Friedmann Models

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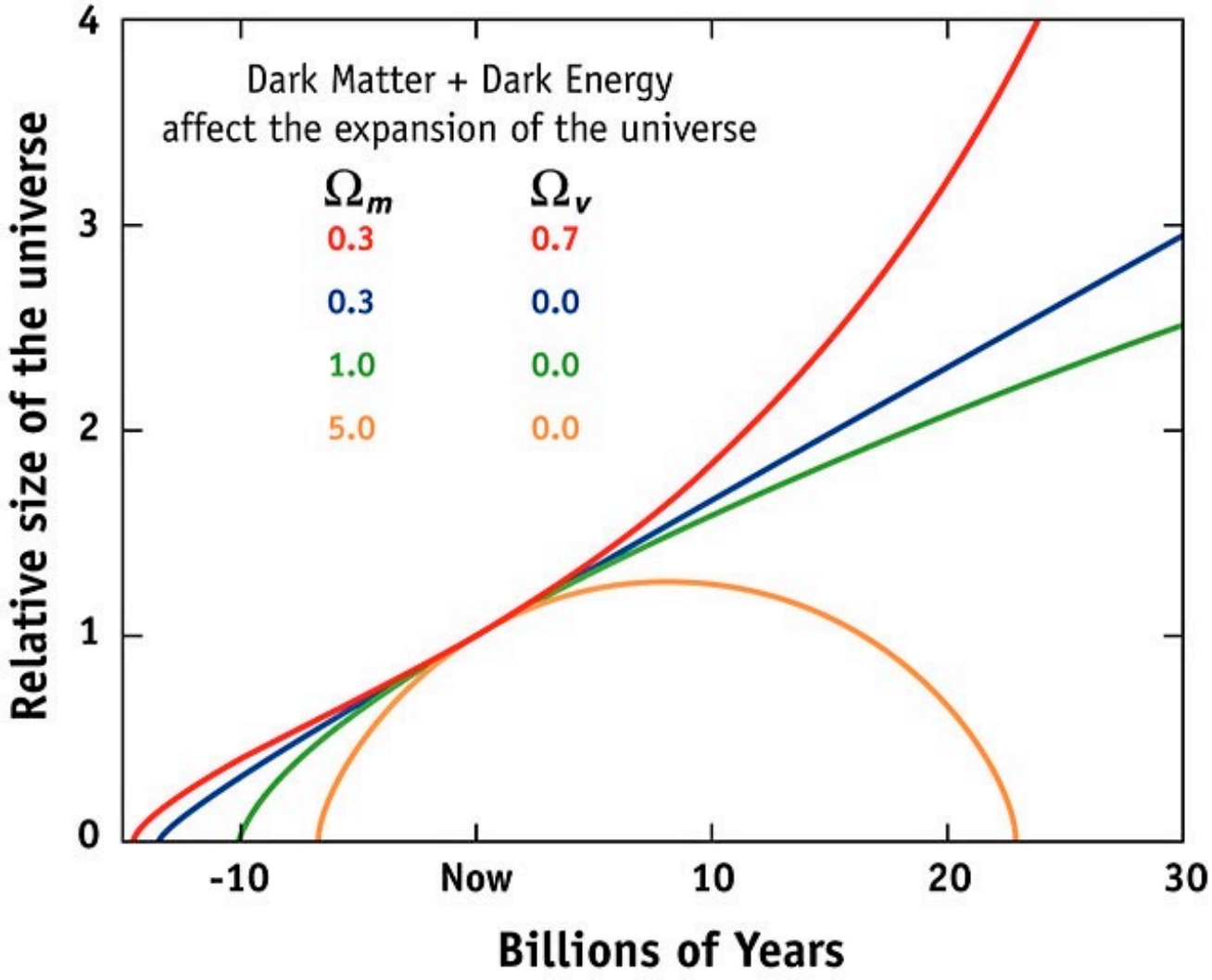
Friedmann Models

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$



Friedmann Models

- Robertson-Walker-Metric

- Friedmann Equations

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- Robertson-Walker-Metric

- Friedmann Equations $\longrightarrow$ Friedmann derived equations for $R(t)$ in 1922, but did not use the most general metric

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- Robertson-Walker-Metric $\longrightarrow$ Robertson (1935) & Walker (1935): derivation of most general metric

- Friedmann Equations $\longrightarrow$ Friedmann derived equations for $R(t)$ in 1922, but did not use the most general metric

- Friedmann Models

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- Robertson-Walker-Metric
  - derivation
  - geodesics
  - redshift
- Friedmann Equations
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  - **derivation**
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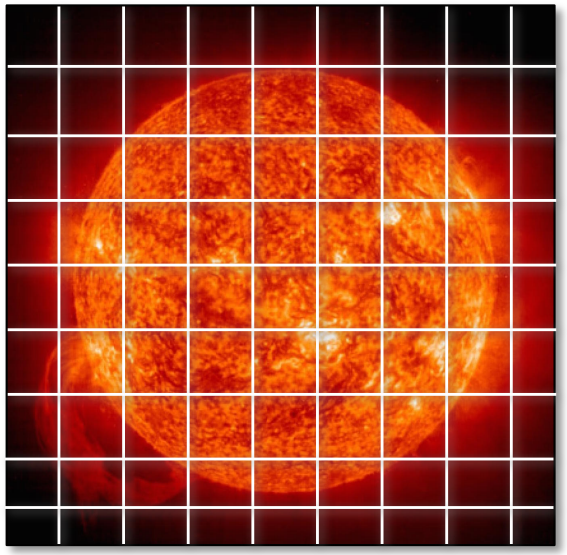
- <u>principle of general relativity</u>

  - the equations describing the laws of physics have the same form *irrespective of the coordinate system*



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    => fulfilled by Einstein's field equations $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$

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we require a metric $g_{\mu\nu}$

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  - space is homogeneous and isotropic

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  - space is *homogeneous* and *isotropic*
    ? ?

- cosmological principle
  - isotropy ?
  - homogeneity ?

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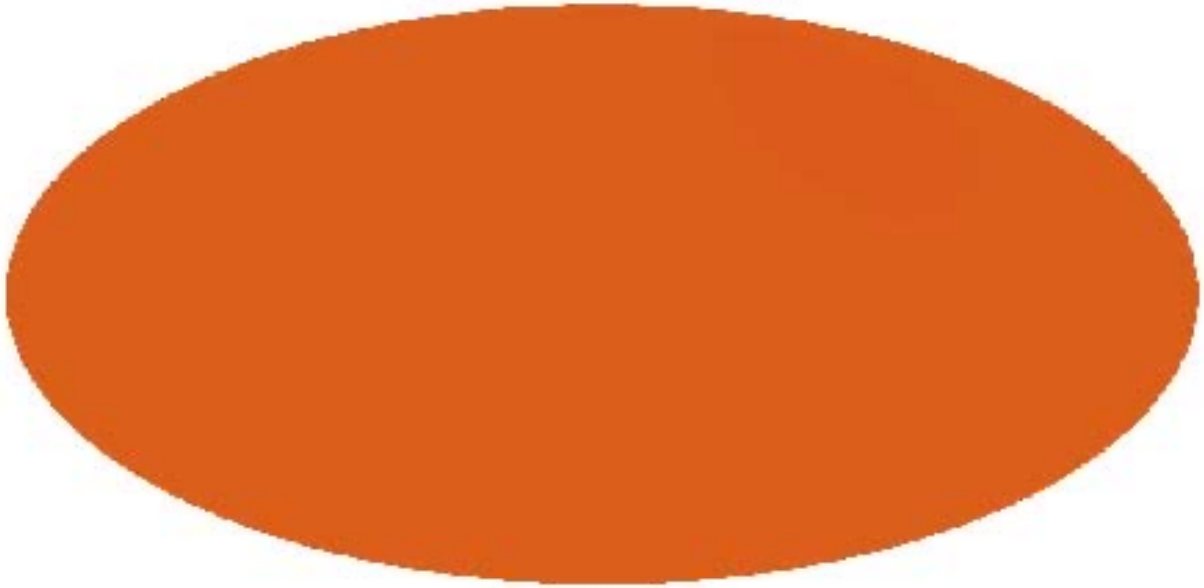
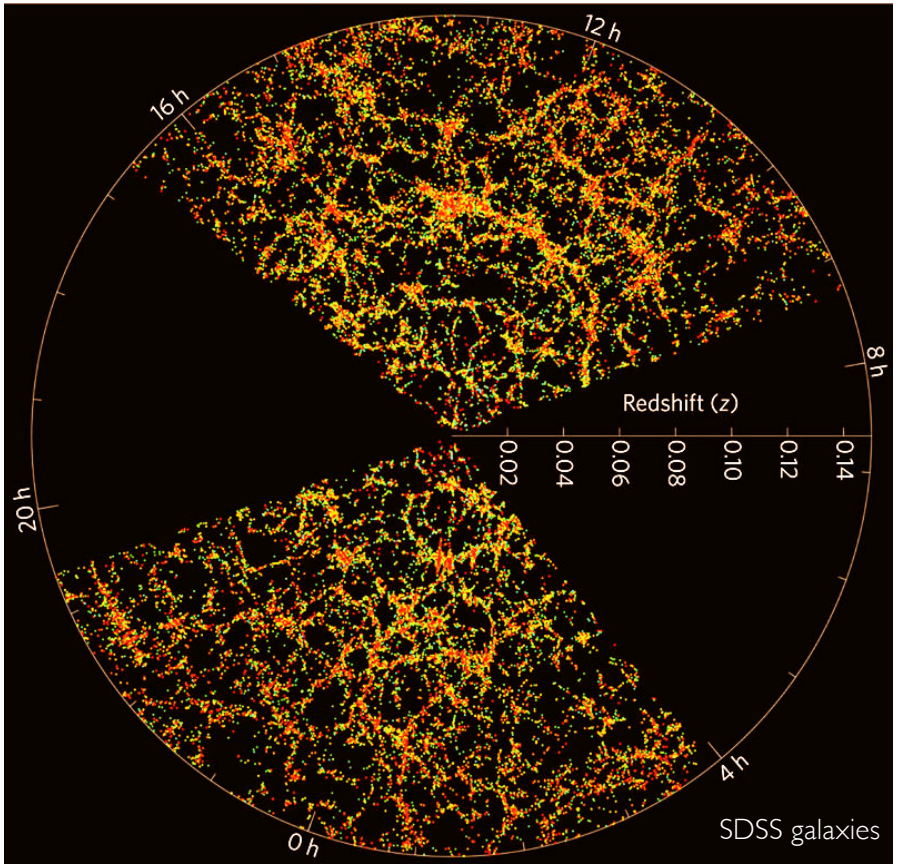
  - isotropy <=> ***CMB***
  - homogeneity ?



T = 2.725K

- cosmological principle
  - isotropy <=> CMB
  - homogeneity <=> ***galaxy surveys****



* for instance, WiggleZ Dark Energy Survey (arXiv:1205.6812) reported R_H~75 Mpc/h

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- <u>cosmological principle</u> => fulfilled by Robertson-Walker-Metric

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- metric in general

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

with x^μ = vector

$g_{\mu\nu}$ = metric tensor

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example for 3D Euclidean space: $ds^2 = dx^2 + dy^2 + dz^2 \quad \Leftrightarrow \quad g_{\mu\nu} = 1$

- metric of curved space-time

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

with $x^\mu = (ct, x^i)$ = space-time vector

$$g_{\mu\nu} = \begin{pmatrix} & & & \\ & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}$$ = metric tensor

Note: Greek-index => four-vector (space-time),
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any non-trivial g_{00} component can be absorbed be re-defining time $dt' = \sqrt{g_{00}} dt$

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$g_{0i} = 0$ (if time depended on space, measurement of time could distinguish one place from another…)

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γ_{ij} should describe a homogeneous and isotropic 3D space

Note: "space" is maximally symmetric, but "space-time" not…

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$\Updownarrow$

maximal symmetry for the 4D hypersurface described by x^i

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  - 1D space ≡ hypersurface in 2D space

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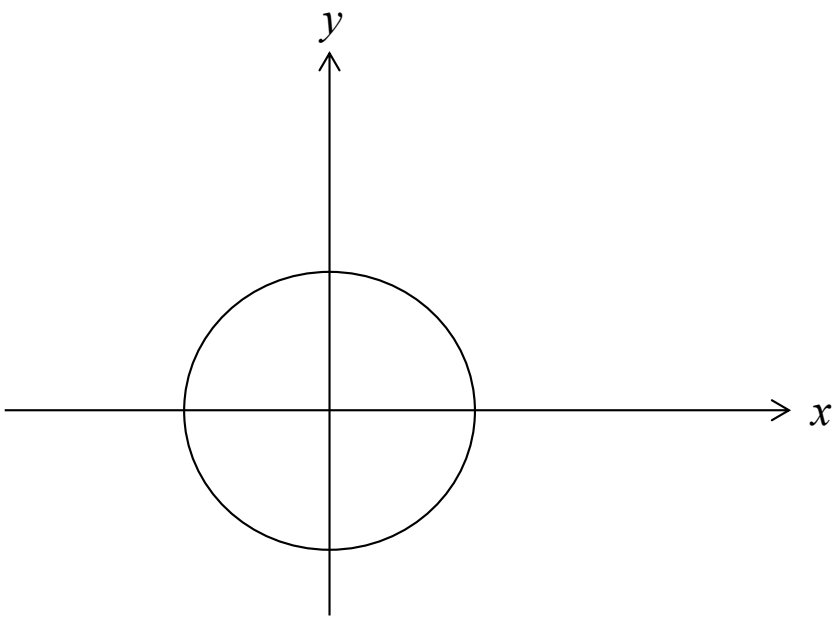

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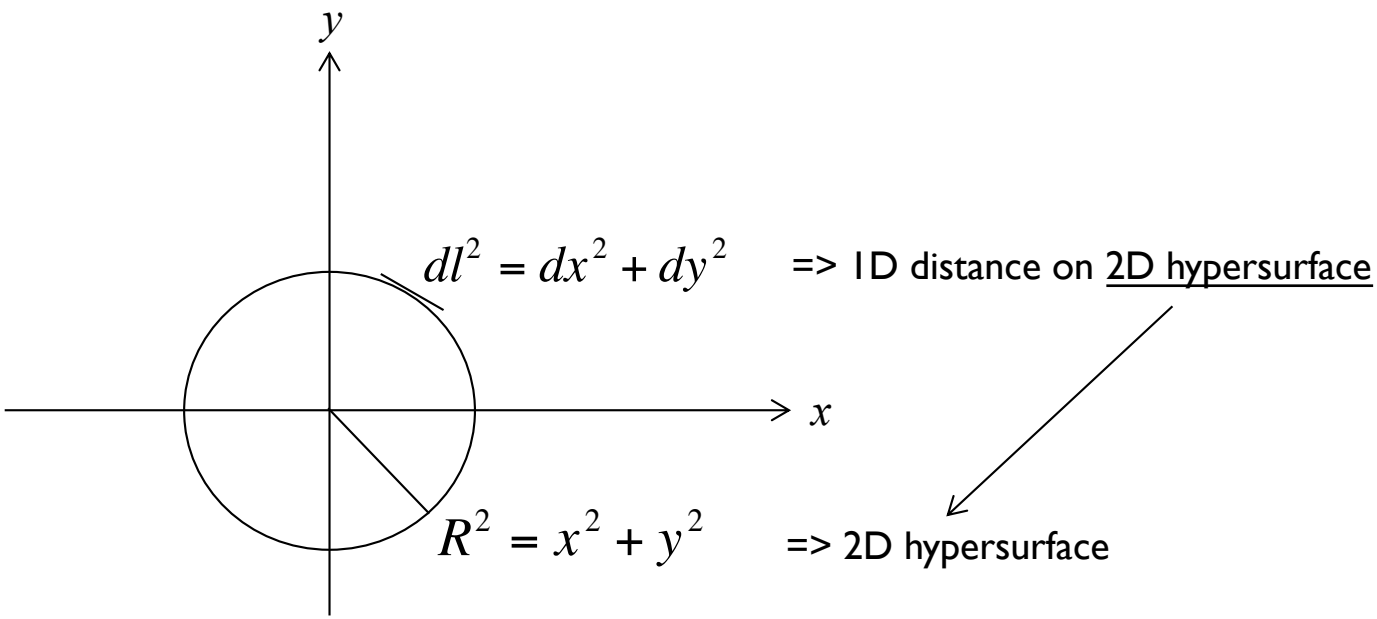

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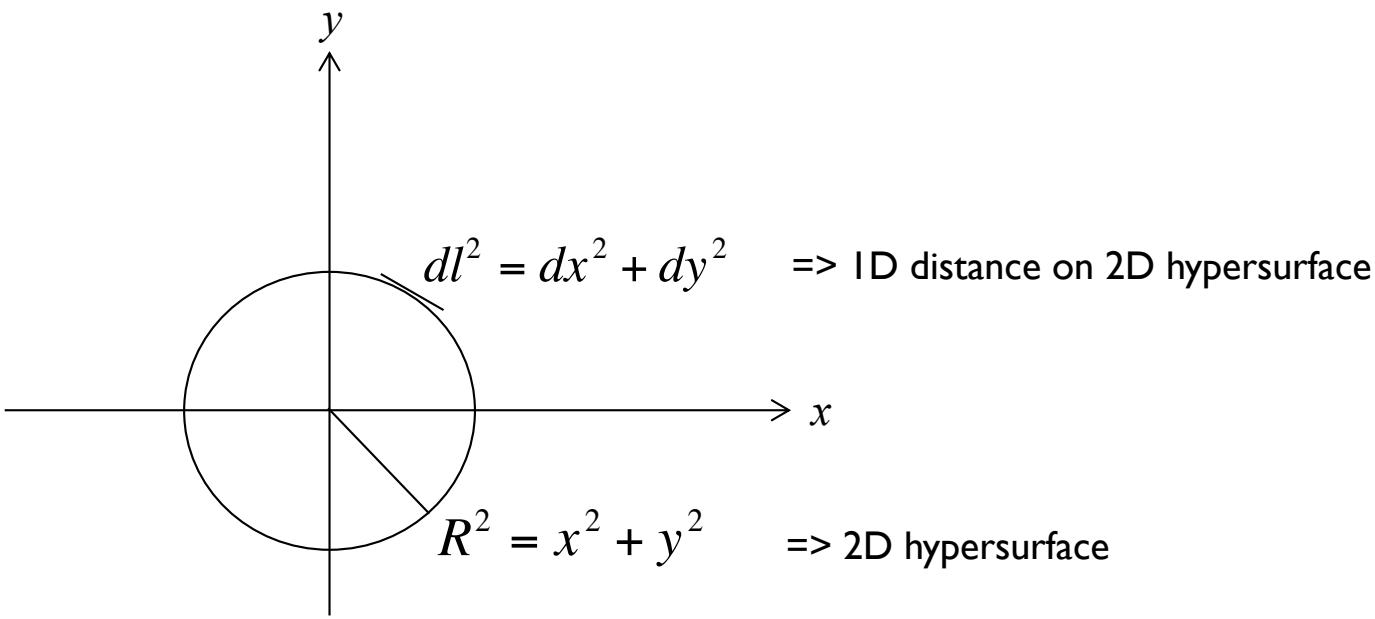

we like to get rid of that 2nd dimension (e.g. y)
as we live in 1D space (e.g. only x)

