, consider a hybrid model at the electroweak scale, where the symmetry breaking field is the SM Higgs field, with a vev $v = 246$ GeV, a relatively strong coupling to the inflaton, $g = 0.4$, and a Higgs self-coupling $\lambda = 0.12$, giving rise to the following masses in the true vacuum

$$m_{\rm inf} = g\, v \sim 100 \text{ GeV}\,, \qquad\qquad m_{_{\rm H}} = \sqrt{2\lambda}\, v \sim 120 \text{ GeV}\,, \tag{324}$$

which are much larger than the rate of expansion at the end of inflation

$$H_{\rm end} = \sqrt{\frac{\pi}{3}}\,\frac{m_{_{\rm H}} v}{M_{\rm P}} \sim 2\times 10^{-5} \text{ eV} \ll m_{_{\rm H}}\,, \tag{325}$$

and therefore we can neglect it during the oscillations of the inflaton and Higgs fields around the minimum of their potential.

The energy density at the end of inflation is

$$\rho_{\rm end} = \frac{1}{8} m_{_{\rm H}}^2 v^2 \sim (10^2 \text{ GeV})^4\,, \tag{326}$$

which is very low indeed.

The couplings of the Higgs to matter could be large, e.g. the top quark Yukawa $h_t \sim 1$, although for such a low mass Higgs there is no phase space for top perturbative production. On the other hand, the inflaton may couple to other particles, so it is expected that their decay widths be similar and both of order $\Gamma \sim 1$ GeV. Naively, using (316), one would thus expect that the reheating temperature be $T_{\rm rh} \sim 10^9$ GeV, but that is impossible because it would correspond to an energy density during inflation much above the actual false vacuum energy, $\rho_{\rm end} \sim (10^2\ {\rm GeV})^4$.

Actually, since the rate of expansion is so low compared with the other scales, we can ignore the decay in energy due to the expansion of the universe, which was so important during chaotic inflation, and use energy conservation to estimate

$$\rho_{\rm end} = \frac{\lambda v^4}{4} = \frac{\pi^2}{30}\, g_*\, T_{\rm rh}^4 \quad \Rightarrow \quad T_{\rm rh} \simeq \left(\frac{15\lambda}{2\pi^2 g_*}\right)^{1/4} v \sim 42\ {\rm GeV}\,, \tag{327}$$

where we have used $g_* = 106.75$ as the effective number of degrees of freedom of the SM particles. Note that this temperature is rather low, but in fact we have no observational evidence that the universe has actually gone through a thermal period with a temperature above this.

We are thus left with three time scales:

$$\left.\begin{array}{r} t_{\rm osc} \sim m_{\rm H}^{-1} \sim 10^{-27}\,{\rm s} \\ t_{\rm exp} \sim H^{-1} \sim 10^{-10}\,{\rm s} \\ t_{\rm dec} \sim \Gamma^{-1} \sim 10^{-23}\,{\rm s} \end{array}\right\} \Rightarrow \qquad t_{\rm osc} \ll t_{\rm dec} \ll t_{\rm exp}\,, \tag{328}$$

so there are many oscillations per Hubble time, but contrary to the case of chaotic inflation models, here the decay time is much smaller than the expansion time, because the universe is already quite old, so once the inflaton and Higgs start oscillating they decay very soon via their usual perturbative decay.

Preheating

The previous discussion falls under the name of *perturbative reheating*, because it assumes that the coherently oscillating inflaton will decay as if it were composed on individual inflaton quanta, each one decaying as described by ordinary QFT, with the perturbative decay rate computed above. This was the standard lore during at least a decade since it was first proposed in 1982. However, it was soon realized that the inflaton at the end of inflation is actually a coherent wave, a zero mode, a condensate made out of many inflaton quanta, all oscillating with the same phase, and non-perturbative effects associated with this condensate were bound to be important for the problem of reheating. In fact, a few years ago, in a seminal paper, Linde, Kofman and Starobinsky proposed a new picture of reheating, which has become known as *preheating*. I will describe these new developments in the following sections. They make use of the well studied problem of particle production in the presence of strong background fields, whose formalism we have already encountered for the analysis of the generation of metric fluctuations during inflation. In this case, instead of a quantum field evolving in a rapidly changing gravitational field (like during inflation), we have a field coupled to the inflaton, which has a rapidly changing frequency or mass due to the inflaton oscillations.

We will first describe the Bogolyubov formalism for a single scalar field with a time-dependent mass and then particularize to the case of the inflaton oscillations after inflation. Later on, we will also extended the formalism to fermions, which can also be produced at preheating.