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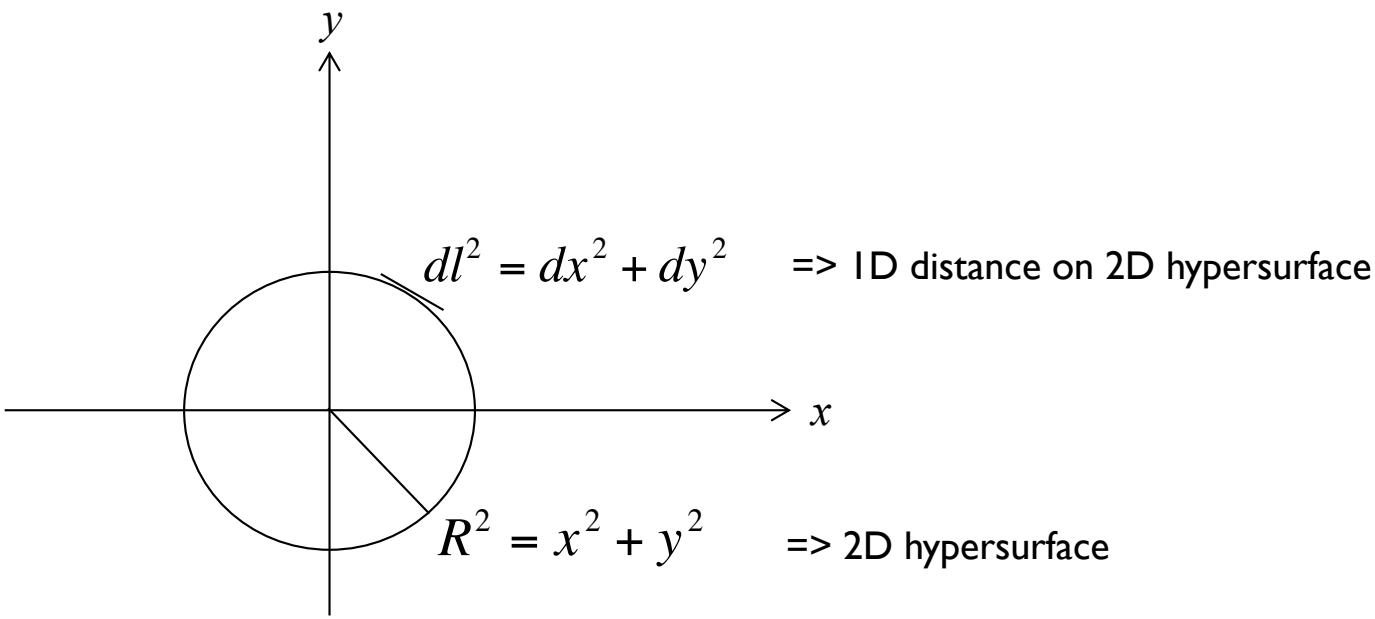

$$dl^2 = dx^2 + dy^2$$

$$R^2 = x^2 + y^2$$

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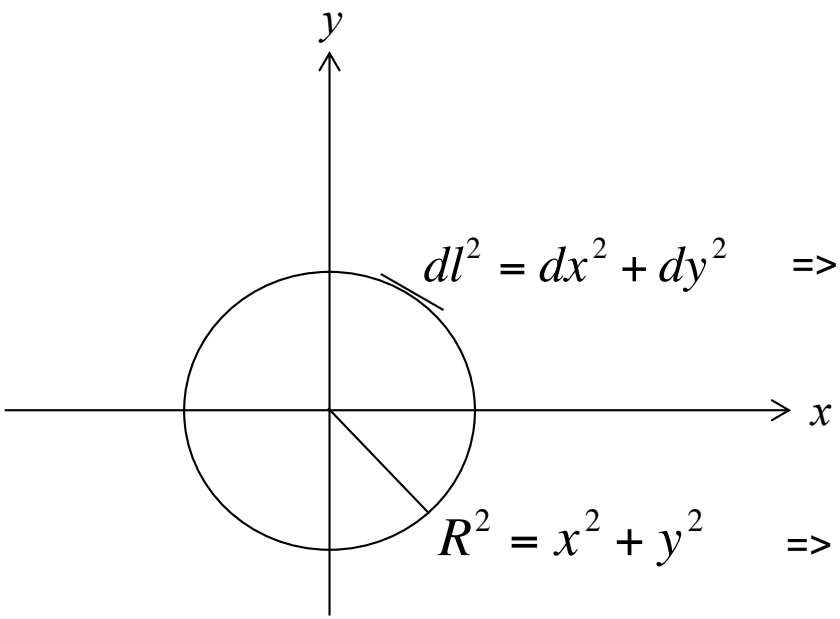

$dl^2 = dx^2 + dy^2$ => 1D distance on 2D hypersurface

$R^2 = x^2 + y^2$ => 2D hypersurface

$$dl^2 = dx^2 + dy^2$$

$$R^2 = x^2 + y^2 \xrightarrow{\text{differentiate}} 0 = xdx + ydy \longrightarrow dy = -xdx / y$$

$$R^2 = x^2 + y^2 \xrightarrow{\text{re-arrange terms}} y^2 = R^2 - x^2$$

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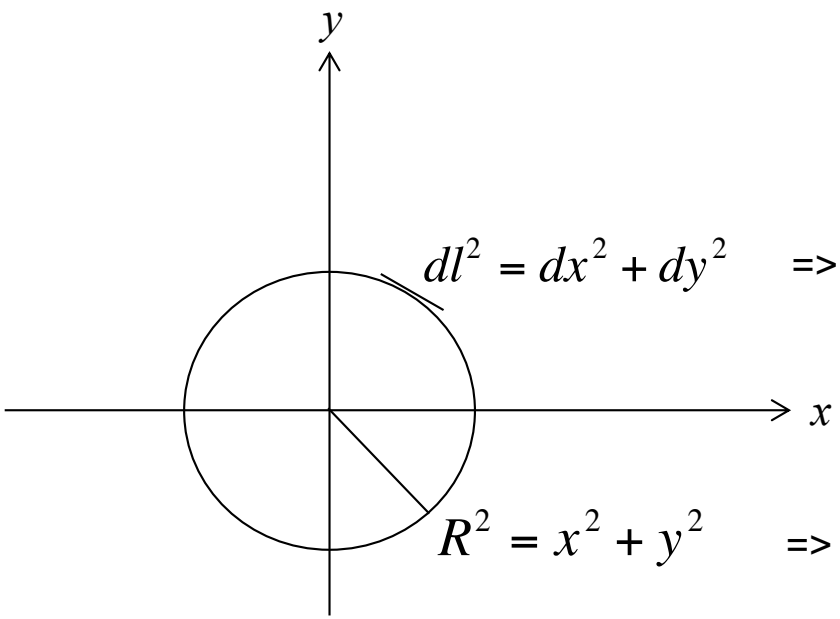

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$$dl^2 = dx^2 + \frac{x^2 dx^2}{R^2 - x^2}$$

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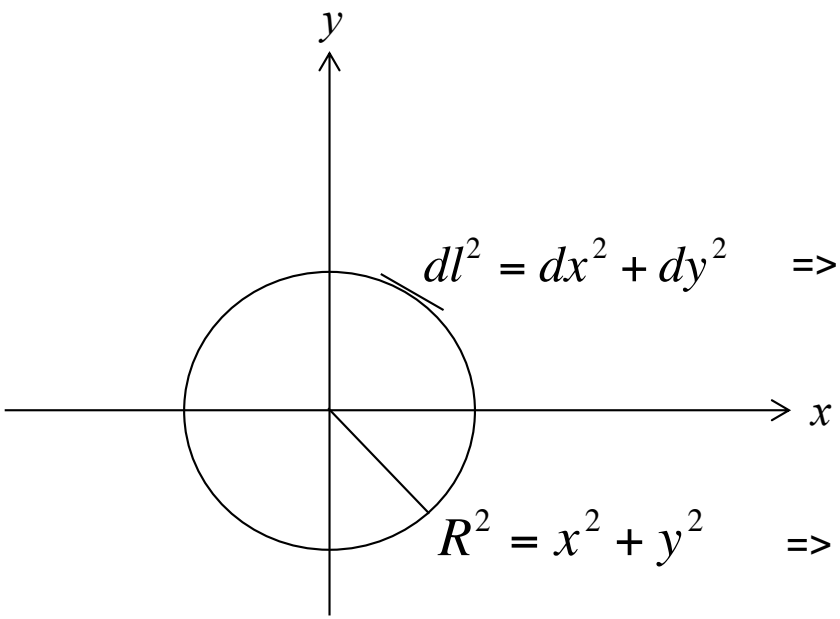

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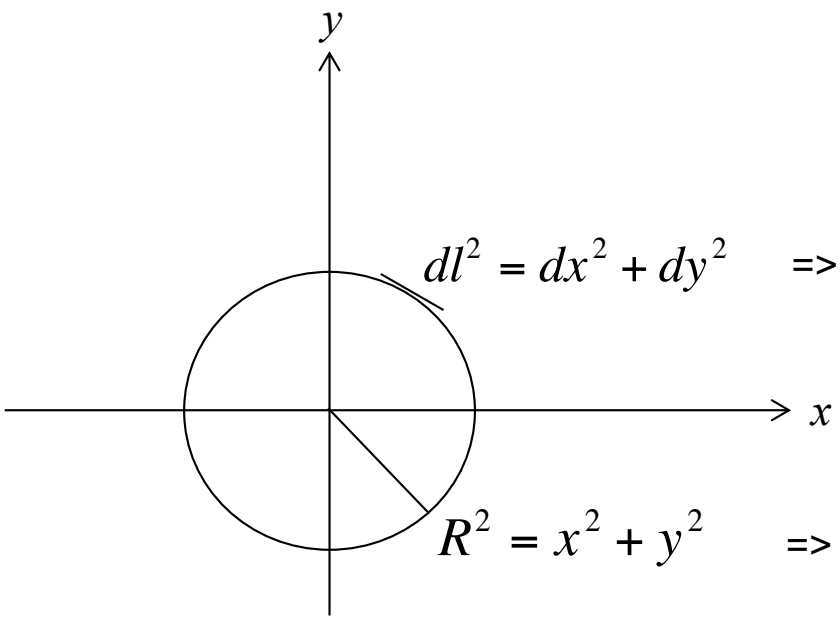

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$$\gamma_{11} = 1 + \frac{x^2}{R^2 - x^2}$$

- metric of curved space
  - 3D space ≡ hypersurface in 4D space

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2$$
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spherical 3D space:
$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2$$
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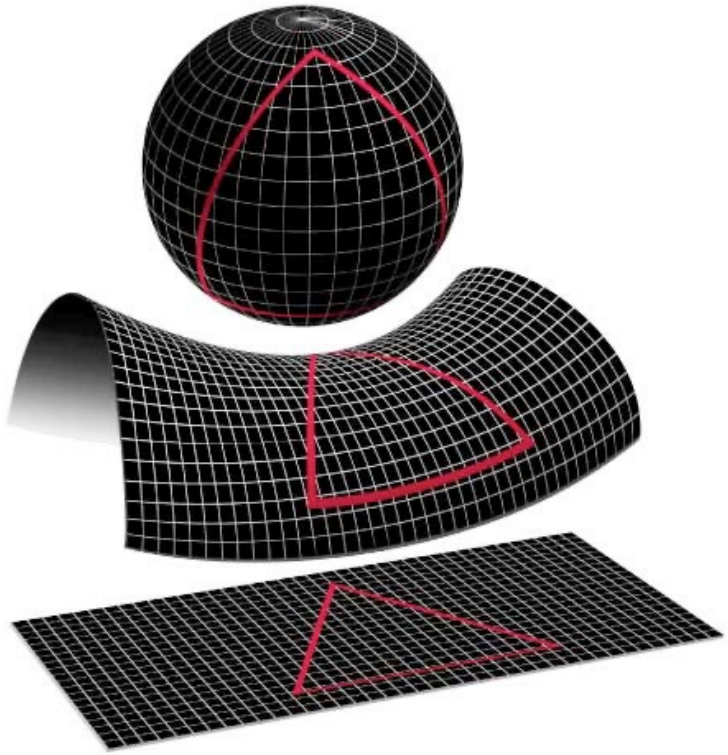
hyperbolic 3D space:
$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2$$
$$-R^2 = \left(x_1^2 + x_2^2 + x_3^2\right) + z^2$$

Euclidean 3D space:
$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2$$



careful: R is **not** the size of the Universe…

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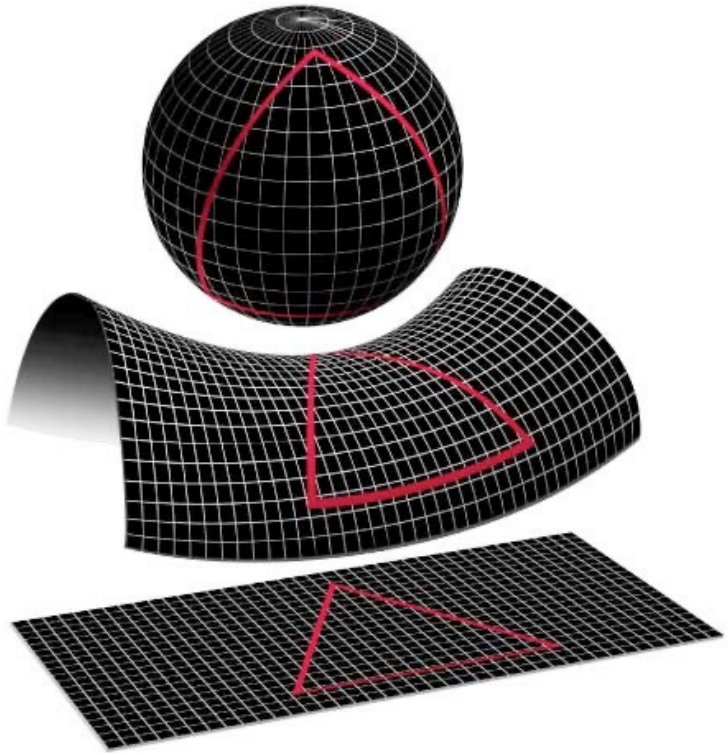

these are the only 3 possible cases
for homogeneous & isotropic metrics!

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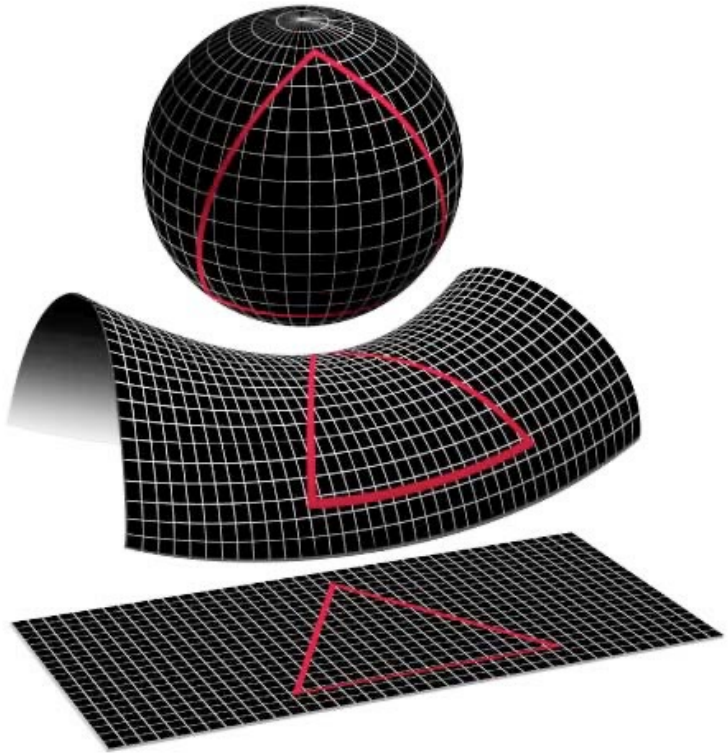

these are the only 3 possible cases
for homogeneous & isotropic metrics:

homogeneity: dl is invariant under translations $x_i \to x_i + D$

isotropy: dl is invariant under rotations $x_i \to \mathcal{R}^i{}_k \; x^k$

3D space:	Euclidean	spherical	hyperbolic
	trivial	*inherited from (trivial) 4D symmetry*	*Sec 13.2 in Weinberg's "Gravitation & Cosmology"*

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$$\gamma_{ij}\,?$$

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$$\boxed{R^2 = x_1^2 + x_2^2 + x_3^2 + z^2}$$

$\downarrow$ differentiation

$$0 = x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + z dz$$

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$\downarrow$ combination with dl^2 to eliminate z

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{\left(x_1 dx_1 + x_2 dx_2 + x_3 dx_3\right)^2}{R^2 - (x_1^2 + x_2^2 + x_3^2)}$$

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introduction of spherical coordinates will diagonlize γ_{ij}

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$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\vartheta\cos\varphi \\ r\sin\vartheta\sin\varphi \\ r\cos\vartheta \end{pmatrix}$$

$$dx_1^2 + dx_2^2 + dx_3^2 = g_r dr^2 + g_\vartheta d\vartheta^2 + g_\varphi d\varphi^2$$

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$$\begin{cases} g_r = \left|\dfrac{\partial\vec{r}}{\partial r}\right| = 1 \\ g_\vartheta = \left|\dfrac{\partial\vec{r}}{\partial\vartheta}\right| = r \\ g_\varphi = \left|\dfrac{\partial\vec{r}}{\partial\varphi}\right| = r\sin\vartheta \end{cases}$$

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$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\vartheta\cos\varphi \\ r\sin\vartheta\sin\varphi \\ r\cos\vartheta \end{pmatrix}$$

$$= dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 + \frac{(\vec{r}\cdot d\vec{r})^2}{R^2 - r^2}$$

$$= dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 + \frac{r^2 dr^2}{R^2 - r^2}$$

$$= dr^2 \left[\frac{r^2}{R^2 - r^2} + 1\right] + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)$$

$$= dr^2 \left[\frac{R^2}{R^2 - r^2}\right] + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)$$

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introduction of rescaled (comoving) coordinates

$$r = Rx$$
$$dr = Rdx$$

$$= R^2\left(dx^2\left[\frac{1}{1-x^2}\right] + x^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right)$$

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generalisation to hyperbolic & Euclidean spaces

$k = +1$: spherical
$k = 0$: Euclidean
$k = -1$: hyperbolic

$$\Rightarrow \quad R^2\left(\frac{dx^2}{1-kx^2} + x^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right)$$

- metric of curved space
  - 3D space ≡ hypersurface in 4D space

$$dl^2 = R^2\left(\frac{dx^2}{1-kx^2} + x^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right)$$

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$k = +1$: spherical
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$k = -1$: hyperbolic

$$\gamma_{ij} = \begin{pmatrix} \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & R^2x^2 & 0 \\ 0 & 0 & R^2x^2\sin^2\vartheta \end{pmatrix}$$

- metric of curved space
  - 3D space ≡ hypersurface in 4D space

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$$\gamma_{ij} = \begin{pmatrix} \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & R^2x^2 & 0 \\ 0 & 0 & R^2x^2\sin^2\vartheta \end{pmatrix} \quad \Rightarrow \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2x^2 & 0 \\ 0 & 0 & 0 & R^2x^2\sin^2\vartheta \end{pmatrix}$$

- Robertson-Walker-Metric

$$ds^2 = -\left(cdt\right)^2 + R^2\left[\frac{dx^2}{1-kx^2} + x^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right]$$

or equivalently:

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2x^2 & 0 \\ 0 & 0 & 0 & R^2x^2\sin^2\vartheta \end{pmatrix}$$

- Robertson-Walker-Metric

$$ds^2 = -\left(cdt\right)^2 + R^2\left[\frac{dx^2}{1-kx^2} + x^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right]$$

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<u>Notes:</u>

- curvature of 3D space: ${}^{(3)}R = 6\frac{k}{R^2}$

- curvature of space-time: ${}^{(4)}R = 6\frac{\ddot{R}}{R} + 6\left(\frac{\dot{R}}{R}\right)^2 + 6\frac{k}{R^2}$

- Robertson-Walker-Metric

$$ds^2 = -(cdt)^2 + R^2\left[\frac{dx^2}{1-kx^2} + x^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right]$$

or equivalently:

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

with

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2x^2 & 0 \\ 0 & 0 & 0 & R^2x^2\sin^2\vartheta \end{pmatrix}$$

can be used to calculate geodesics…

- Robertson-Walker-Metric
  - derivation
  - **geodesics**
  - redshift
- Friedmann Equations
- Friedmann Models

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \dfrac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2 \vartheta \end{pmatrix}$$

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2\vartheta \end{pmatrix}$$

geodesic equation (absence of external forces):

$$\frac{d^2 x^\mu}{ds^2} = -\Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds}$$

with $\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\kappa} \left[\frac{\partial g_{\kappa\nu}}{\partial x^\lambda} + \frac{\partial g_{\kappa\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\kappa} \right]$

(Christoffel symbols/affine connection)

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \dfrac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2\vartheta \end{pmatrix}$$

geodesic equation (absence of external forces):

$$0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds} \quad \text{and} \quad \Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\kappa} \left[\frac{\partial g_{\kappa\nu}}{\partial x^\lambda} + \frac{\partial g_{\kappa\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\kappa} \right]$$

- geodesics

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$$\Gamma^0_{ij} = \frac{\dot{R}}{R} \gamma_{ij} \qquad \Gamma^i_{0j} = \dot{R}^2 R^2 \delta_{ij}$$

$$\Gamma^i_{jk} = \frac{1}{2} \gamma^{il} \left(\frac{\partial \gamma_{kl}}{\partial x^j} + \frac{\partial \gamma_{jl}}{\partial x^k} - \frac{\partial \gamma_{jk}}{\partial x^l} \right)$$

(all other components are zero)

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \dfrac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2\vartheta \end{pmatrix}$$

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- $\mu = 0$ component: $\qquad \Gamma^0_{ij} = \dfrac{\dot{R}}{R} \gamma_{ij}$

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2\vartheta \end{pmatrix}$$

geodesic equation (absence of external forces):

$$\underline{\underline{0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda}} \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds}$$

- $\mu = 0$ component:

$$\Gamma^0_{ij} = \frac{\dot{R}}{R} \gamma_{ij}$$

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- geodesics

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$$0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds}$$

- $\mu = 0$ component in FRW metric: $\Gamma^0_{ij} = \frac{\dot{R}}{R}\gamma_{ij}$

"(all other components are zero)"

$$0 = \frac{du^0}{ds} + \Gamma^0_{\nu\lambda} u^\nu u^\lambda = \frac{du^0}{ds} + \Gamma^0_{ij} u^i u^j$$

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2\vartheta \end{pmatrix}$$

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- $\mu = 0$ component in FRW metric: $\Gamma^0_{ij} = \frac{\dot{R}}{R}\gamma_{ij}$

$$0 = \frac{du^0}{ds} + \Gamma^0_{\nu\lambda} u^\nu u^\lambda = \frac{du^0}{ds} + \Gamma^0_{ij} u^i u^j = \frac{du^0}{ds} + \frac{\dot{R}}{R}\gamma_{ij} u^i u^j$$

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2\vartheta \end{pmatrix}$$

geodesic equation (absence of external forces):

$$0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds}$$

- $\mu = 0$ component in FRW metric:

physical 3D velocity: $|u|^2 = \gamma_{ij} u^i u^j$

$$0 = \frac{du^0}{ds} + \Gamma^0_{\nu\lambda} u^\nu u^\lambda = \frac{du^0}{ds} + \Gamma^0_{ij} u^i u^j = \frac{du^0}{ds} + \frac{\dot{R}}{R} \gamma_{ij} u^i u^j = \frac{du^0}{ds} + \frac{\dot{R}}{R} |u|^2$$

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2\vartheta \end{pmatrix}$$

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$$0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds}$$

- $\mu = 0$ component in FRW metric:

$$0 = \boxed{\frac{du^0}{ds}} + \frac{\dot{R}}{R} |u|^2$$

get rid of ds (in favour of dt)!?

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2\vartheta \end{pmatrix}$$

geodesic equation (absence of external forces):

$$0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds}$$

- $\mu = 0$ component in FRW metric:

$$u^0 = -c\frac{dt}{ds}, u^i = \frac{dx^i}{ds}$$

$$0 = \boxed{\frac{du^0}{ds}} + \frac{\dot{R}}{R}|u|^2 \qquad \frac{du^0}{ds} = \frac{du^0}{dt}\frac{dt}{ds} = -\frac{du^0}{dt}\frac{u^0}{c}$$

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2\vartheta \end{pmatrix}$$

geodesic equation (absence of external forces):

$$0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds}$$

- $\mu = 0$ component in FRW metric:

$$-c^2 = u_\nu u^\nu \;\; => \;\; u^0\, du^0 = -|u| d|u|$$

$$0 = \boxed{\frac{du^0}{ds}} + \frac{\dot{R}}{R}|u|^2$$

$$\frac{du^0}{ds} = \frac{du^0}{dt}\frac{dt}{ds} = -\frac{du^0}{dt}\frac{u^0}{c} = -\frac{u^0 du^0}{cdt} = \frac{|u| d|u|}{cdt}$$

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2\vartheta \end{pmatrix}$$

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$$0 = \boxed{\frac{du^0}{ds}} + \frac{\dot{R}}{R}|u|^2$$

$$\frac{du^0}{ds} = \frac{du^0}{dt}\frac{dt}{ds} = -\frac{du^0}{dt}\frac{u^0}{c} = -\frac{u^0 du^0}{cdt} = \frac{|u|d|u|}{cdt} = \frac{|u|}{c}|\dot{u}|$$

- geodesics

$$x^\mu = (ct, x^i)$$

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- $\mu = 0$ component in FRW metric:

$$0 = \boxed{\frac{du^0}{ds}} + \frac{\dot{R}}{R}|u|^2$$

$$\frac{du^0}{ds} = \frac{du^0}{dt}\frac{dt}{ds} = -\frac{du^0}{dt}\frac{u^0}{c} = -\frac{u^0 du^0}{cdt} = \frac{|u|d|u|}{cdt} = \frac{|u|}{c}|\dot{u}|$$

- geodesics

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geodesic equation (absence of external forces):

$$0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds}$$

- $\mu = 0$ component in FRW metric:

$$0 = \frac{|u|}{c} |\dot{u}| + \frac{\dot{R}}{R} |u|^2$$

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2\vartheta \end{pmatrix}$$

geodesic equation (absence of external forces):

$$0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds}$$

- $\mu = 0$ component in FRW metric:

$$\frac{|\dot{u}|}{|u|} \propto -\frac{\dot{R}}{R}$$

- geodesics

$$x^\mu = (ct, x^i)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad \text{with} \qquad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2\vartheta \end{pmatrix}$$

geodesic equation (absence of external forces):

$$0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds}$$

- $\mu = 0$ component in FRW metric:

magnitude of physical 3D velocity

$$\frac{|\dot{u}|}{|u|} \propto -\frac{\dot{R}}{R} \xrightarrow{\frac{d|u|}{dt} = \frac{d|u|}{dR}\dot{R}} \frac{1}{|u|}\frac{d|u|}{dR}\dot{R} \propto -\frac{\dot{R}}{R} \longrightarrow \frac{d|u|}{|u|} \propto -\frac{dR}{R} \longrightarrow |u| \propto \frac{1}{R}$$

- geodesics in FRW metric:

$$|p| \propto \frac{1}{R}$$

the momentum of particles scale like $1/R$

- Robertson-Walker-Metric
  - derivation
  - geodesics
  - **redshift**
- Friedmann Equations
- Friedmann Models

- Robertson-Walker-Metric

$$ds^2 = -(cdt)^2 + R^2(t)\left[\frac{dx^2}{1-kx^2} + x^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right]$$

Note: $R(t)$ can be an arbitrary function of time

- Robertson-Walker-Metric

$$ds^2 = -(cdt)^2 + R^2(t)\left[\frac{dx^2}{1-kx^2} + x^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right]$$

any connection to observables?

- Robertson-Walker-Metric

$$ds^2 = -(cdt)^2 + R^2(t)\left[\frac{dx^2}{1-kx^2} + x^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right]$$

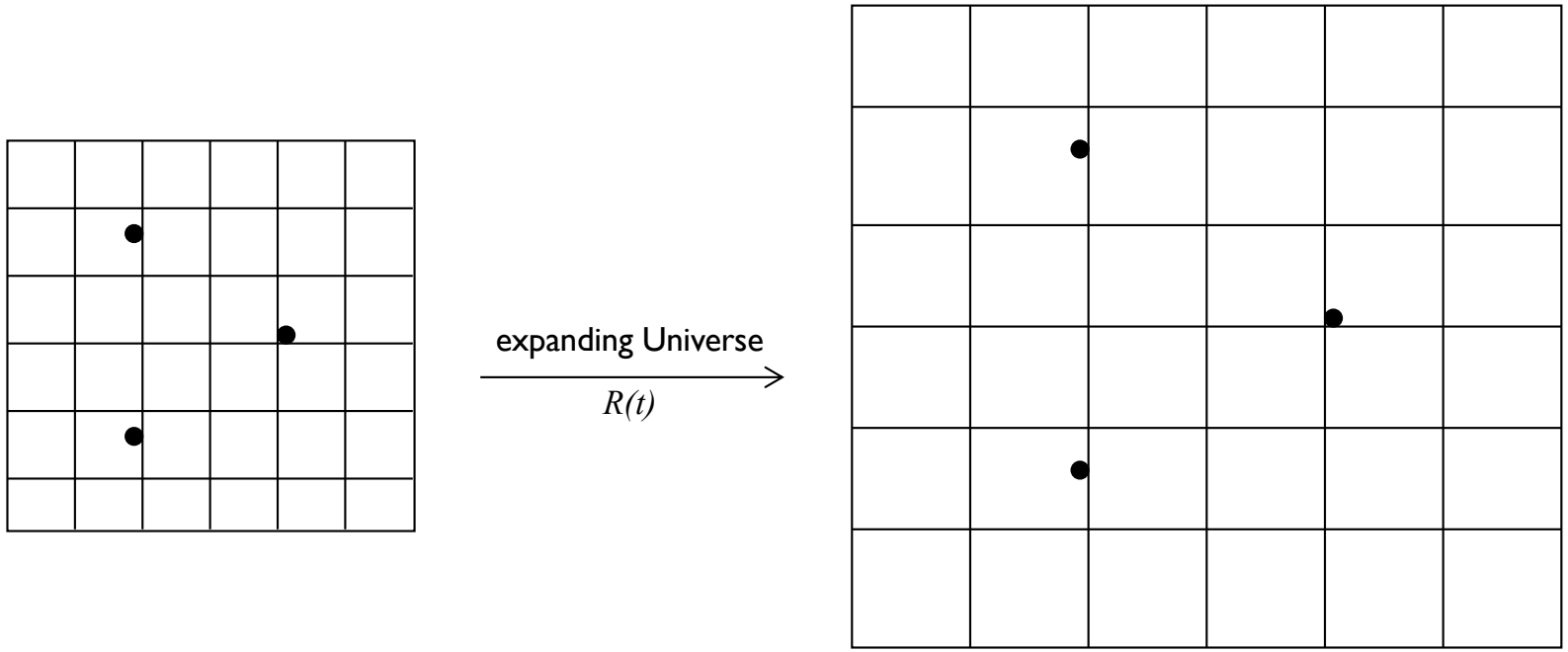
comoving coordinates: $\vec{r}(t) = R(t)\vec{x}$

any connection to observables?

- cosmological expansion

all length scales scale like $R(t)$

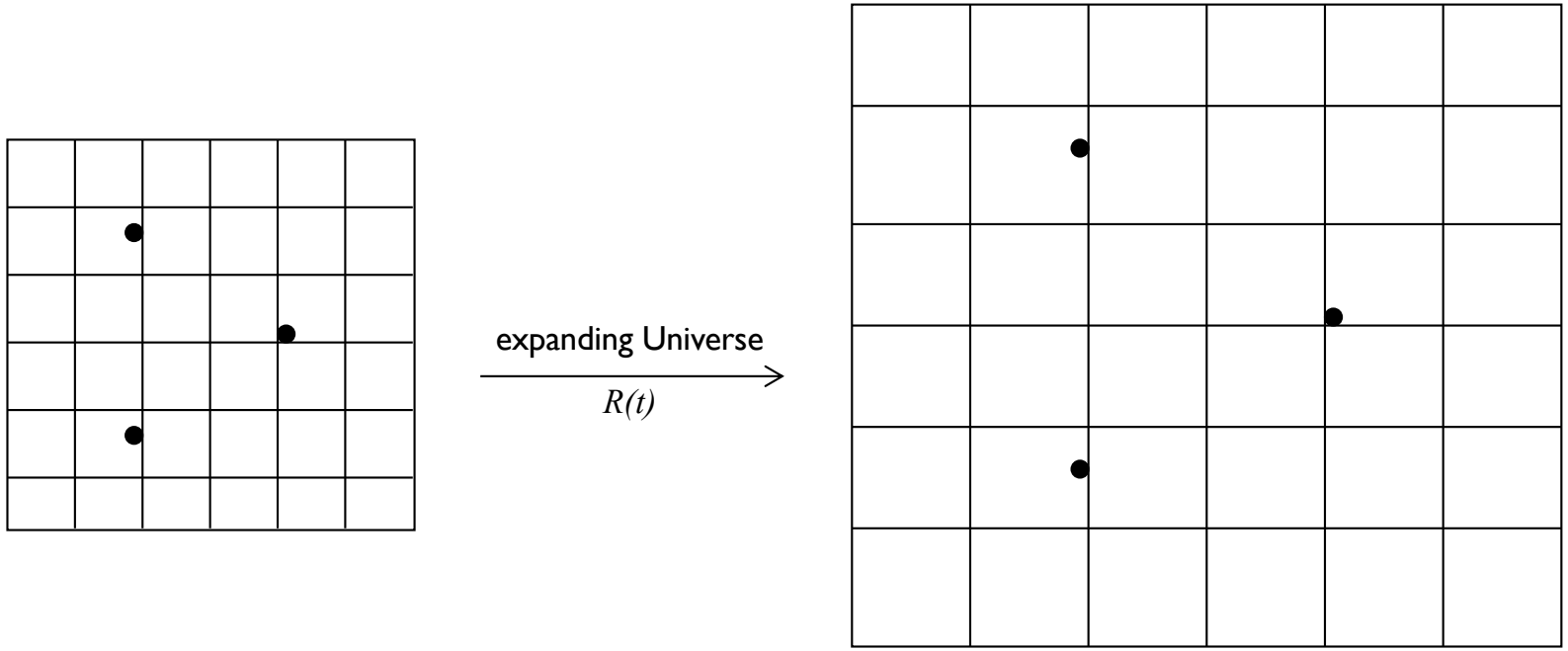
$$\vec{r}(t) = R(t)\vec{x}$$



- cosmological expansion

all length scales scale like $R(t)$

$$\vec{r}(t) = R(t)\vec{x}$$



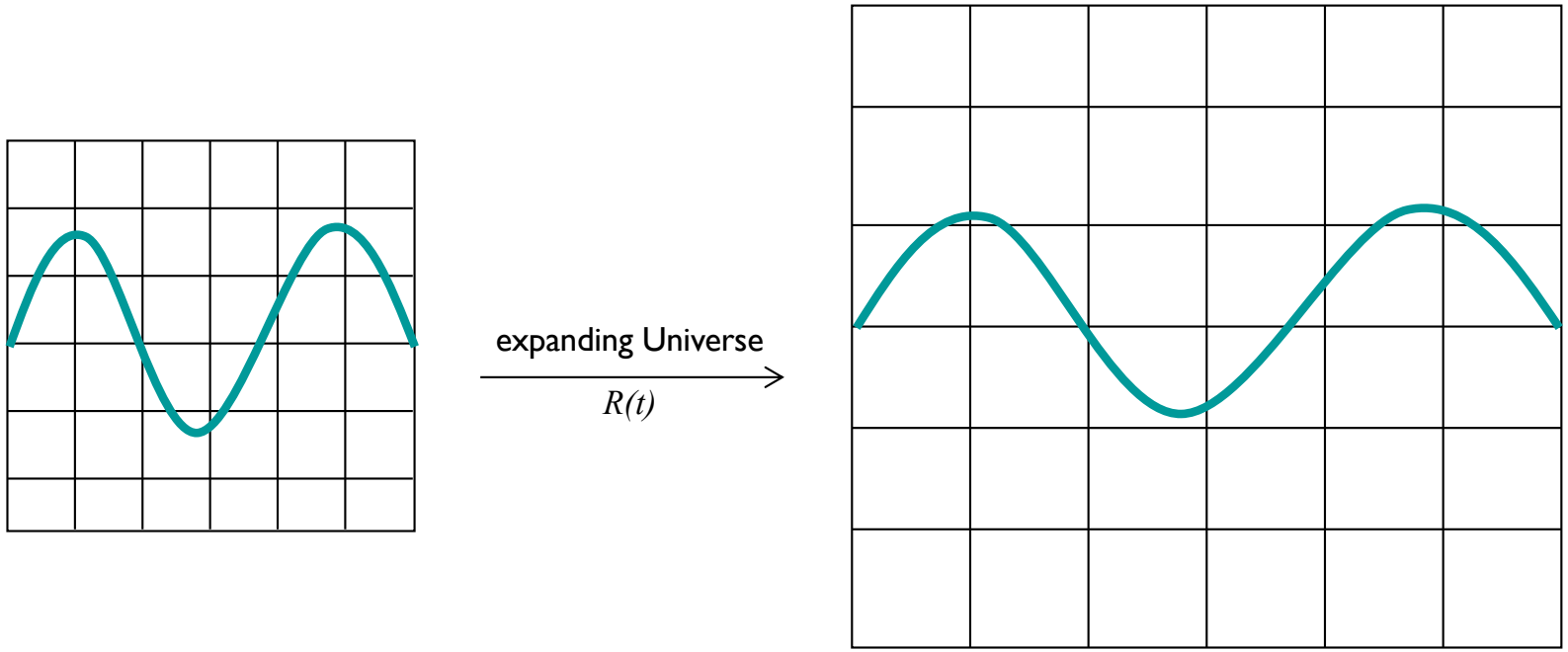
Note:

- space can be curved and is not necessarily Euclidean
- even though we call it 'expanding' we simply consider $R(t) \neq \text{const.}$

- cosmological expansion

all length scales scale like $R(t)$

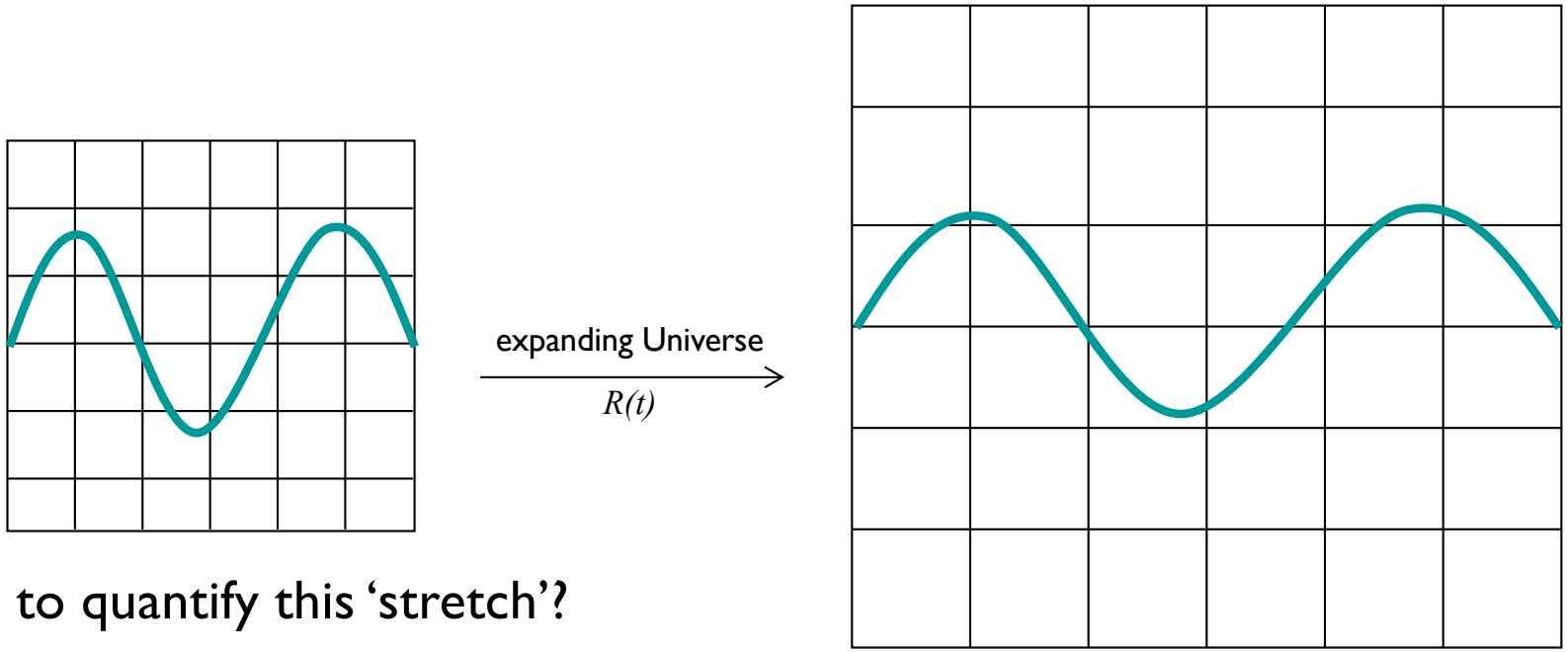
$$\vec{r}(t) = R(t)\vec{x}$$



- cosmological expansion

all length scales scale like $R(t)$

$$\vec{r}(t) = R(t)\vec{x}$$



how to quantify this 'stretch'?

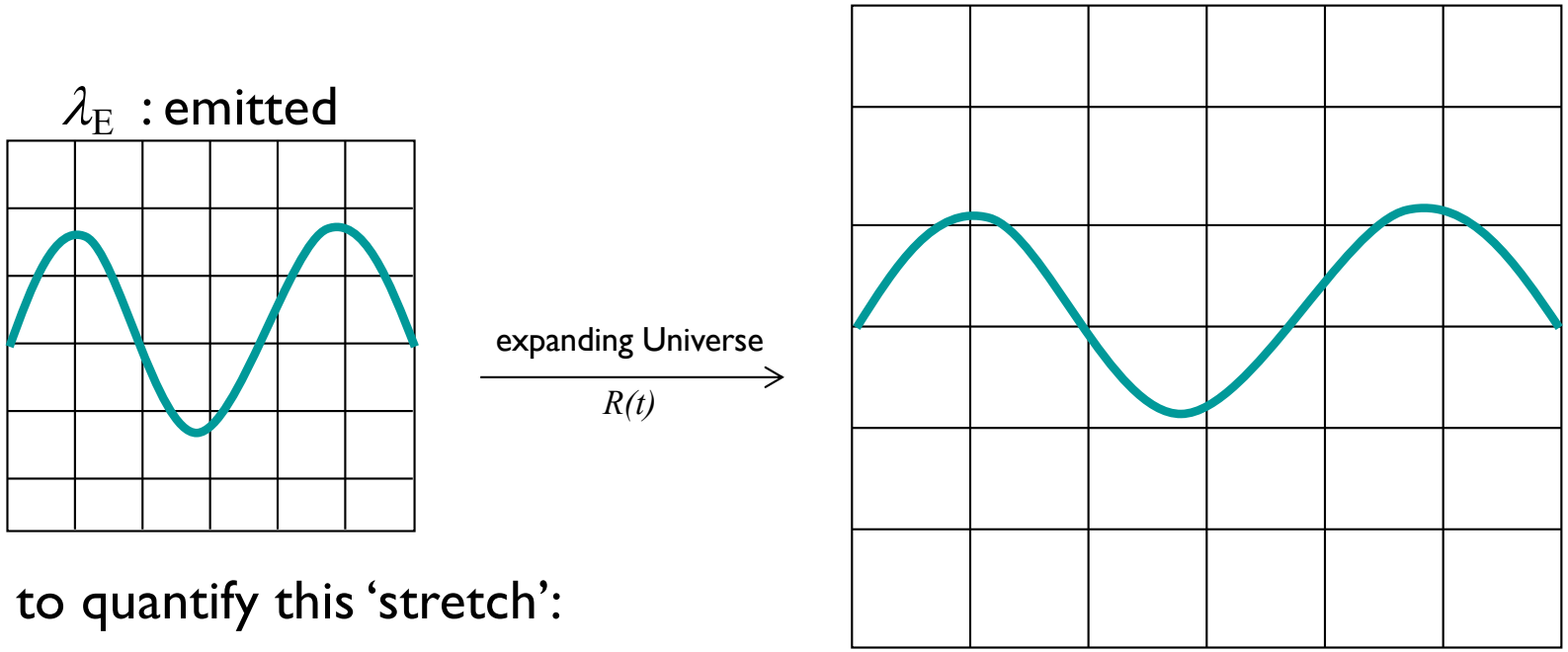
- cosmological expansion and redshift

all length scales scale like $R(t)$

$$\vec{r}(t) = R(t)\vec{x}$$

λ_0 : observed

λ_E : emitted



how to quantify this 'stretch':

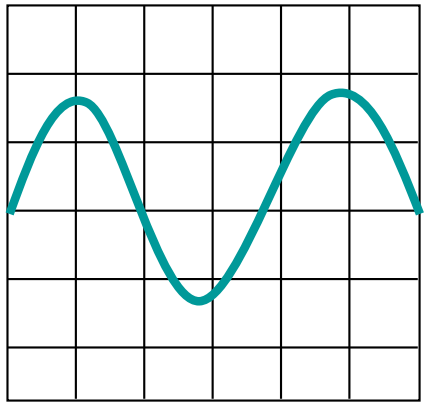
redshift $$z = \frac{\lambda_0 - \lambda_E}{\lambda_E}$$

- cosmological expansion and redshift

all length scales scale like $R(t)$

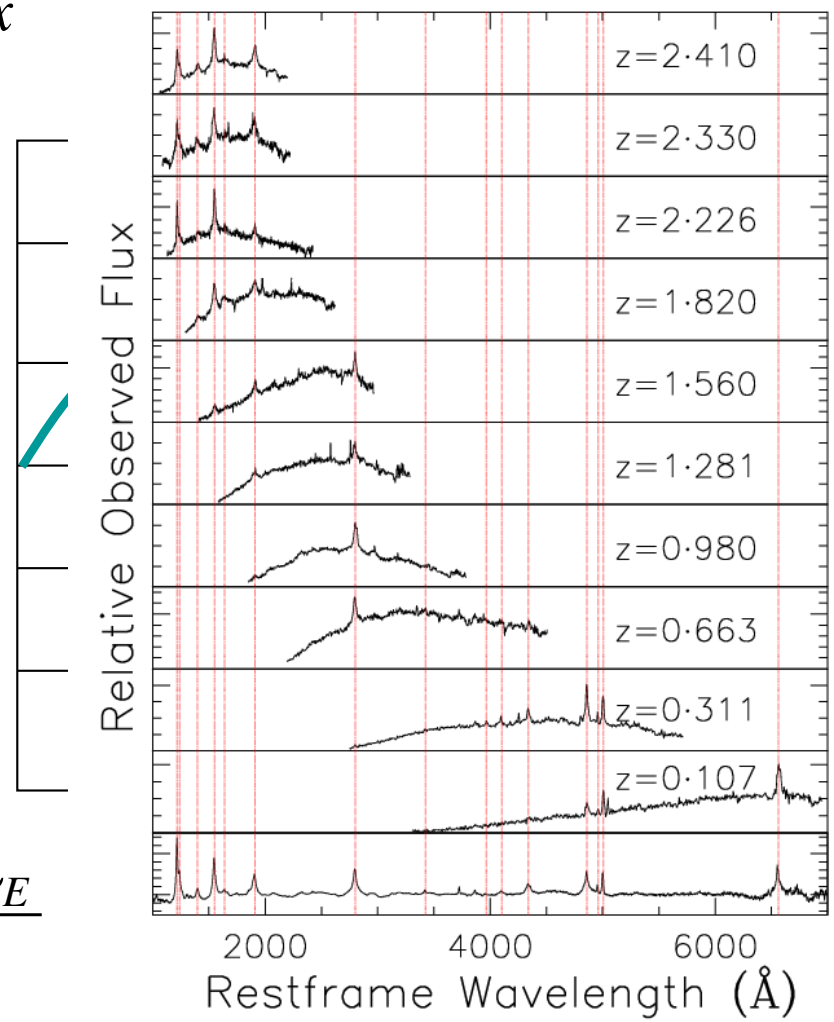
$$\vec{r}(t) = R(t)\vec{x}$$

λ_E : emitted



expanding Universe

$R(t)$



how to quantify this 'stretch':

redshift

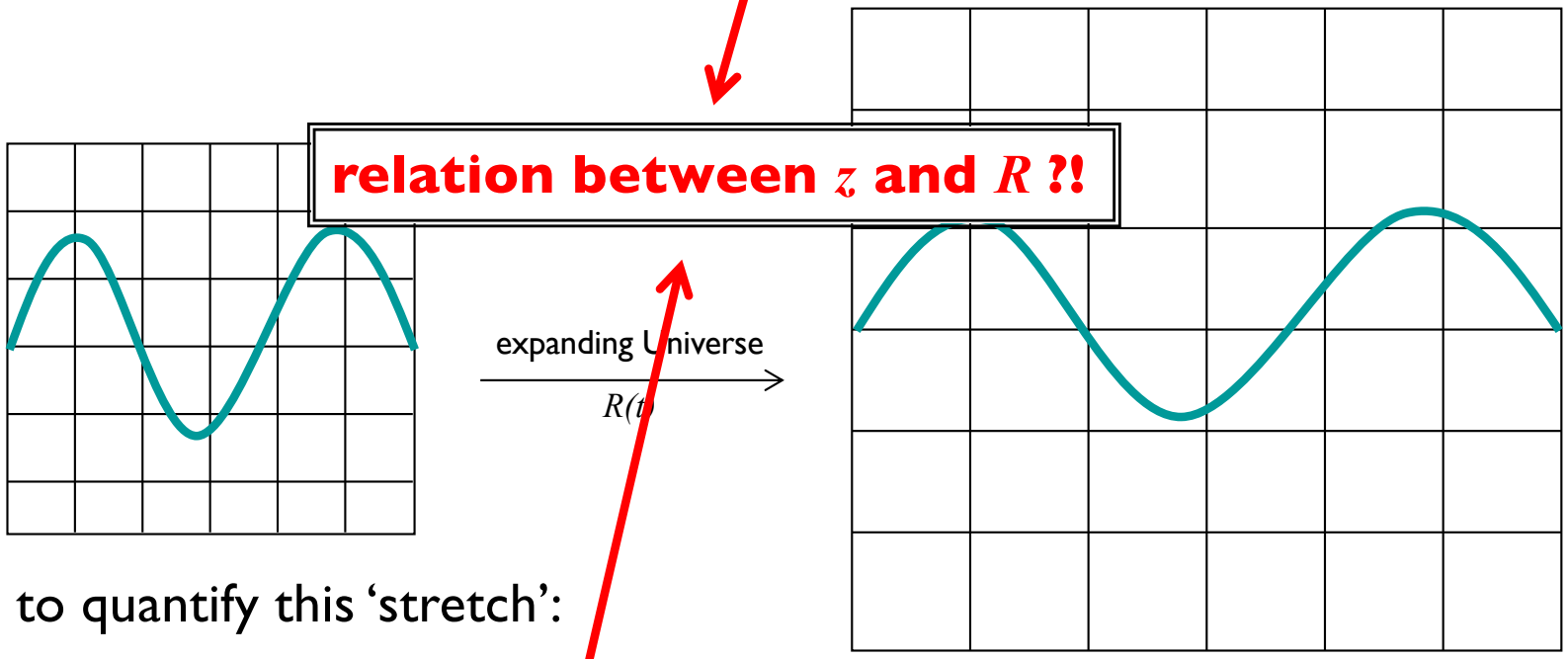
$$z = \frac{\lambda_0 - \lambda_E}{\lambda_E}$$

- cosmological expansion and redshift

all length scales scale like $R(t)$

$$\vec{r}(t) = R(t)\vec{x}$$

relation between z and R ?!



how to quantify this 'stretch':

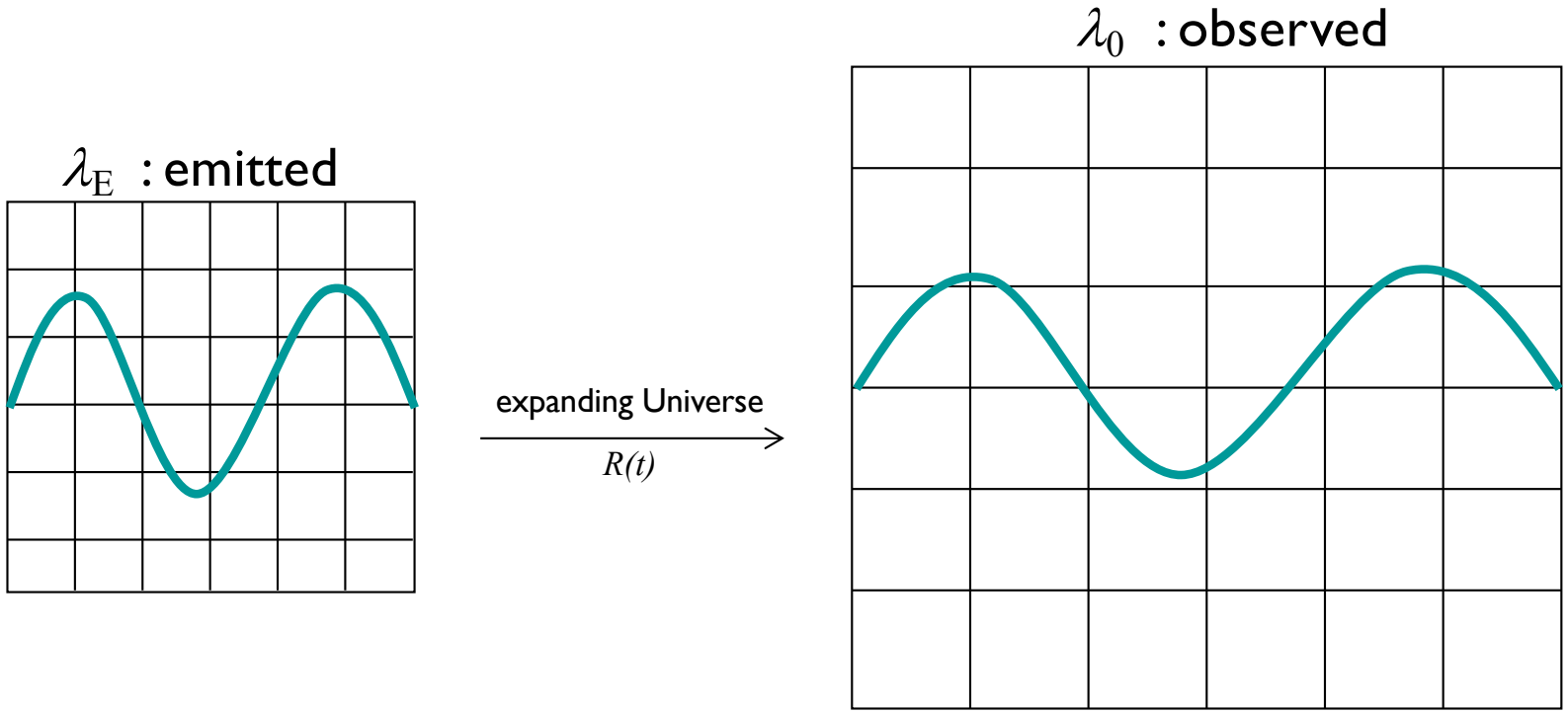
redshift

$$z = \frac{\lambda_0 - \lambda_E}{\lambda_E}$$

λ_E : emitted
λ_0 : observed

- expansion factor and redshift

$$z = \frac{\lambda_0 - \lambda_E}{\lambda_E} \overset{?}{\Longleftrightarrow} \vec{r}(t) = R(t)\vec{x}$$