Consider a massive scalar field ϕ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\dot\phi^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\,, \qquad (329)$$

which gives a canonically conjugate momentum $\pi = \dfrac{\delta \mathcal{L}}{\delta \dot{\phi}} = \dot{\phi}$, and the Hamiltonian

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \,. \tag{330}$$

We can treat the fields as quantum fields and define the usual equal time commutation relation

$$[\phi(\mathbf{x}, t),\ \pi(\mathbf{x}', t)] = i\,\delta^{(3)}(\mathbf{x} - \mathbf{x}')\,, \tag{331}$$

as well as expand in Fourier components,

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}}\,\phi_{\mathbf{k}}(t)\,e^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{332}$$

The field mode $\phi_{\mathbf{k}}(t)$ satisfies the harmonic oscillator equation

$$\ddot{\phi}_k + \omega_k^2\,\phi_k = 0\,, \tag{333}$$

$$\omega_k^2(t) = k^2 + m^2(t)\,, \tag{334}$$

where the time dependence of the oscillation frequency comes through that of the mass. We will assume that the field is real, so we should impose the constraint $\phi_{\mathbf{k}}(t) = \phi^*_{-\mathbf{k}}(t)$. Following the quantization condition (331), we can write the field and momentum operators in terms of time-dependent creation and annihilation operators,

$$\begin{aligned} \phi_{\mathbf{k}}(t) &= \frac{1}{\sqrt{2\omega_k}}\Big(a_{\mathbf{k}}(t) + a^\dagger_{-\mathbf{k}}(t)\Big)\,, \\ \pi_{\mathbf{k}}(t) &= -i\sqrt{\frac{\omega_k}{2}}\Big(a_{\mathbf{k}}(t) + a^\dagger_{-\mathbf{k}}(t)\Big)\,, \end{aligned} \tag{335}$$

satisfying the usual commutation relation, $\forall t$,

$$[a_{\mathbf{k}}(t),\ a^\dagger_{\mathbf{k}'}(t)] = \delta^{(3)}(\mathbf{k} - \mathbf{k}')\,,$$

and in terms of which the Hamiltonian becomes

$$\begin{aligned} H &= \frac{1}{2}\int d^3k \left[\pi_{\mathbf{k}}\pi_{\mathbf{k}}^\dagger + \omega_k^2\, \phi_{\mathbf{k}}\phi_{\mathbf{k}}^\dagger\right] \\ &= \frac{1}{2}\int d^3k\, \omega_k \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{\mathbf{k}} a_{\mathbf{k}}^\dagger\right) \equiv H_{\rm part} + H_{\rm vac}(t)\,, \end{aligned} \tag{336}$$

where

$$H_{\rm part} = \int d^3k\, \omega_k\, a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\,, \tag{337}$$

$$H_{\rm vac}(t) = \frac{V}{(2\pi)^3}\int d^3k\, \frac{\omega_k}{2}\,. \tag{338}$$

We can then define a number operator for these fields

$$N = \int d^3k\, a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\,, \tag{339}$$

and a Fock space with vacuum state defined as

$$a_{\mathbf{k}}(t)|0_t\rangle = 0\,, \qquad \langle 0_t|0_t\rangle = 1\,, \tag{340}$$

and particle states $|n_k\rangle \propto (a_{\mathbf{k}}^\dagger)^n|0_t\rangle$ satisfying

$$H_{\rm part}|n_k\rangle = n_k\omega_k|n_k\rangle \equiv E_k|n_k\rangle\,, \tag{341}$$

$$N|n_k\rangle = n_k|n_k\rangle\,. \tag{342}$$

In the vacuum state $|0_t\rangle$, the energy takes its lowest possible value, $H_{\rm vac}(t) = \langle 0_t|H|0_t\rangle$.

We can compute the equations of motion as usual with

$$\frac{d}{dt}a_{\mathbf{k}} = \frac{\partial a_{\mathbf{k}}}{\partial t} + i\,[H, a_{\mathbf{k}}]$$

where we can invert the relations (335)

$$\begin{aligned} a_{\mathbf{k}}(t) &= \sqrt{\frac{\omega_k}{2}}\phi_{\mathbf{k}}(t) + \frac{i}{\sqrt{2\omega_k}}\pi_{\mathbf{k}}(t)\,, \\ a_{-\mathbf{k}}^\dagger(t) &= \sqrt{\frac{\omega_k}{2}}\phi_{\mathbf{k}}(t) - \frac{i}{\sqrt{2\omega_k}}\pi_{\mathbf{k}}(t)\,. \end{aligned} \tag{343}$$