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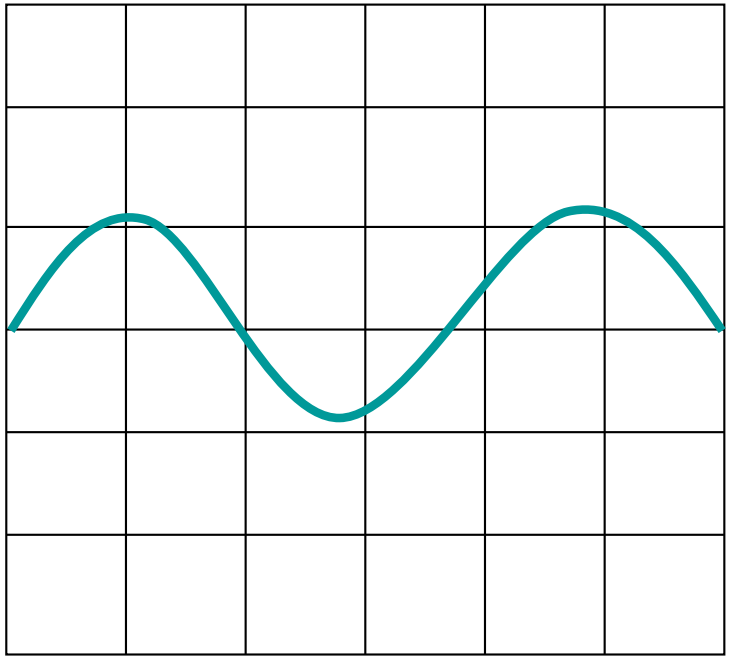
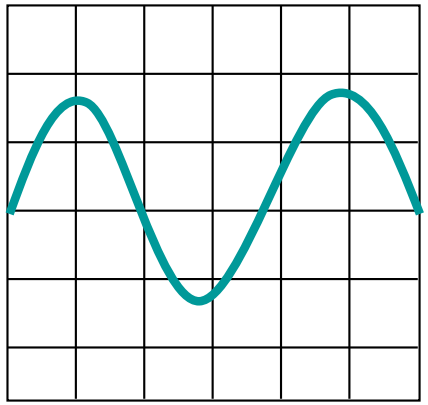


λ_E : emitted



expanding Universe

$R(t)$

- de Broglie relation: $p = \frac{h}{\lambda}$

- expansion factor and redshift

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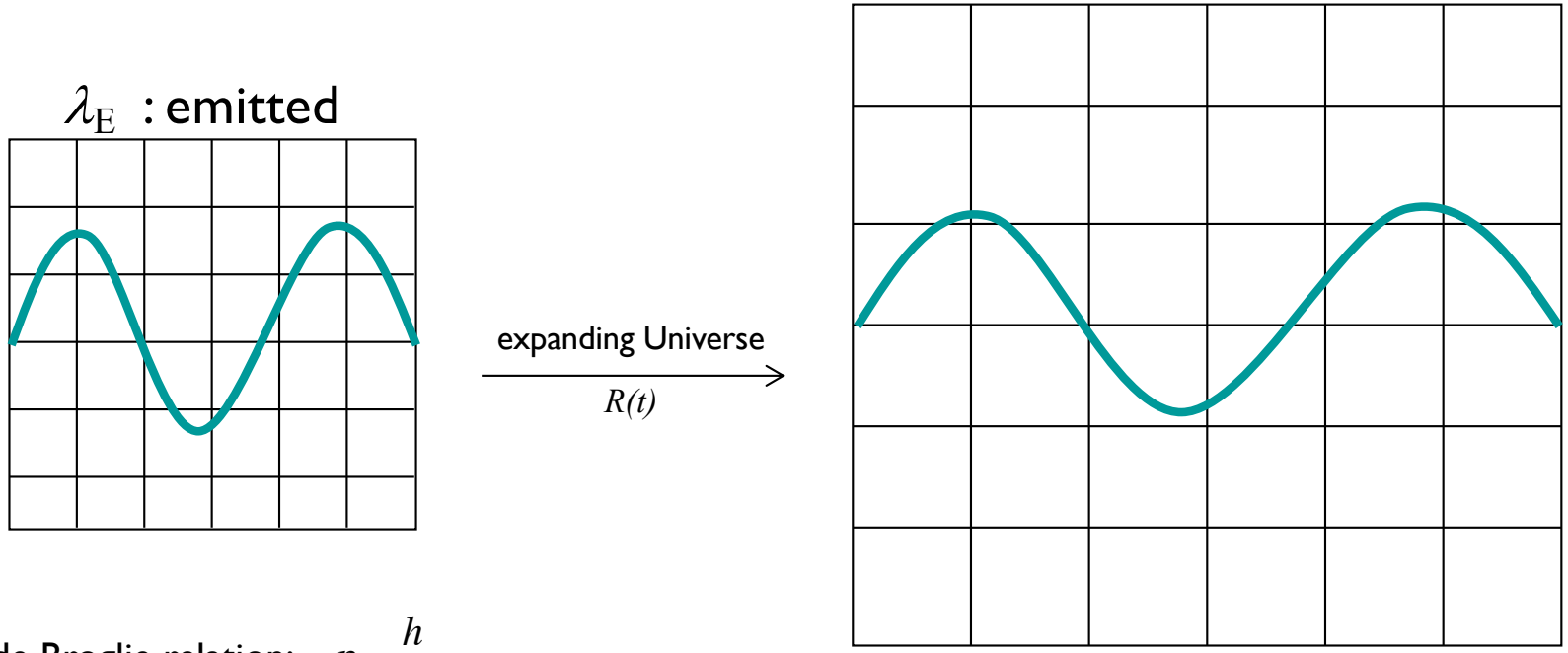
λ_E : emitted



- de Broglie relation: $p = \frac{h}{\lambda}$
- FRW geodesics: $p \propto \frac{1}{R}$

- expansion factor and redshift

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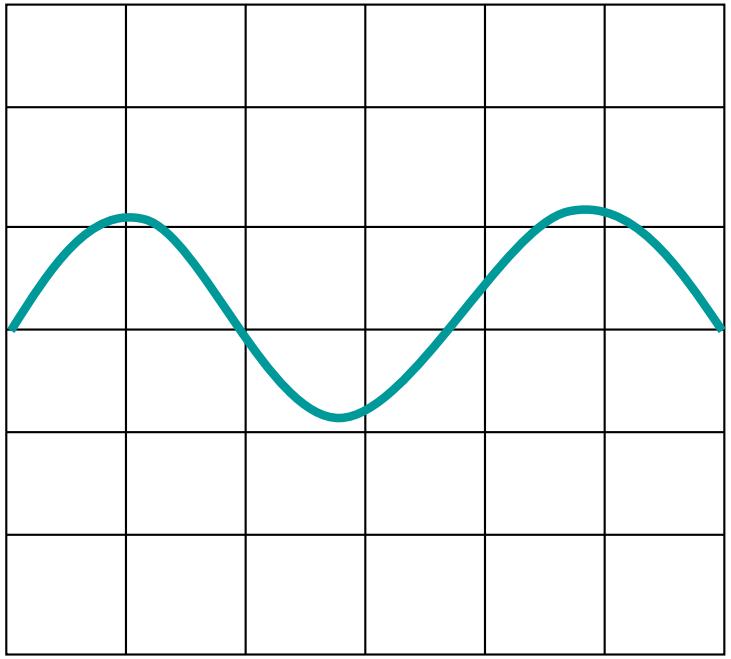
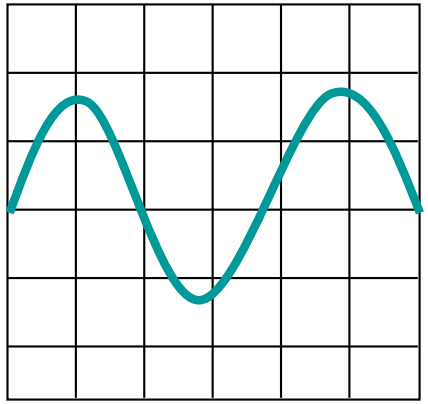


λ_E : emitted



expanding Universe

$R(t)$

- de Broglie relation: $p = \frac{h}{\lambda}$
- FRW geodesics: $p \propto \frac{1}{R}$

$$\left.\right\} \quad \frac{\lambda_0}{R_0} = \frac{\lambda_E}{R_E}$$

- expansion factor and redshift

$$z = \frac{\lambda_0 - \lambda_E}{\lambda_E} \qquad \& \qquad \frac{\lambda_0}{R_0} = \frac{\lambda_E}{R_E}$$

$$=> \qquad \frac{R_E}{R_0} = \frac{\lambda_E}{\lambda_0} = \frac{1}{\lambda_0/\lambda_E} = \frac{1}{\lambda_0/\lambda_E - 1 + 1} = \frac{1}{z+1}$$

$$=> \qquad \boxed{\frac{R(t_E)}{R(t_0)} = \frac{1}{1+z}}$$

(very important relation; does **not** depend on cosmology)

- expansion factor and redshift

$$\boxed{a(t) = \frac{1}{1+z}}$$

cosmic expansion factor: $a(t) = \dfrac{R(t)}{R(t_0)}$

(very important relation; does **not** depend on cosmology)

- expansion factor and redshift

$$\boxed{a(t) = \frac{1}{1+z}}$$

observable

cosmic expansion factor: $a(t) = \dfrac{R(t)}{R(t_0)}$

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- expansion factor and redshift

cosmology

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cosmic expansion factor: $a(t) = \dfrac{R(t)}{R(t_0)}$

(very important relation; does **not** depend on cosmology)

- expansion factor and redshift

...but how to calculate?

$$\boxed{a(t) = \frac{1}{1+z}}$$

cosmic expansion factor: $a(t) = \dfrac{R(t)}{R(t_0)}$

(very important relation; does **not** depend on cosmology)

- Robertson-Walker-Metric

- Friedmann Equations
  - **derivation**
  - equation of state
  - adiabatic expansion
  - density parameters

- Friedmann Models

- derivation
  - Einstein's Field Equations

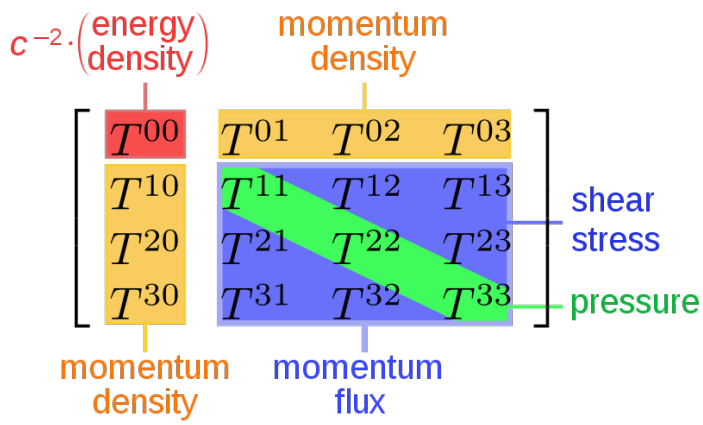
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} \quad = \quad 8\pi G T_{\mu\nu}$$

- derivation

  - Einstein's Field Equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = \underbrace{8\pi G T_{\mu\nu}}_{\text{properties of matter}}$$

…as described by the energy-stress tensor $T_{\mu\nu}$



- derivation

  - Einstein's Field Equations

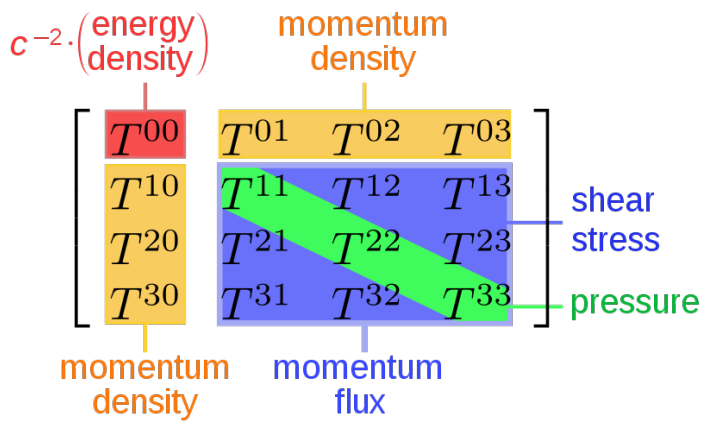
$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu}}_{\text{pure geometry}} \quad = \quad \underbrace{8\pi G T_{\mu\nu}}_{\text{properties of matter}}$$

…as described by the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R

…as described by the energy-stress tensor $T_{\mu\nu}$

$$R_{\mu\nu} = f\left(g_{\mu\nu}\right)$$

$$R = h\left(R_{\mu\nu}\right)$$



- derivation
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$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu}}_{\text{pure geometry}} = \underbrace{8\pi G T_{\mu\nu}}_{\text{properties of matter}}$$

"matter tells space-time how to curve,
and curved space-time tells matter how to move."
(J.A. Wheeler)

- derivation

  - Einstein's Field Equations

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu}}_{\text{pure geometry}} \quad = \quad \underbrace{8\pi G T_{\mu\nu}}_{\text{properties of matter}}$$

isotropy/homogeneity $\Downarrow$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2 \vartheta \end{pmatrix}$$

perfect fluid $\Downarrow$

$$T_{\mu\nu} = -p g^{\mu\nu} + (\rho c^2 + p) u^\mu u^\nu$$

<u>Careful</u>: R in the metric tensor $g_{\mu\nu}$ is *not* the Ricci scalar R

- derivation

  - Einstein's Field Equations

$$\underbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu}}_{\text{pure geometry}} = \underbrace{8\pi G T_{\mu\nu}}_{\text{properties of matter}}$$

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Friedmann-Equations

- derivation

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{c^2\Lambda}{3}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}$$

- derivation

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$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}$$

- critical density

$$k = 0, \quad \Lambda = 0 \quad \rightarrow \quad \rho \mathrel{\hat{=}} \rho_{crit} = \frac{3H^2}{8\pi G} \;, H = \frac{\dot{R}}{R}$$

- derivation

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- critical density → will be our favourite unit for densities later on…

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- equation of state

$$\boxed{p = p(\rho)}$$

- Robertson-Walker-Metric

- Friedmann Equations
  - derivation
  - **equation of state**
  - adiabatic expansion
  - density parameters

- Friedmann Models

- equation of state
  - solving the Friedmann Equations still requires an equation-of-state

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number density

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  - solving the Friedmann Equations still requires an equation-of-state
  - we will restrict ourselves to barotropic fluids (i.e. linear relation between p and ρ):

$$p = \omega(T)\rho c^2$$

$$p = nk_B T = \frac{k_B T}{m_p c^2}\rho_m c^2$$

number density

mass density

- equation of state
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  - we will restrict ourselves to barotropic fluids (i.e. linear relation between p and ρ):

$$p = \omega(T)\rho c^2$$

energy density

$$p = nk_BT = \frac{k_BT}{m_pc^2}\rho_m c^2 = \frac{k_BT}{m_pc^2}\frac{\rho c^2}{(1+\frac{k_BT}{(\gamma-1)m_pc^2})}$$

number density mass density

$p = (\gamma-1)\varepsilon$; γ = adiabatic coefficient (see next sub-section…)

mass density → energy density:

$$\rho c^2 = \rho_m c^2 + \varepsilon = \rho_m + \frac{p}{(\gamma-1)} = \rho_m c^2 + \frac{nk_BT}{(\gamma-1)} = \rho_m c^2 + \frac{\rho_m c^2 k_BT}{(\gamma-1)m_pc^2} = \rho_m c^2\left(1+\frac{k_BT}{(\gamma-1)m_pc^2}\right)$$

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number density mass density

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  - solving the Friedmann Equations still requires an equation-of-state

  - we will restrict ourselves to barotropic fluids (i.e. linear relation between p and ρ):

$$p = \omega(T)\rho c^2$$

  - the main types are:

    - radiation $w = ?$

    - collisionless matter $w = ?$

- equation of state

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    - radiation $w = 1/3$

    - collisionless matter $w = 0$

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$$p = \omega(T)\rho c^2$$

  - the main types are:

    - radiation $w = 1/3$

    - collisionless matter $w = 0$

    - vacuum energy $w = -1$
    - curvature $w = -1/3$

can be cast into energy densities, too

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{c^2\Lambda}{3}$$

- Robertson-Walker-Metric

- Friedmann Equations
  - derivation
  - equation of state
  - **adiabatic expansion**
  - density parameters

- Friedmann Models

- adiabatic expansion
  - energy conservation: $$0 = \dot{\rho} + 3\frac{\dot{R}}{R}(\rho + \frac{p}{c^2})$$

- adiabatic expansion
  - energy conservation: $$0 = \dot{\rho} + 3\frac{\dot{R}}{R}(\rho + \frac{p}{c^2})$$ follows directly from $T^{\mu\nu}_{;\nu} = 0$

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follows from combination of 1. and 2. Friedmann equation:

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1. Friedmann Equation

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 + \frac{c^2\Lambda}{3}R^2$$

- adiabatic expansion
  - energy conservation: $$0 = \dot{\rho} + 3\frac{\dot{R}}{R}(\rho + \frac{p}{c^2})$$

follows from combination of 1. and 2. Friedmann equation:

1. Friedmann Equation

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 + \frac{c^2\Lambda}{3}R^2$$

d/dt

$$2\dot{R}\ddot{R} = \frac{8\pi G}{3}(\dot{\rho}R^2 + 2\rho R\dot{R}) + 2R\dot{R}\frac{c^2\Lambda}{3}$$

$$\ddot{R} = \frac{4\pi G}{3}(\dot{\rho}\frac{R^2}{\dot{R}} + 2\rho R) + R\frac{c^2\Lambda}{3}$$

$$\frac{\ddot{R}}{R} = \frac{4\pi G}{3}(\dot{\rho}\frac{R}{\dot{R}} + 2\rho) + \frac{c^2\Lambda}{3}$$

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2. Friedmann Equation

$$\boxed{\frac{\ddot{R}}{R}} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}$$

- adiabatic expansion
  - energy conservation: $$0 = \dot{\rho} + 3\frac{\dot{R}}{R}(\rho + \frac{p}{c^2})$$

follows from combination of 1. and 2. Friedmann equation:

1. Friedmann Equation

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 + \frac{c^2\Lambda}{3}R^2$$

d/dt

$$2\dot{R}\ddot{R} = \frac{8\pi G}{3}(\dot{\rho}R^2 + 2\rho R\dot{R}) + 2R\dot{R}\frac{c^2\Lambda}{3}$$

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2. Friedmann Equation

$$\boxed{\frac{\ddot{R}}{R}} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}$$

$$\frac{4\pi G}{3}(\dot{\rho}\frac{R}{\dot{R}} + 2\rho) + \frac{c^2\Lambda}{3} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}$$

$$(\dot{\rho}\frac{R}{\dot{R}} + 2\rho) = -\left(\rho + \frac{3p}{c^2}\right)$$

$$\dot{\rho}\frac{R}{\dot{R}} + 2\rho = -\rho - \frac{3p}{c^2}$$

$$0 = \dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right)$$

- adiabatic expansion
  - energy conservation: $$0 = \dot{\rho} + 3\frac{\dot{R}}{R}(\rho + \frac{p}{c^2})$$
  - 2nd law of thermodynamics: $$TdS = dU + pdV$$

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?

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  - energy conservation: $$0 = \dot{\rho} + 3\frac{\dot{R}}{R}(\rho + \frac{p}{c^2})$$
  - 2nd law of thermodynamics: $$TdS = dU + pdV$$

$$dV = d(R^3)$$
$$dU = d(V\rho c^2) = d(R^3\rho c^2)$$

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combine with energy conservation → requires time derivatives...

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  - energy conservation: $$0 = \dot{\rho} + 3\frac{\dot{R}}{R}(\rho + \frac{p}{c^2})$$
  - 2nd law of thermodynamics: $$TdS = dU + pdV$$

$$dV = d(R^3)$$
$$dU = d(R^3\rho c^2)$$

$$\frac{dU}{dt} + p\frac{dV}{dt} = \frac{d\left(R^3\rho c^2\right)}{dt} + p\frac{d\left(R^3\right)}{dt} = 3R^2\dot{R}\rho c^2 + R^3\dot{\rho}c^2 + 3pR^2\dot{R}$$

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  - energy conservation: $$0 = \dot{\rho} + 3\frac{\dot{R}}{R}(\rho + \frac{p}{c^2})$$
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$$0 = R^3\dot{\rho}c^2 + 3\dot{R}R^2\rho c^2 + 3\dot{R}R^2 p$$

$$dV = d(R^3)$$
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$$\frac{dU}{dt} + p\frac{dV}{dt} = \frac{d\left(R^3\rho c^2\right)}{dt} + p\frac{d\left(R^3\right)}{dt} = 3R^2\dot{R}\rho c^2 + R^3\dot{\rho}c^2 + 3pR^2\dot{R} = 0$$

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$$\Rightarrow \quad T\frac{dS}{dt} = 0 \quad \Leftrightarrow \quad TdS = 0$$

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during cosmic expansion entropy is conserved
and hence the expansion is adiabatic:

$$dU + pdV = 0$$

- adiabatic expansion
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- adiabatic expansion
  - cosmic expansion: $$dU + pdV = 0$$

- implication for barotropic fluids $p = \omega \rho c^2$:

- adiabatic expansion
  - cosmic expansion: $$dU + pdV = 0$$

- implication for barotropic fluids $p = \omega \rho c^2$:

$$\begin{aligned} dU &= d(R^3 \rho c^2) \\ dV &= d(R^3) \end{aligned} \quad \Rightarrow \quad d(R^3 \rho c^2) = -p d(R^3) = -\omega \rho c^2 d(R^3)$$

$$\begin{aligned} 0 &= R^3 d\rho + \rho d(R^3) + \omega \rho d(R^3) \\ &= R^3 d\rho + (1+\omega)\rho d(R^3) \end{aligned}$$

$$\frac{1}{\rho} d\rho = -(1+\omega)\frac{d(R^3)}{R^3}$$

$$\boxed{\rho R^{3(1+\omega)} = const.}$$

- adiabatic expansion
  - cosmic expansion: $$dU + pdV = 0$$

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  - radiation $w = 1/3 \quad \Rightarrow \quad \rho_r \propto R^{-4}$
  - collisionless matter $w = 0 \quad \Rightarrow \quad \rho_m \propto R^{-3}$
  - vacuum energy $w = -1 \quad \Rightarrow \quad \rho_\Lambda = const.$
  - curvature $w = -1/3 \quad \Rightarrow \quad \rho_k \propto R^{-2}$

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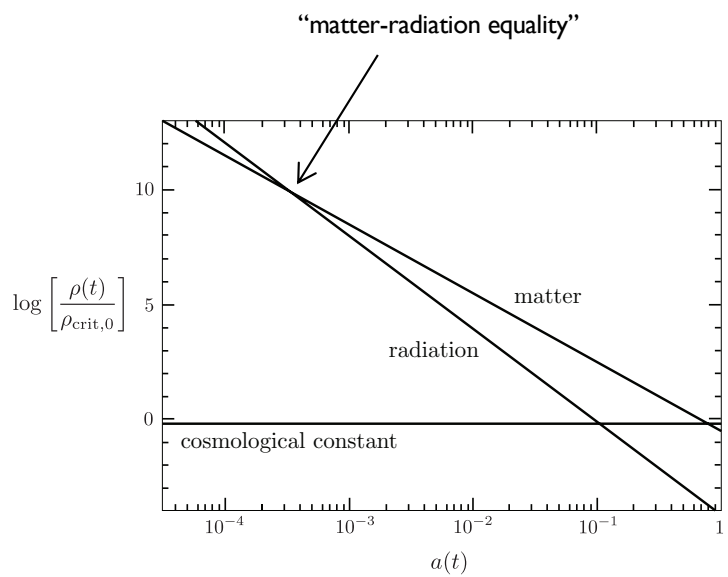
  - curvature $w = -1/3 \Rightarrow \rho_k \propto R^{-2}$

different components dominate at different times!

$$\boxed{\rho R^{3(1+\omega)} = const.}$$

- adiabatic expansion
  - cosmic expansion: $$dU + pdV = 0$$

- implication for barotropic fluids $p = \omega \rho c^2$:
  - radiation $w = 1/3 \quad \Rightarrow \quad \rho_r \propto R^{-4}$
  - collisionless matter $w = 0 \quad \Rightarrow \quad \rho_m \propto R^{-3}$
  - vacuum energy $w = -1 \quad \Rightarrow \quad \rho_\Lambda = const.$
  - curvature $w = -1/3 \quad \Rightarrow \quad \rho_k \propto R^{-2}$



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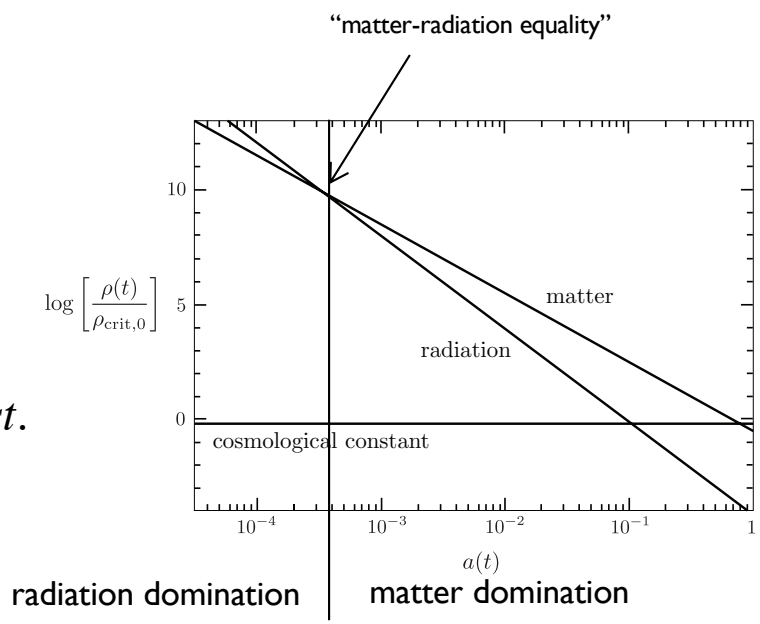

$$\boxed{\rho R^{3(1+\omega)} = const.}$$

- Robertson-Walker-Metric

- Friedmann Equations
  - derivation
  - equation of state
  - adiabatic expansion
  - **density parameters**

- Friedmann Models

- density parameters

the introduction of density parameters

leads to the most memorable form for the Friedmann equations…

- density parameters

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{c^2\Lambda}{3}$$

- density parameters

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{c^2\Lambda}{3}$$

$$\rho = \sum_w \rho_w\left(R(t)\right)$$

- density parameters

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- radiation $w = 1/3 \quad \Rightarrow \quad \rho(R) \propto R^{-4}$
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- density parameters

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{c^2\Lambda}{3}$$

$$\rho = \sum_w \rho_w\left(R(t)\right)$$

- radiation $w = 1/3 \quad \Rightarrow \quad \rho(R) \propto R^{-4}$
- collisionless matter $w = 0 \quad \Rightarrow \quad \rho(R) \propto R^{-3}$

Note:

we can also include

- the Λ-term (w=-1) and
- the curvature (w=-1/3)

in the total energy density….

- density parameters

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{c^2\Lambda}{3}$$

$$\rho = \sum_w \rho_w\left(R(t)\right)$$

- radiation $w = 1/3 \quad \Rightarrow \quad \rho(R) \propto \boxed{R^{-4}}$
- collisionless matter $w = 0 \quad \Rightarrow \quad \rho(R) \propto \boxed{R^{-3}}$

$$\Rightarrow \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left(\rho_{m,0}\boxed{\left(\frac{R}{R_0}\right)^{-3}} + \rho_{r,0}\boxed{\left(\frac{R}{R_0}\right)^{-4}}\right) - \frac{kc^2}{R_0^2}\left(\frac{R}{R_0}\right)^{-2} + \frac{c^2\Lambda}{3}$$

- density parameters

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left(\rho_{m,0}\left(\frac{R}{R_0}\right)^{-3} + \rho_{r,0}\left(\frac{R}{R_0}\right)^{-4}\right) - \frac{kc^2}{R_0^2}\left(\frac{R}{R_0}\right)^{-2} + \frac{c^2\Lambda}{3}$$

- density parameters

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left(\rho_{m,0}\left(\frac{R}{R_0}\right)^{-3} + \rho_{r,0}\left(\frac{R}{R_0}\right)^{-4}\right) - \frac{kc^2}{R_0^2}\left(\frac{R}{R_0}\right)^{-2} + \frac{c^2\Lambda}{3}$$

- critical density $\rho_{crit,0} = \dfrac{3H_0^2}{8\pi G}$

- density parameters

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left(\rho_{m,0}\left(\frac{R}{R_0}\right)^{-3} + \rho_{r,0}\left(\frac{R}{R_0}\right)^{-4}\right) - \frac{kc^2}{R_0^2}\left(\frac{R}{R_0}\right)^{-2} + \frac{c^2\Lambda}{3}$$

<u>the cosmological parameters</u>

- matter density $\Omega_{m,0} = \frac{8\pi G}{3H_0^2}\rho_{m,0}$
- radiation density $\Omega_{r,0} = \frac{8\pi G}{3H_0^2}\rho_{r,0}$
- vacuum energy $\Omega_{\Lambda,0} = \frac{c^2}{3H_0^2}\Lambda$
- curvature $\Omega_{k,0} = -\frac{c^2}{H_0^2 R_0^2}k$
- expansion rate $H_0 = \left(\frac{\dot{R}}{R}\right)_0$

- critical density $\rho_{crit,0} = \frac{3H_0^2}{8\pi G}$

- density parameters

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left(\rho_{m,0}\left(\frac{R}{R_0}\right)^{-3} + \rho_{r,0}\left(\frac{R}{R_0}\right)^{-4}\right) - \frac{kc^2}{R_0^2}\left(\frac{R}{R_0}\right)^{-2} + \frac{c^2\Lambda}{3}$$

<u>the cosmological parameters</u>

- matter density $\Omega_{m,0} = \boxed{\frac{8\pi G}{3H_0^2}}\rho_{m,0}$
- radiation density $\Omega_{r,0} = \frac{8\pi G}{3H_0^2}\rho_{r,0}$
- vacuum energy $\Omega_{\Lambda,0} = \frac{c^2}{3H_0^2}\Lambda$
- curvature $\Omega_{k,0} = -\frac{c^2}{H_0^2 R_0^2}k$
- expansion rate $H_0 = \left(\frac{\dot{R}}{R}\right)_0$

- critical density $\rho_{crit,0} = \frac{3H_0^2}{8\pi G}$

- density parameters

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\underbrace{\left(\rho_{m,0}\left(\frac{R}{R_0}\right)^{-3} + \rho_{r,0}\left(\frac{R}{R_0}\right)^{-4}\right)} - \frac{kc^2}{R_0^2}\left(\frac{R}{R_0}\right)^{-2} + \frac{c^2\Lambda}{3}$$

introduction of $\rho_{crit,0}$ here leads to $8\pi G/3$ factor, allowing for a combination with the other two energy densities...

- critical density $\rho_{crit,0} = \dfrac{3H_0^2}{8\pi G}$

<u>the cosmological parameters</u>

- matter density $\Omega_{m,0} = \dfrac{8\pi G}{3H_0^2}\rho_{m,0}$
- radiation density $\Omega_{r,0} = \dfrac{8\pi G}{3H_0^2}\rho_{r,0}$
- vacuum energy $\Omega_{\Lambda,0} = \dfrac{c^2}{3H_0^2}\Lambda$
- curvature $\Omega_{k,0} = -\dfrac{c^2}{H_0^2 R_0^2}k$
- expansion rate $H_0 = \left(\dfrac{\dot{R}}{R}\right)_0$

- density parameters

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\left(\rho_{m,0}\left(\frac{R}{R_0}\right)^{-3} + \rho_{r,0}\left(\frac{R}{R_0}\right)^{-4}\right) - \frac{kc^2}{R_0^2}\left(\frac{R}{R_0}\right)^{-2} + \frac{c^2\Lambda}{3}$$

<u>the cosmological parameters</u>

- matter density $\Omega_{m,0} = \dfrac{\rho_{m,0}}{\rho_{crit,0}}$
- radiation density $\Omega_{r,0} = \dfrac{\rho_{r,0}}{\rho_{crit,0}}$
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- curvature $\Omega_{k,0} = -\dfrac{c^2}{H_0^2 R_0^2}k$
- expansion rate $H_0 = \left(\dfrac{\dot{R}}{R}\right)_0$

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

derivation uses $R/R_0=1/(1+z)$ and otherwise (hopefully) obvious…

<u>the cosmological parameters</u>

- matter density $\Omega_{m,0} = \dfrac{\rho_{m,0}}{\rho_{crit,0}}$
- radiation density $\Omega_{r,0} = \dfrac{\rho_{r,0}}{\rho_{crit,0}}$
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<u>the cosmological parameters</u>

- matter density $\Omega_{m,0} = \dfrac{\rho_{m,0}}{\rho_{crit,0}}$
- radiation density $\Omega_{r,0} = \dfrac{\rho_{r,0}}{\rho_{crit,0}}$
- vacuum energy $\Omega_{\Lambda,0} = \dfrac{c^2}{3H_0^2}\Lambda$
- curvature $\Omega_{k,0} = -\dfrac{c^2}{H_0^2 R_0^2}k$
- expansion rate $H_0 = \left(\dfrac{\dot{R}}{R}\right)_0$

<u>cosmic sum rule: (z=0)</u>

$$1 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{k,0} + \Omega_{\Lambda,0}$$

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

<u>the cosmological parameters</u>

- matter density $\Omega_{m,0} = \dfrac{\rho_{m,0}}{\rho_{crit,0}}$
- radiation density $\Omega_{r,0} = \dfrac{\rho_{r,0}}{\rho_{crit,0}}$
- vacuum energy $\Omega_{\Lambda,0} = \dfrac{c^2}{3H_0^2}\Lambda$
- curvature $\Omega_{k,0} = -\dfrac{c^2}{H_0^2 R_0^2}k$
- expansion rate $H_0 = \left(\dfrac{\dot{R}}{R}\right)_0$

<u>cosmic sum rule:</u>

$$1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda$$

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

<u>the cosmological parameters</u>

- matter density $\Omega_{m,0} = \dfrac{\rho_{m,0}}{\rho_{crit,0}}$
- radiation density $\Omega_{r,0} = \dfrac{\rho_{r,0}}{\rho_{crit,0}}$
- vacuum energy $\Omega_{\Lambda,0} = \dfrac{c^2}{3H_0^2}\Lambda$
- curvature $\Omega_{k,0} = -\dfrac{c^2}{H_0^2 R_0^2}k$
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$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

derivation:

$$\frac{R}{R_0} = \frac{1}{1+z}$$

$$\begin{aligned}
\frac{\ddot{R}}{R} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3} \\
&= -\frac{4\pi G}{3}\left(\sum\rho_i + \frac{3\sum\omega_i\rho_i c^2}{c^2}\right) + \frac{c^2\Lambda}{3} \\
&= -\frac{4\pi G}{3}\left(\sum\rho_i + 3\sum\omega_i\rho_i\right) + \frac{c^2\Lambda}{3} \\
&= -\frac{4\pi G}{3}\sum(1+3\omega_i)\rho_i + \frac{c^2\Lambda}{3} \\
&= -\frac{H^2}{2}\left(\sum(1+3\omega_i)\Omega_i(z)\right) + H^2\Omega_\Lambda(z) \\
&= -\frac{H^2}{2}\left(\sum(1+3\omega_i)\Omega_i(z)\right) - H^2\Omega_\Lambda(z) \\
&= -\frac{H^2}{2}\left(\Omega_m(z) + 2\Omega_r(z)\right) - H^2\Omega_\Lambda(z) \\
-\frac{\ddot{R}R}{\dot{R}^2} &= \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)
\end{aligned}$$

only radiation and matter enter as for curvature (1+3(-1/3))=0

$$-\frac{1}{H^2}$$

the cosmological parameters

- matter density $\Omega_{m,0} = \dfrac{\rho_{m,0}}{\rho_{crit,0}}$
- radiation density $\Omega_{r,0} = \dfrac{\rho_{r,0}}{\rho_{crit,0}}$
- vacuum energy $\Omega_{\Lambda,0} = \dfrac{c^2}{3H_0^2}\Lambda$
- curvature $\Omega_{k,0} = -\dfrac{c^2}{H_0^2 R_0^2}k$
- expansion rate $H_0 = \left(\dfrac{\dot{R}}{R}\right)_0$

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$ → uses density parameters at present time

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$ → uses density parameters at redshift z

<u>cosmic sum rule:</u>

$$1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda$$

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$ → uses density parameters at present time

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$ → uses density parameters at redshift z

$$\Omega(z)\ ?$$

<u>cosmic sum rule:</u>

$$1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda$$

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2 \left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0} \right)$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

- evolution of density parameters:

$$\Omega(z) = \frac{8\pi G}{3H^2(z)}\rho(z)$$

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

- evolution of density parameters:

$$\Omega(z) = \frac{8\pi G}{3H^2(z)}\rho(z) = \frac{8\pi G}{3H^2(z)}\rho_0\left(1+z\right)^{3(1+\omega)}$$

barotropic matter

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

- evolution of density parameters:

$$\Omega(z) = \frac{8\pi G}{3H^2(z)}\rho(z) = \frac{8\pi G}{3H_0^2}\cdot\frac{\rho_0\left(1+z\right)^{3(1+\omega)}}{(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0})}$$