In the Heisenberg picture, the original canonical operators $\{\phi_k, \pi_k\}$ may have no explicit time-dependence, but ω_k is indeed time-dependent, so

$$\frac{d}{dt}a_{\mathbf{k}} = -i\omega_k a_{\mathbf{k}} + \frac{\dot{\omega}_k}{2\omega_k}a^{\dagger}_{-\mathbf{k}}\,. \tag{344}$$

The solution to the equations of motion is

$$\begin{pmatrix} a_{\mathbf{k}}(t) \\ a^{\dagger}_{-\mathbf{k}}(t) \end{pmatrix} = \begin{pmatrix} u_k(t) & v_k(t) \\ v_k^*(t) & u_k^*(t) \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}}(0) \\ a^{\dagger}_{-\mathbf{k}}(0) \end{pmatrix} \tag{345}$$

The unitary evolution preserves the commutation relation (331) iff

$$|u_k|^2 - |v_k|^2 = 1\,, \tag{346}$$

$$\text{with initial condition :} \qquad |u_k|^2 = 1\,, \qquad |v_k|^2 = 0\,. \tag{347}$$

If the initial state is the vacuum, $|0\rangle \equiv |0_{t=0}\rangle$, then

$$a_{\mathbf{k}}(0)|0\rangle = 0 \qquad \Rightarrow \qquad a_{\mathbf{k}}(t)|0\rangle = v_k(t)\, a^{\dagger}_{-\mathbf{k}}(0)|0\rangle \neq 0 \tag{348}$$

In particular, the number density of particles created from the vacuum is

$$n(t) = \frac{1}{V}\langle 0|N|0\rangle = \int \frac{d^3k}{(2\pi)^3}\,|v_k|^2(t)\,. \tag{349}$$

In order to find the function $n(t)$ explictly, we have to solve for u_k and v_k as a solution of

$$\begin{pmatrix} \dot{u}_k(t) \\ \dot{v}_k^*(t) \end{pmatrix} = \begin{pmatrix} -i\omega_k & \dfrac{\dot{\omega}_k}{2\omega_k} \\ \dfrac{\dot{\omega}_k}{2\omega_k} & i\omega_k \end{pmatrix} \begin{pmatrix} u_k(t) \\ v_k^*(t) \end{pmatrix} \tag{350}$$

It is customary to write the mode functions u_k and v_k is terms of the usual Bogolyubov coefficients, $\{\alpha_k,\ \beta_k\}$,

$$u_k = \alpha_k\, e^{-i\displaystyle\int^t \omega_k dt}\,, \qquad v_k^* = \beta_k\, e^{+i\displaystyle\int^t \omega_k dt}\,, \tag{351}$$

then the evolution equations (350) become

$$
\begin{aligned}
\dot\alpha_k &= \frac{\dot\omega_k}{2\omega_k}\,\beta_k\, e^{+2i\int^t \omega_k dt}\,,\\
\dot\beta_k &= \frac{\dot\omega_k}{2\omega_k}\,\alpha_k\, e^{-2i\int^t \omega_k dt}\,,
\end{aligned}
\tag{352}
$$

which can be integrated in the adiabatic approximation, to give

$$
n(t) = \int \frac{d^3k}{(2\pi)^3}\, n_k(t) = \int \frac{d^3k}{(2\pi)^3}\, |\beta_k|^2(t)\,, \tag{353}
$$

the number density of particles produced due to the time-dependent background field.

Alternatively, one can introduce the $|in\rangle$ and $|out\rangle$ states, and make the field decomposition over time-independent creation and annihilation operators $\{a_{\mathbf{k}},\ a^\dagger_{-\mathbf{k}}\}$,

$$
\begin{aligned}
\phi(\mathbf{x},t) &= \int \frac{d^3k}{(2\pi)^{3/2}}\Big[f_k(t)\,a_{\mathbf{k}}\,e^{i\mathbf{k}\cdot\mathbf{x}} + h.c.\Big]\\
&= \int \frac{d^3k}{(2\pi)^{3/2}}\Big(f_k(t)\,a_{\mathbf{k}} + f_k^*(t)\,a^\dagger_{-\mathbf{k}}\Big)\, e^{i\mathbf{k}\cdot\mathbf{x}}\,,
\end{aligned}
\tag{354}
$$

$$
\pi(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}}\Big(g_k(t)\,a_{\mathbf{k}} + g_k^*(t)\,a^\dagger_{-\mathbf{k}}\Big)\, e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{355}
$$

where the mode functions $f_k(t)$ and $g_k(t)$ depend only on the modulus $k = |\mathbf{k}|$, thanks to the homogeneity and isotropy of the background fields. These functions satisfy the equations of motion

$$
\ddot f_k + \omega_k^2\, f_k = 0\,, \qquad g_k = i\dot f_k\,. \tag{356}
$$

Comparing with the former decomposition (335), we find the relation

$$
\begin{aligned}
u_k &= \frac{1}{\sqrt{2\omega_k}}(\omega_k f_k + g_k)\,,\\
v_k &= \frac{1}{\sqrt{2\omega_k}}(\omega_k f_k - g_k)\,,
\end{aligned}
\tag{357}
$$

and viceversa

$$
\begin{aligned}
f_k &= \frac{1}{\sqrt{2\omega_k}}(u_k + v_k^*)\,, \\
g_k &= \sqrt{\frac{\omega_k}{2}}(u_k - v_k^*)\,,
\end{aligned}
\tag{358}
$$

which gives for the occupation number

$$
n_k(t) = |\beta_k|^2 = \frac{1}{2\omega_k}|\dot{f}_k|^2 + \frac{\omega_k}{2}|f_k|^2 - \frac{1}{2}\,, \tag{359}
$$

where we have used the Wronskian

$$
i(\dot{f}_k f_k^* - \dot{f}_k^* f_k) = 2\,\mathrm{Re}\,(f_k^* g_k) = 1 \quad \Leftrightarrow \quad |u_k|^2 - |v_k|^2 = 1\,. \tag{360}
$$

Diagonalization of the Hamiltonian

With the above decomposition, we can write the Hamiltonian as

$$H = \int d^3k \, \Big[E_k(t) \Big(a^\dagger_{\mathbf{k}} \, a_{\mathbf{k}} + a_{\mathbf{k}} \, a^\dagger_{\mathbf{k}} \Big) \\ + F_k(t) \, a_{\mathbf{k}} \, a_{-\mathbf{k}} + F^*_k(t) \, a^\dagger_{\mathbf{k}} \, a^\dagger_{-\mathbf{k}} \Big] \,, \tag{361}$$

where

$$E_k(t) = \frac{1}{2}(|\dot{f}_k|^2 + \omega_k^2 |f_k|^2) = \omega_k \left(n_k + \frac{1}{2} \right) , \tag{362}$$

$$F_k(t) = \frac{1}{2}(\dot{f}_k^2 + \omega_k^2 f_k^2) \,, \tag{363}$$

$$E_k^2(t) - |F_k(t)|^2 = \frac{\omega_k^2}{4} \,. \tag{364}$$

Let us now introduce a canonical Bogolyubov transformation

$$\begin{pmatrix} a_{\mathbf{k}} \\ a^\dagger_{-\mathbf{k}} \end{pmatrix} = \begin{pmatrix} u_k(t) & v_k(t) \\ v^*_k(t) & u^*_k(t) \end{pmatrix} \begin{pmatrix} b_{\mathbf{k}} \\ b^\dagger_{-\mathbf{k}} \end{pmatrix} \tag{365}$$

Then

$$n_k = |\beta_k|^2 = \frac{2E_k - \omega_k}{2\omega_k} \,, \tag{366}$$

$$\frac{u_k}{v_k} = \frac{2E_k + \omega_k}{2F^*_k} \,, \tag{367}$$

and the Hamiltonian becomes diagonal

$$H = \int d^3k \, \frac{\omega_k}{2} \left(b^\dagger_{\mathbf{k}} \, b_{\mathbf{k}} + b_{\mathbf{k}} \, b^\dagger_{\mathbf{k}} \right) , \tag{368}$$

which can be decomposed into $H_{\rm part}$ and $H_{\rm vac}$, as before, see (337).

The Schrödinger picture

We can define the unitary evolution operator $U^\dagger(t) = U^{-1}(t)$, where $i\hbar\,\partial_t U(t) = H\,U(t)$, such that time evolution determines

$$a_{\mathbf{k}}(t) = U^\dagger(t) a_{\mathbf{k}}(0) U(t)\,. \tag{369}$$

The solution of the Schrödinger equation is the squeezed state

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle\,. \tag{370}$$

The vacuum at time t is given by $|0_t\rangle = U^\dagger(t)|0\rangle$. Let $|\psi(0)\rangle$ be the initial vacuum state $|0\rangle$. Then the operator

$$b_{\mathbf{k}}(t) = U(t) a_{\mathbf{k}}(0) U^\dagger(t) = u_k^*\, a_{\mathbf{k}}(0) - v_k\, a^\dagger_{-\mathbf{k}}(0) \tag{371}$$

annihilates the state $|\psi(t)\rangle$. Now let us use (343) to evaluate

$$a_{\mathbf{k}}(0) = \sqrt{\frac{\omega_k}{2}}\,\phi_k(0) + \frac{i}{\sqrt{2\omega_k}}\,\pi_k(0)\,, \tag{372}$$

and substitute into $b_{\mathbf{k}}(t)|\psi(t)\rangle = 0$,

$$\frac{1}{\sqrt{2\omega_k}}\left[(u_k^* - v_k)\omega_k\,\phi_k(0) + i(u_k^* + v_k)\,\pi_k(0)\right]|\psi(t)\rangle = 0.$$