1st Friedmann equation

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

- evolution of density parameters:

$$\Omega(z) = \frac{8\pi G}{3H^2(z)}\rho(z) = \frac{\Omega_0\left(1+z\right)^{3(1+\omega)}}{(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0})}$$

$\rho_0 \rightarrow \Omega_0$

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

- evolution of density parameters:

$$\Omega(z) = \frac{\Omega_0}{(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0})}\left(1+z\right)^{3(1+\omega)}$$

- radiation $w = 1/3$
- matter $w = 0$

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

- evolution of density parameters:

$$\Omega(z) = \frac{\Omega_0}{(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0})}\left(1+z\right)^{3(1+\omega)}$$

- radiation $w = 1/3$
- matter $w = 0$
- vacuum energy $w = -1$

($\Omega_\Lambda(z) = \frac{c^2\Lambda}{3H^2(z)} = \frac{\Omega_\Lambda}{(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0})}$; same results as for $\omega = -1$)

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2 \boxed{\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)}$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

- evolution of density parameters:

$$\Omega(z) = \frac{\Omega_0}{\boxed{(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0})}}(1+z)^{3(1+\omega)}$$

- radiation $w = 1/3$
- matter $w = 0$
- vacuum energy $w = -1$

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

- evolution of density parameters:

$$\Omega(z) = \frac{\Omega_0}{(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0})}\left(1+z\right)^{3(1+\omega)}$$

- radiation $w = 1/3$
- matter $w = 0$
- vacuum energy $w = -1$

<u>Further useful:</u>

$$\frac{\Omega_i}{\Omega_j} = \frac{\Omega_{i,0}}{\Omega_{j,0}}\left(1+z\right)^{(n_i - n_j)} \quad \text{with} \quad n = 3(1+\omega)$$

- density parameters → Friedmann Equations in most memorizable form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

- evolution of density parameters:

$$\Omega(z) = \frac{\Omega_0}{(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0})}\left(1+z\right)^{3(1+\omega)}$$

- radiation $w = 1/3$
- matter $w = 0$
- vacuum energy $w = -1$

Further useful:

$$\frac{\Omega_i}{\Omega_j} = \frac{\Omega_{i,0}}{\Omega_{j,0}}\left(1+z\right)^{(n_i - n_j)}$$

- cosmic sum rule:

$$1 = \Omega_r(z) + \Omega_m(z) + \Omega_k(z) + \Omega_\Lambda(z)$$

- density parameters → Friedmann Equations in most **memorizable** form

$$H^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2}\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)$$

- evolution of density parameters:

$$\Omega(z) = \frac{\Omega_0}{(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0})}(1+z)^{3(1+\omega)}$$

- radiation $w = 1/3$
- matter $w = 0$
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Further useful:

$$\frac{\Omega_i}{\Omega_j} = \frac{\Omega_{i,0}}{\Omega_{j,0}}(1+z)^{(n_i - n_j)}$$

- cosmic sum rule:

$$1 = \Omega_r(z) + \Omega_m(z) + \Omega_k(z) + \Omega_\Lambda(z)$$

- interplay scale factor – redshift – time

$$H^2 = H_0^2 \left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0} \right)$$

$\Rightarrow$ solution is $a(t)$

$\Rightarrow$ relation to redshift: $a(t) = \dfrac{1}{1+z}$

for standard cosmological models we have a unique mapping between

$$a \leftrightarrow t \leftrightarrow z$$

(further, the constancy of the speed of light then also relates those to distance)

- Robertson-Walker-Metric

- Friedmann Equations

- Friedmann Models
  - **general trends**
  - special solutions for flat and open world models
  - horizons

Friedmann Models

- Friedmann Equations

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 + \frac{c^2\Lambda}{3}R^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}R$$

Friedmann Models

- Friedmann Equations - general behaviour

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right)$$

- $\Lambda = 0$, $p = \omega\rho c^2$

Friedmann Models

- Friedmann Equations - general behaviour

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right)$$

- $\Lambda = 0,\ p = \omega\rho c^2$ $\qquad \dot{R}^2 \propto R^{2-3(1+\omega)} - kc^2$ $\qquad$ as $\rho \propto R^{-3(1+\omega)}$

Note : $2 - 3(1+\omega) < 0$

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as $\rho \propto R^{-3(1+\omega)}$

$$\ddot{R} < 0$$

as $\rho + \frac{3p}{c^2} > 0 \quad \left(p = \omega\rho c^2,\ \omega > 0\right)$

Note : $2 - 3(1+\omega) < 0$

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- Friedmann Equations - general behaviour

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as $\rho + \frac{3p}{c^2} > 0 \quad \left(p = \omega\rho c^2,\ \omega > 0\right)$

we observe: $\left(\dot{R}/R\right)_0 > 0$

Note : $2 - 3(1+\omega) < 0$

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- Friedmann Equations - general behaviour

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we observe: $\left(\dot{R}/R\right)_0 > 0 \quad \underset{R \neq 0}{\Rightarrow} \quad \left(\dot{R}^2\right)_0 > 0$

Note : $2 - 3(1+\omega) < 0$

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- Friedmann Equations - general behaviour

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

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as $\rho + \frac{3p}{c^2} > 0 \ \left(p = \omega\rho c^2,\ \omega > 0\right)$

we observe: $\left(\dot{R}/R\right)_0 > 0 \quad \Rightarrow \quad \left(\dot{R}^2\right)_0 > 0$

$\dot{R} > 0,\ \ddot{R} < 0$

=> $R(t)$ has been monotonically increasing! => $\exists\ t_c$ with $R(t_c) = 0$ => **singularity**!

Note : $2 - 3(1+\omega) < 0$

Friedmann Models

- Friedmann Equations - general behaviour

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as $\rho \propto R^{-3(1+\omega)}$

$$\ddot{R} < 0$$

as $\rho + \frac{3p}{c^2} > 0 \ \left(p = \omega\rho c^2,\ \omega > 0\right)$

<u>future:</u>

$k = 0$: $\dot{R}^2 \propto R^{2-3(1+\omega)} \longrightarrow R \propto t^{2/3(1+\omega)}$ => indefinite expansion!

$$R^{3(1+\omega)/2-1}dR = dt$$

Note : $2 - 3(1+\omega) < 0$

Friedmann Models

- Friedmann Equations - general behaviour

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right)$$

- $\Lambda = 0,\ p = \omega\rho c^2$

$$\dot{R}^2 \propto R^{2-3(1+\omega)} - kc^2 \qquad \text{as } \rho \propto R^{-3(1+\omega)}$$

$$\ddot{R} < 0 \qquad \text{as } \rho + \frac{3p}{c^2} > 0 \ \left(p = \omega\rho c^2,\ \omega > 0\right)$$

<u>future:</u>

$k = 0$: $\quad \dot{R}^2 \propto R^{2-3(1+\omega)} \longrightarrow R \propto t^{2/3(1+\omega)}$ => indefinite expansion!

$k > 0$: $\quad \dot{R}^2 \propto R^{2-3(1+\omega)} - |k|c^2 \longrightarrow \exists \dot{R}(t_s) = 0$ => recollapse!

$$\left.\begin{array}{l}\dot{R}(0) = +\infty \\ \dot{R}(\infty) = -|k|c^2\end{array}\right\} \dot{R} \text{ changes sign (sometime)!}$$

Note : $2 - 3(1+\omega) < 0$

Friedmann Models

- Friedmann Equations - general behaviour

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$$\ddot{R} < 0 \qquad \text{as } \rho + \frac{3p}{c^2} > 0 \ \left(p = \omega\rho c^2,\ \omega > 0\right)$$

<u>future:</u>

$k = 0$: $\dot{R}^2 \propto R^{2-3(1+\omega)} \longrightarrow R \propto t^{2/3(1+\omega)}$ => indefinite expansion!

$k > 0$: $\dot{R}^2 \propto R^{2-3(1+\omega)} - |k|c^2 \longrightarrow \exists \dot{R}(t_s) = 0$ => recollapse!

$k < 0$: $\dot{R}^2 \propto R^{2-3(1+\omega)} + |k|c^2 \longrightarrow \dot{R}^2 > 0 \rightarrow R(t)\uparrow$ => increased indefinite expansion!

Note : $2 - 3(1+\omega) < 0$

Friedmann Models

- Friedmann Equations - general behaviour

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 + \frac{c^2\Lambda}{3}R^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}R$$

- $\Lambda < 0$

Friedmann Models

- Friedmann Equations - general behaviour

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 + \frac{c^2\Lambda}{3}R^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}R$$

- $\Lambda < 0$

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 - \frac{c^2|\Lambda|}{3}R^2 = \frac{8\pi G}{3}\rho_{m+\Lambda}R^2 - kc^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) - \frac{c^2|\Lambda|}{3}R = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2} + \frac{c^2|\Lambda|}{12\pi G}\right) < 0$$

Friedmann Models

- Friedmann Equations - general behaviour

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"+" => attractive force!

Friedmann Models

- Friedmann Equations - general behaviour

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 + \frac{c^2\Lambda}{3}R^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}R$$

- $\Lambda < 0$

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 - \frac{c^2|\Lambda|}{3}R^2 = \frac{8\pi G}{3}\rho_{m+\Lambda}R^2 - kc^2$$

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"+" => attractive force!

=> $R(t)$ starts monotonically increasing & $\exists \dot{R}(t_s) = 0$

=> recollapse happens even earlier due to "attractive force"!

Friedmann Models

- Friedmann Equations - general behaviour

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 + \frac{c^2\Lambda}{3}R^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}R$$

- $\Lambda > 0$

Friedmann Models

- Friedmann Equations - general behaviour

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 + \frac{c^2\Lambda}{3}R^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2\Lambda}{3}R$$

- $\Lambda > 0$

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2 - \frac{c^2|\Lambda|}{3}R^2 = \frac{8\pi G}{3}\rho_{m+\Lambda}R^2 - kc^2$$

$$\ddot{R} = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2}\right) + \frac{c^2|\Lambda|}{3}R = -\frac{4\pi G}{3}R\left(\rho + \frac{3p}{c^2} - \frac{c^2|\Lambda|}{12\pi G}\right) < 0$$

"–" => repulsive force!

$\Rightarrow$ forever expanding!*

(Note: ρ drops while Λ remains constant…)

***careful**: there are (ρ, Λ) combinations possible where this is not necessarily true…

- Robertson-Walker-Metric

- Friedmann Equations

- Friedmann Models
  - general trends
  - **special solutions for flat and open world models**
  - horizons

Friedmann Models

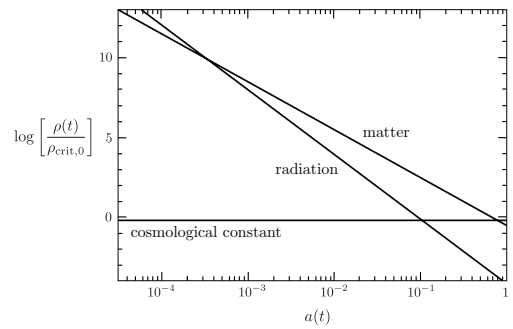
- Friedmann Equations - solutions

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

Friedmann Models

- Friedmann Equations - solutions

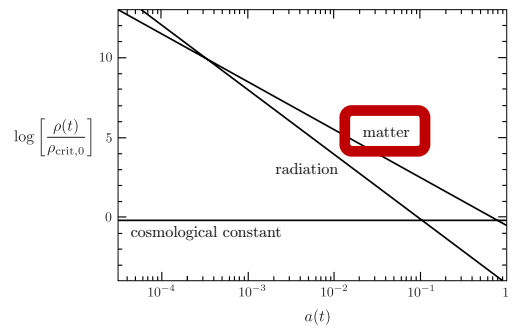
$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$



Friedmann Models

- Friedmann Equations - solutions

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$



Friedmann Models

- Friedmann Equations - solutions

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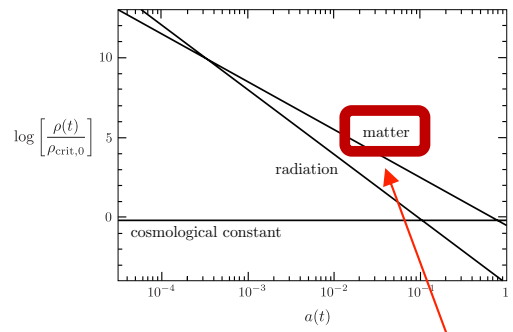
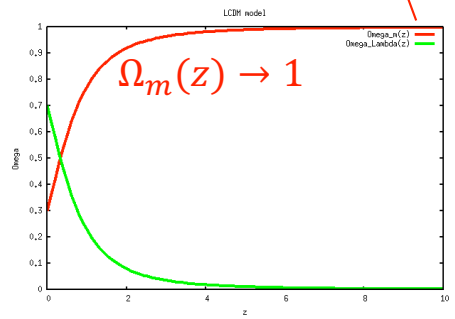


$\Omega_m \to 1$

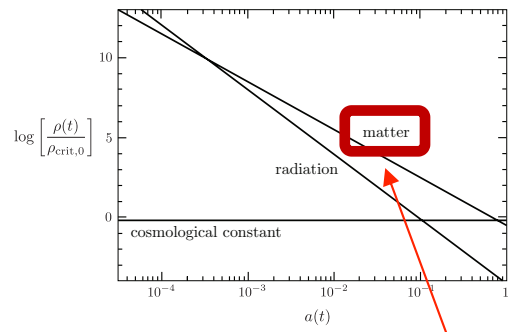


Friedmann Models

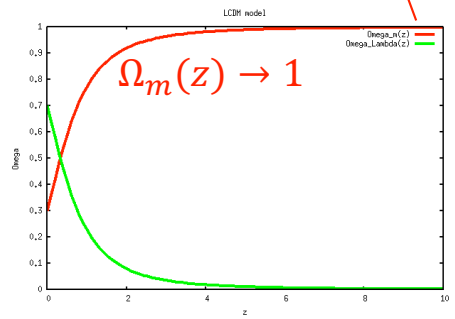
- Friedmann Equations - solutions

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \underline{\Omega_{m,0}(1+z)^3} + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

- flat model with $\Omega_m = 1$



$\Omega_m \to 1$



Note: this solution is important *right after* matter-radiation equality!

Friedmann Models

- Friedmann Equations - solutions

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \underline{\Omega_{m,0}(1+z)^3} + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

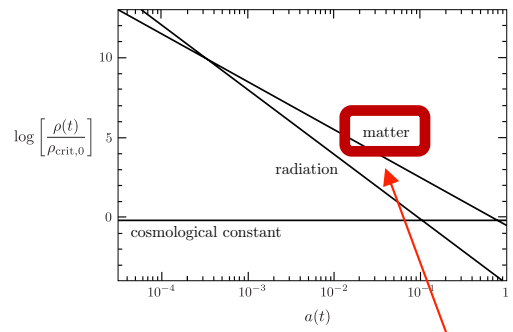
- flat model with $\Omega_m = 1$

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\underline{\left(\frac{R_0}{R}\right)^3}$$

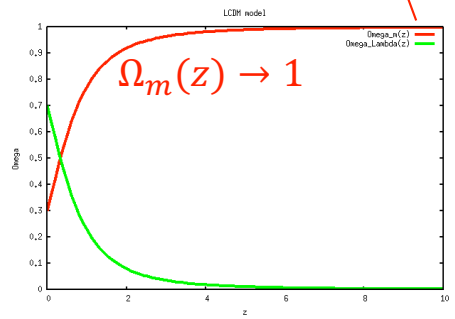
$$\frac{\dot{R}}{R} = H_0\left(\frac{R_0}{R}\right)^{3/2}$$

$$R^{3/2}\frac{\dot{R}}{R} = H_0 R_0^{3/2}$$

$$R^{1/2}dR = H_0 R_0^{3/2}dt \quad => \quad \frac{R(t)}{R_0} = \left(\frac{t}{t_0}\right)^{2/3} \quad t_0 = \frac{2}{3H_0}$$



$\Omega_m \to 1$



Friedmann Models

- Friedmann Equations - solutions

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\underline{\Omega_{r,0}(1+z)^4} + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

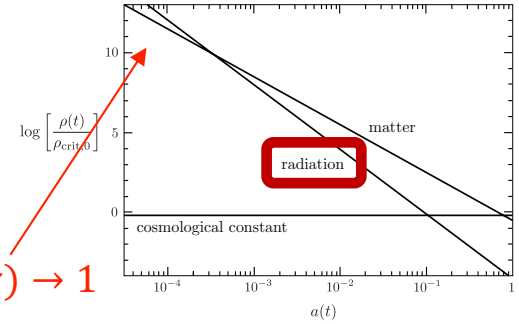
- flat model with $\Omega_r = 1$

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\underline{\left(\frac{R_0}{R}\right)^4}$$

$$\frac{\dot{R}}{R} = H_0\left(\frac{R_0}{R}\right)^2$$

$$R^2\frac{\dot{R}}{R} = H_0 R_0^2$$

$$RdR = H_0 R_0^2 dt \qquad => \qquad \frac{R(t)}{R_0} = \left(\frac{t}{t_0}\right)^{1/2} \qquad t_0 = \frac{1}{2H_0}$$



$\Omega_r(z) \to 1$

Friedmann Models

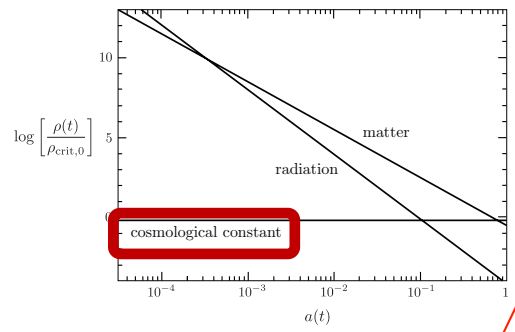
- Friedmann Equations - solutions

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \underline{\Omega_{\Lambda,0}}\right)$$

- flat model with $\Omega_\Lambda = 1$

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2$$

$$\frac{\dot{R}}{R} = H_0$$



$\Omega_\Lambda(z) \to 1$

$$\frac{dR}{R} = H_0 dt \qquad => \qquad R(t) = e^{H_0 t}$$

inflationary Universe!