Therefore, the evolved state satisfies the Schrödinger equation

$$[\Omega_k(t)\phi_k(0) + i\pi_k(0)]|\psi(t)\rangle = 0\,, \tag{373}$$

$$\Omega_k(t) \equiv \omega_k \frac{u_k^* - v_k}{u_k^* + v_k} = \frac{g_k^*}{f_k^*} = \frac{1 - 2iF_k(t)}{2|f_k(t)|^2}\,, \tag{374}$$

where we have used $g_k^* f_k = \mathrm{Re}\,(g_k^* f_k) - i\,\mathrm{Im}\,(f_k^* g_k) = \frac{1}{2} - i\,\mathrm{Re}\,(f_k^* \dot{f}_k)$. Using the operator definition $\pi_k = -i\frac{\partial}{\partial\phi_{-\mathbf{k}}} = -i\frac{\partial}{\partial\phi_k^*}$, we find the solution

$$\psi(\phi_k, \phi_k^*, t) \sim e^{-\Omega_k(t)|\phi_k|^2}\,, \tag{375}$$

$$P(\phi_k, \phi_k^*, t) = |\psi(\phi_k, \phi_k^*, t)|^2 \sim e^{-\frac{1}{|f_k(t)|^2}|\phi_k|^2}\,. \tag{376}$$

The phase $F_k(t) = \mathrm{Re}\,(f_k^* \dot{f}_k) \gg 1$ quickly becomes very large during preheating, which ensures that the state becomes a squeezed state, with large occupation numbers, described by the Gaussian distribution (376).

Parametric resonance

We will consider here the case of a scalar field χ coupled to the inflaton ϕ with coupling $\frac{1}{2}g^2\phi^2\chi^2$, which induces an oscillating mass term

$$m_\chi^2(t) = m_\chi^2 + g^2\phi^2(t)\,. \tag{377}$$

The inflaton is assumed to oscillate like (301) with a frequency given by its mass m, not necessarily much larger than the "bare" mass of the field χ. In that case, the frequency can be written as

$$\omega_k^2(t) = k^2 + m_\chi^2 + g^2\Phi^2(t)\sin^2 mt\,, \tag{378}$$

and the mode equation (356) can be written as a Mathieu equation, where $z = mt$, and primes denote differentiation w.r.t. z,

$$f_k'' + (A_k - 2q\,\cos 2z)\,f_k = 0\,, \tag{379}$$

$$A_k = \frac{k^2 + m_\chi^2}{m^2} + 2q\,, \qquad\qquad q = \frac{g^2\Phi^2(t)}{4m^2}\,. \tag{380}$$

The Mathieu equation is part of a large class of Hill equations (which includes also the Lamé equation and many others) that present unstable solutions for certain values of the momenta for a given set of parameters $\{A_k, q\}$, with $A \geq 2q$. These solutions fall into bands of instability that are narrow for small values of the resonant parameter $q \leq 1$, but can be very broad for larger values of q.

The solutions to the Mathieu eq. have the form $f_k(z) = e^{\mu_k z}\,p(z)$, with μ_k the Floquet index, characterizing the exponential instability, and typically much smaller than one, although it could be as large as $\mu_{\rm max} = 0.28055$; and where $p(z)$ is a periodic function of z. The occupation number can then be computed to be

$$n_k(t) \sim e^{2\mu_k\,mt}\,, \tag{381}$$

which can grow significantly in a few oscillations, if the growth index μ_k is not totally negligible.

The effect of parametric resonance is similar to the lasing effect (or light amplification by stimulated emission of radiation), where a large population of particles is produced from oscillations of a coherent source.

Narrow resonance

Let us consider first the case where $m_\chi,\ g\Phi \ll m$, or $q \ll 1$. Then the Mathieu equation instability chart shows that the resonance occurs only in some narrow bands around $A_k \simeq l^2\,,\ l = 1, 2, \ldots$, with widths in momentum space of order $\Delta k \sim m\, q^l$; so, for $q < 1$, the most important band is the first one, $A_k \sim 1 \pm q$, centered around $k = m/2$.

The growth factor μ_k for the first instability band is given by

$$\mu_k = \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{2k}{m} - 1\right)^2}\,. \tag{382}$$

The resonance occurs for k within the range $\frac{m}{2}(1 \pm \frac{q}{2})$. The index μ_k vanishes at the edges of the resonance band and takes its maximum value $\mu_k = \frac{q}{2}$ at $k = \frac{m}{2}$. The corresponding modes grow at a maximal rate $\chi_k \sim \exp(qz/2)$. This leads to a growth of the occupation numbers (381) as $n_k \sim \exp(qmt)$.

We can interpret this as follows. In the limit $q \ll 1$, the effective mass of the χ particles is much smaller than m, and each decaying ϕ particle creates two χ particles with momentum $k \sim m/2$. The difference with respect to the perturbative decay $\Gamma(\phi \to \chi\chi)$ is that, in the regime of parametric resonance, the rate of production of χ particles is proportional to the number of particles produced earlier (which gives rise to an exponential growth in time). This is a non-perturbative effect, as we will discuss later, and we could not have obtained it by using the methods described in the previous section, at any finite order of perturbation theory with respect to the interaction term $g^2\Phi^2 \sin^2 mt$. It is by solving the mode equation (379) *exactly* that we have found this result.

Note that only a very narrow range of modes grow exponentially with time, so the spectrum of particles is dominated by these modes, while the rest are still in the vacuum, produced only through the ordinary perturbative decay process. Of course, the exponential production does not last for ever: the universe expansion is going to affect the resonant production of particles in two ways, leading to the end of the narrow resonance regime.

First, the time-dependent amplitude of oscillations $\Phi(t)$, which determines q, see (380), not only decays ($\sim t^{-1}$) due to the expansion of the universe, but also due to the perturbative decay of the inflaton field, $\Phi(t) \sim \exp(\Gamma_\phi t/2)$. Therefore, the narrow resonance will end when the usual perturbative decay becomes important, i.e. when $q\,m < \Gamma_\phi$.

Second, in the evolution equation (379), the momenta k are actually physical momenta, which redshift with the scale factor as $k_{\rm phys} = k/a$, and therefore, even if a given mode is initially within the narrow band, $\Delta k \sim q\,m$, it will very quickly redshift away from it, within the time scale $\Delta t \sim q\,H^{-1}$, preventing its occupation numbers (381) from growing exponentially. Thus, the narrow resonance will end when $q^2\,m < H$.

Therefore, if the amplitude of inflaton oscillations decays like $\Phi \sim 1/t$, there will always be a time (typically a dozen oscillations) for which one of the two conditions above will hold and the narrow resonance will end.

Broad resonance

If the *initial* amplitude of oscillations is very large, like in models of chaotic inflation, in which $\Phi_0 \sim M_{\rm P}/10$ and $m \sim 10^{-6} M_{\rm P}$, then the initial q-parameter could be very large,

$$q_0 = \frac{g^2 \Phi_0^2}{4m^2} \sim g^2\, 10^{10} \;\lesssim\; 10^4 \,, \tag{383}$$

where we have used the constraint due to radiative corrections (319). In this case, the χ particle production due to stimulated emission by the oscillating inflaton field can be very efficient as it enters into the broad resonance regime.

Particles are produced only at the instances of maximum acceleration of the inflaton field, when $\phi(t) \sim 0$, and

$$\left|\frac{\dot\omega_k}{\omega_k^2}\right| \gg 1 \,, \tag{384}$$

a relation known as the *non-adiabaticity condition*. When it holds, we cannot define a proper Fock space for the χ particles, and the occupation numbers of those particles grow very quickly. We thus associate (384) with particle production.

We will now describe how to compute the growth of modes and the Floquet index in this regime, using the formalism developed above. We can expand the quantum field χ in Fourier components f_k satisfying the mode equation (356) with time-dependent frequency (378) and initial conditions

$$f_k(0) = \frac{1}{\sqrt{2\omega_k}}\, e^{-i\omega_k t} \,, \qquad g_k(0) = i\dot f_k(0) = \omega_k\, f_k(0) \,, \tag{385}$$

whose evolution in terms of the Bogolyubov coefficients is

$$f_k(t) = \frac{\alpha_k(t)}{\sqrt{2\omega_k}}\, e^{-i\int \omega_k dt} + \frac{\beta_k(t)}{\sqrt{2\omega_k}}\, e^{+i\int \omega_k dt}\,, \tag{386}$$

$$\alpha_k(0) = 1\,, \qquad\qquad \beta_k(0) = 0\,. \tag{387}$$

And the occupation numbers are

$$n_k(t) = |\beta_k(t)|^2 = \frac{1}{2\omega_k}|\dot{f}_k|^2 + \frac{\omega_k}{2}|f_k|^2 - \frac{1}{2}\,. \tag{388}$$

The inflaton field has maximum acceleration at $t = t_j = j\pi/m$, such that $\sin mt_j = 0$. Between t_j and t_{j+1}, the amplitude $\phi(t) \approx \phi_0 =$ const, so that the frequency $\omega_k(t)$ is approximately constant between succesive zeros of the inflaton, and we can properly define a Fock space for χ. At t_j, the amplitude changes rapidly, such that (384) is satisfied and we cannot define an adiabatic invariant like the occupation number (388). Therefore, let us study the behaviour of the modes χ_k precisely at those instances $t = t_j$. We can expand the time-dependent frequency (378) around those points (where the frequency has a minimum) as

$$\omega_k^2(t) = \omega_k^2(t_j) + \frac{1}{2}\omega_k^{2\prime\prime}(t_j)(t-t_j)^2 + \cdots \tag{389}$$

and make the change of variables

$$\eta \equiv [2\omega_k^{2\prime\prime}(t_j)]^{1/4}(t-t_j)\,, \tag{390}$$

$$\kappa^2 \equiv \frac{\omega_k^2(t_j)}{\sqrt{2\omega_k^{2\prime\prime}(t_j)}} = \frac{k^2+m_\chi^2}{2gm\Phi} = \frac{A_k - 2q}{4\sqrt{q}}\,. \tag{391}$$

The mode equation (356) around $t = t_j$ then becomes

$$\frac{d^2 f_k}{d\eta^2} + \left(\kappa^2 + \frac{\eta^2}{4}\right) f_k = 0\,, \tag{392}$$

which can be interpreted as a Schrödinger equation for a wave function scattering in an inverted parabolic potential. The exact solutions are parabolic cylinder functions, $W(-\kappa^2, \pm\eta)$, whose asymptotic expressions are well known. Thus we have substituted the problem of parametric resonance after chaotic inflation with that of partial waves scattering off successive inverted parabolic potentials.

Let the wave $f_k(t)$ have the form of the adiabatic solution (386) before scattering at t_j,

$$f_k^j(t) = \frac{\alpha_k^j}{\sqrt{2\omega_k}}\, e^{-i\int \omega_k dt} + \frac{\beta_k^j}{\sqrt{2\omega_k}}\, e^{+i\int \omega_k dt}\,, \tag{393}$$

where the coefficients $\{\alpha_k^j,\ \beta_k^j\}$ are constant, for $t_{j-1} < t < t_j$. After scattering off the potential at t_j, the wave $f_k(t)$ takes the form

$$f_k^{j+1}(t) = \frac{\alpha_k^{j+1}}{\sqrt{2\omega_k}}\, e^{-i\int \omega_k dt} + \frac{\beta_k^{j+1}}{\sqrt{2\omega_k}}\, e^{+i\int \omega_k dt}\,, \tag{394}$$

where the coefficients $\{\alpha_k^{j+1},\ \beta_k^{j+1}\}$ are again constant, for $t_j < t < t_{j+1}$. These are essentially the asymptotic expressions for the incoming and the outgoing waves, scattered at t_j. Therefore, the outgoing amplitudes $\{\alpha_k^{j+1},\ \beta_k^{j+1}\}$ can be expressed in terms of the incoming amplitudes $\{\alpha_k^j,\ \beta_k^j\}$ with the help of the reflection R_k and transmission D_k coefficients of scattering at t_j,

$$\begin{pmatrix} \alpha_k^{j+1} e^{-i\theta_k^j} \\ \\ \beta_k^{j+1} e^{+i\theta_k^j} \end{pmatrix} = \begin{pmatrix} \frac{1}{D_k} & \frac{R_k^*}{D_k^*} \\ \\ \frac{R_k}{D_k} & \frac{1}{D_k^*} \end{pmatrix} \begin{pmatrix} \alpha_k^j e^{-i\theta_k^j} \\ \\ \beta_k^j e^{+i\theta_k^j} \end{pmatrix} \tag{395}$$

where $\theta_k^j = \displaystyle\int_0^{t_j} \omega_k(t) dt$, and

$$\left.\begin{aligned} R_k &= -i\, e^{-i\phi_k}\, [1 + e^{2\pi\kappa^2}]^{-1/2}\,, \\ D_k &= e^{-i\phi_k}\, [1 + e^{-2\pi\kappa^2}]^{-1/2}\,, \end{aligned}\right\} \qquad |R_k|^2 + |D_k|^2 = 1\,. \tag{396}$$

The k-dependent angle of scattering is

$$\phi_k = \mathrm{Arg}\,\Gamma\left[\frac{1}{2} + i\kappa^2\right] + \kappa^2(1 - \ln\kappa^2)\,. \tag{397}$$

Simplifying (395), we find

$$\begin{pmatrix} \alpha_k^{j+1} \\ \\ \beta_k^{j+1} \end{pmatrix} = \begin{pmatrix} [1 + e^{-2\pi\kappa^2}]^{1/2} e^{i\phi_k} & i\, e^{-\pi\kappa^2 + 2i\theta_k} \\ \\ -i\, e^{-\pi\kappa^2 - 2i\theta_k} & [1 + e^{-2\pi\kappa^2}]^{1/2} e^{-i\phi_k} \end{pmatrix} \begin{pmatrix} \alpha_k^j \\ \\ \beta_k^j \end{pmatrix} \tag{398}$$

and therefore, using $n_k^j = |\beta_k^j|^2$ and $|\alpha_k^j|^2 |\beta_k^j|^2 = n_k^j(n_k^j + 1)$, we have

$$\begin{aligned} n_k^{j+1} &= e^{-2\pi\kappa^2} + (1 + 2e^{-2\pi\kappa^2})\, n_k^j \\ &- 2e^{-\pi\kappa^2}[1 + e^{-2\pi\kappa^2}]^{1/2}[n_k^j(n_k^j + 1)]^{1/2} \sin\theta_{\rm tot}^j\,, \end{aligned} \tag{399}$$

where $\theta_{\rm tot}^j = 2\theta_k^j - \phi_k + \mathrm{Arg}\,\beta_k^j - \mathrm{Arg}\,\alpha_k^j$.

This expression is very enlightening. Let us describe its properties:

- **Step-like**. The number of created particles is a step-like function of time. The occupation number between successive scatterings is constant. In the first scattering (when $n_k^0 = 0$), we have

$$n_k = e^{-2\pi\kappa^2} = e^{-\frac{\pi k^2}{gm\Phi}} < 1\,. \tag{400}$$

- **Non-perturbative**. The occupation number (400) cannot be expanded perturbatively, for small coupling, because the function $e^{-1/g}$ is non-analytical at $g = 0$. This is the form that most non-perturbative effects take in Quantum Mechanics.

- **Infrared effect**. For large momenta, the occupation number decays exponentially, so even if there are bands at low momenta, i.e. in the IR region, the high momentum modes will not be populated,

$$\kappa^2 \gg \pi^{-1} \quad \Rightarrow \quad n_k^{j+1} \simeq n_k^j \simeq 0\,. \tag{401}$$

- **Non-linear**. For small momenta one may have production of particles with mass *greater* than that of the inflaton:

$$\kappa^2 = \frac{k^2 + m_\chi^2}{2gm\Phi_0} \lesssim \pi^{-1} \;\Rightarrow\; n_k \text{ large if } m^2 < m_\chi^2 \ll gm\Phi_0 \quad (402)$$

- **Exponential boson production**. In the case of bosons (we will discuss the fermionic case later), the occupation number can grow exponentially due to Bose-Einstein statistics, $n_k \sim \exp(2\mu_k z) \gg 1$,

$$n_k^{j+1} \simeq [(1+2e^{-2\pi\kappa^2}) - 2e^{-\pi\kappa^2}[1+e^{-2\pi\kappa^2}]^{1/2} \sin\theta_{\rm tot}^j]\, n_k^j \equiv e^{2\pi\mu_k^j}\, n_k^j$$

which allows one to estimate the Floquet index μ_k.

- **Resonant production**. Valid only for periodic sources. If scattering occurs in phase, the incoming and outgoing waves add up constructively, and we can have resonant effects. This occurs when $\theta_{\rm tot}^j$ is a semi-integer multiple of π. In that case, it is possible that, for some modes, $n_k^{j+1} > n_k^j$. This gives rise to a particular band structure.

- **Stochastic preheating**. It may happen that the phase a mode has acquired in a given scattering exactly compensates for the universe expansion in that interval and the phases destructively interfere, *decreasing* the number of particles in that mode. This gives rise to a stochastic growth of particles, where approximately 3/4 of the time the particle number increases.

- **Band structure**. Different models of inflation give rise to different evolution laws for the amplitude of inflaton oscillations, and therefore to different mode equations (356). The corresponding Hill equations (linear second order differential equations with periodic coefficients) can have quite different band structures, e.g. those of Mathieu or Lamé equations.

Even if we compute the complete band structure of the Mathieu or Lamé equation and we determine the growth factors μ_k with great accuracy, the universe expansion will shift any given mode from one band to the next, as the mode redshifts and the amplitude of inflaton oscillations decreases: A mode starts in a given band, its occupation numbers increase exponentially through several oscillations, and suddenly it falls out of the band, until the expansion makes it fall into the next band, and so on until it reaches the narrow resonance regime described above.