Friedmann Models

- Friedmann Equations - solutions

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

- general model with $\Omega_m \neq 0,\ \Omega_\Lambda = 0,\ \ \Omega_r = 0$

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{m,0}\left(\frac{R_0}{R}\right)^3 + \Omega_{k,0}\left(\frac{R_0}{R}\right)^2\right)$$

$$\dot{R}^2 = R^2 H_0^2\left(\frac{R_0}{R}\right)^2\left(\Omega_{m,0}\left(\frac{R_0}{R}\right) + \Omega_{k,0}\right)$$

$$\dot{R} = H_0 R_0\sqrt{\Omega_{m,0}\left(\frac{R_0}{R}\right) + \left(1 - \Omega_{m,0}\right)}$$

$$\dot{R} = H_0 R_0\sqrt{\Omega_{m,0}\left(\frac{R_0}{R} - 1\right) + 1}$$

Friedmann Models

- Friedmann Equations - solutions

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

- general model with $\Omega_m \neq 0,\ \Omega_\Lambda = 0,\ \ \Omega_r = 0$

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{m,0}\left(\frac{R_0}{R}\right)^3 + \Omega_{k,0}\left(\frac{R_0}{R}\right)^2\right)$$

$$\dot{R}^2 = R^2 H_0^2\left(\frac{R_0}{R}\right)^2\left(\Omega_{m,0}\left(\frac{R_0}{R}\right) + \Omega_{k,0}\right) \quad =>$$

$$\dot{R} = H_0 R_0\sqrt{\Omega_{m,0}\left(\frac{R_0}{R}\right) + \left(1-\Omega_{m,0}\right)}$$

$$\dot{R} = H_0 R_0\sqrt{\Omega_{m,0}\left(\frac{R_0}{R} - 1\right) + 1}$$

$\underline{\Omega_{m,0} > 1}$

$$R = a\left(1-\cos\vartheta\right) \qquad a = \frac{\Omega_{m,0}}{2(\Omega_{m,0}-1)}$$

$$t = b(\vartheta - \sin\vartheta) \qquad b = \frac{\Omega_{m,0}}{2H_0(\Omega_{m,0}-1)^{3/2}}$$

$\underline{\Omega_{m,0} < 1}$

$$R = A\left(\cosh\theta - 1\right) \qquad A = \frac{\Omega_{m,0}}{2(1-\Omega_{m,0})}$$

$$t = B(\sinh\theta - \theta) \qquad B = \frac{\Omega_{m,0}}{2H_0(1-\Omega_{m,0})^{3/2}}$$

- Friedmann Equations - solutions

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

- general model with $\Omega_m \neq 0,\ \Omega_\Lambda = 0,\ \ \Omega_r = 0$

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{m,0}\left(\frac{R_0}{R}\right)^3 + \Omega_{k,0}\left(\frac{R_0}{R}\right)^2\right)$$

$$\dot{R}^2 = R^2 H_0^2\left(\frac{R_0}{R}\right)^2\left(\Omega_{m,0}\left(\frac{R_0}{R}\right) + \Omega_{k,0}\right) \quad \Rightarrow$$

$$\dot{R} = H_0 R_0\sqrt{\Omega_{m,0}\left(\frac{R_0}{R}\right) + \left(1-\Omega_{m,0}\right)}$$

$$\dot{R} = H_0 R_0\sqrt{\Omega_{m,0}\left(\frac{R_0}{R} - 1\right) + 1}$$

important for analytical structure formation

$\underline{\Omega_{m,0} > 1}$

$$R = a(1-\cos\vartheta) \qquad a = \frac{\Omega_{m,0}}{2(\Omega_{m,0}-1)}$$

$$t = b(\vartheta - \sin\vartheta) \qquad b = \frac{\Omega_{m,0}}{2H_0(\Omega_{m,0}-1)^{3/2}}$$

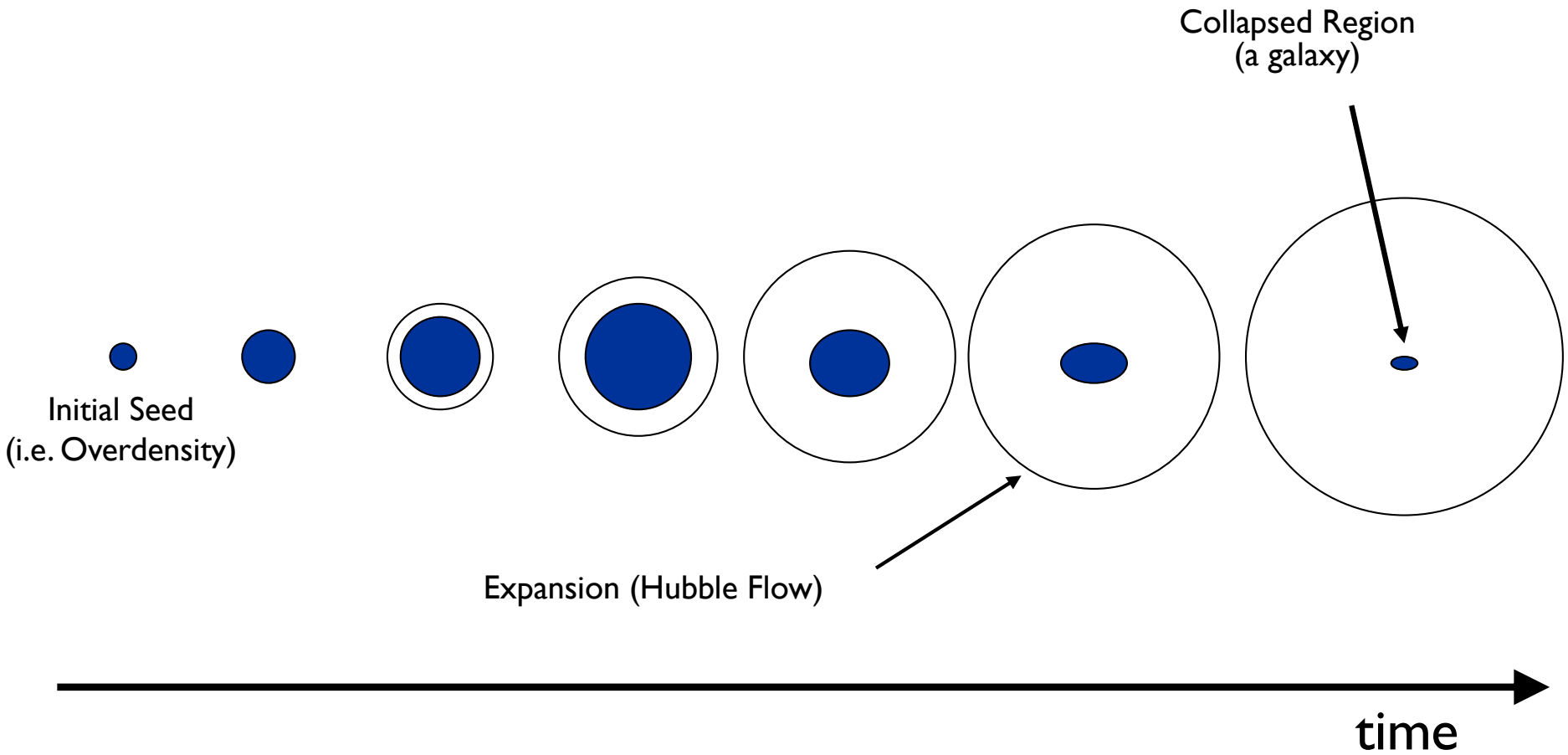
$\underline{\Omega_{m,0} < 1}$

$$R = A(\cosh\theta - 1) \qquad A = \frac{\Omega_{m,0}}{2(1-\Omega_{m,0})}$$

$$t = B(\sinh\theta - \theta) \qquad B = \frac{\Omega_{m,0}}{2H_0(1-\Omega_{m,0})^{3/2}}$$

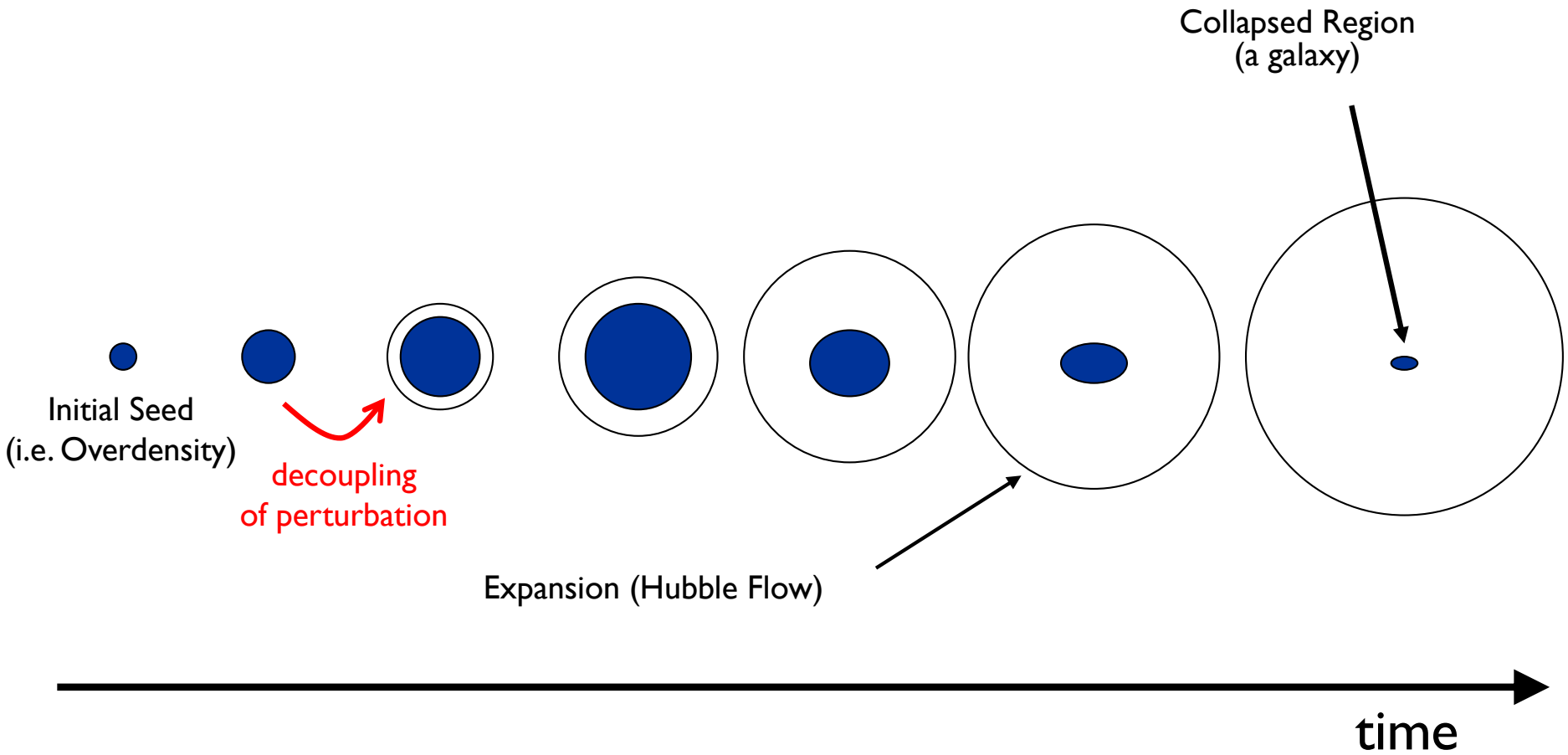
Friedmann Models

- Friedmann Equations – structure formation



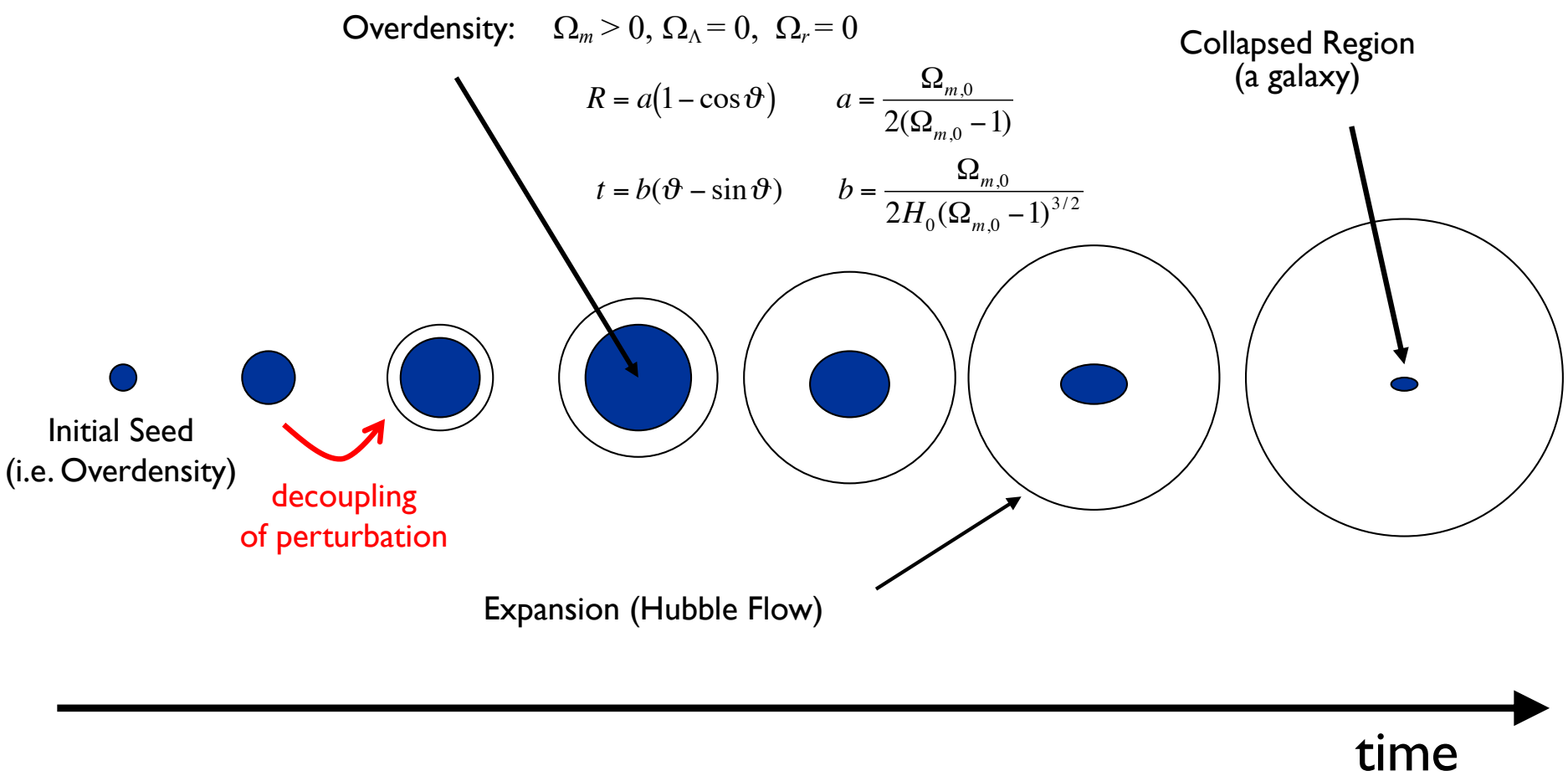
Friedmann Models

- Friedmann Equations – structure formation



Friedmann Models

- Friedmann Equations – structure formation



Friedmann Models

- Friedmann Equations - solutions

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

- flat model with $\Omega_m + \Omega_\Lambda = 1,\ \Omega_r = 0$

$$t(z) = \frac{2}{3H_0}\frac{1}{\sqrt{\Omega_{\Lambda,0}}}\ln\left(\sqrt{\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\left(\frac{1}{1+z}\right)^3} + \sqrt{\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\left(\frac{1}{1+z}\right)^3 + 1}\right)$$

Friedmann Models

- Friedmann Equations - solutions

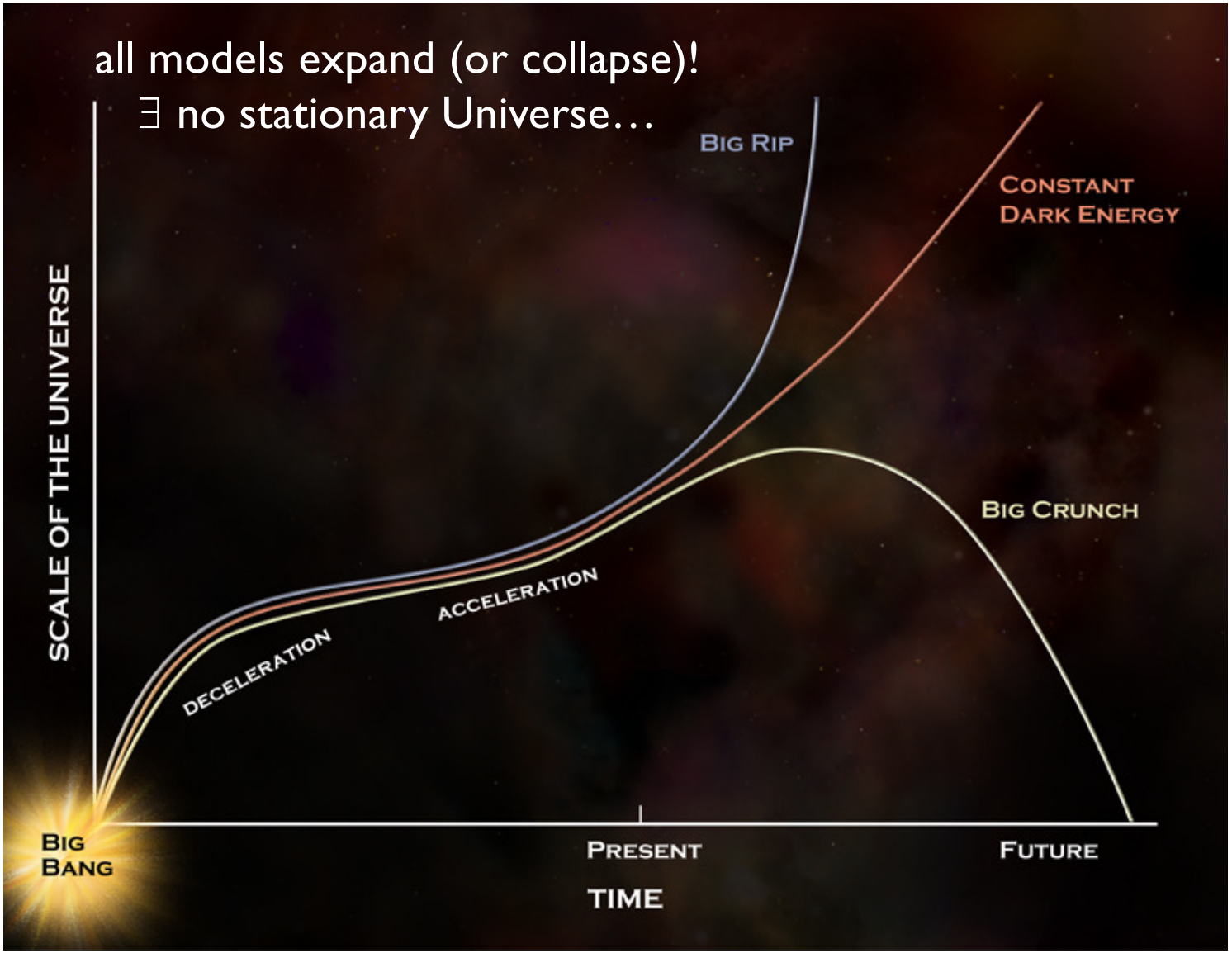
$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}\right)$$

- general model with $\Omega_m \neq 0,\ \Omega_\Lambda \neq 0,\ \ \Omega_r = 0$

no analytical solution for the general case!

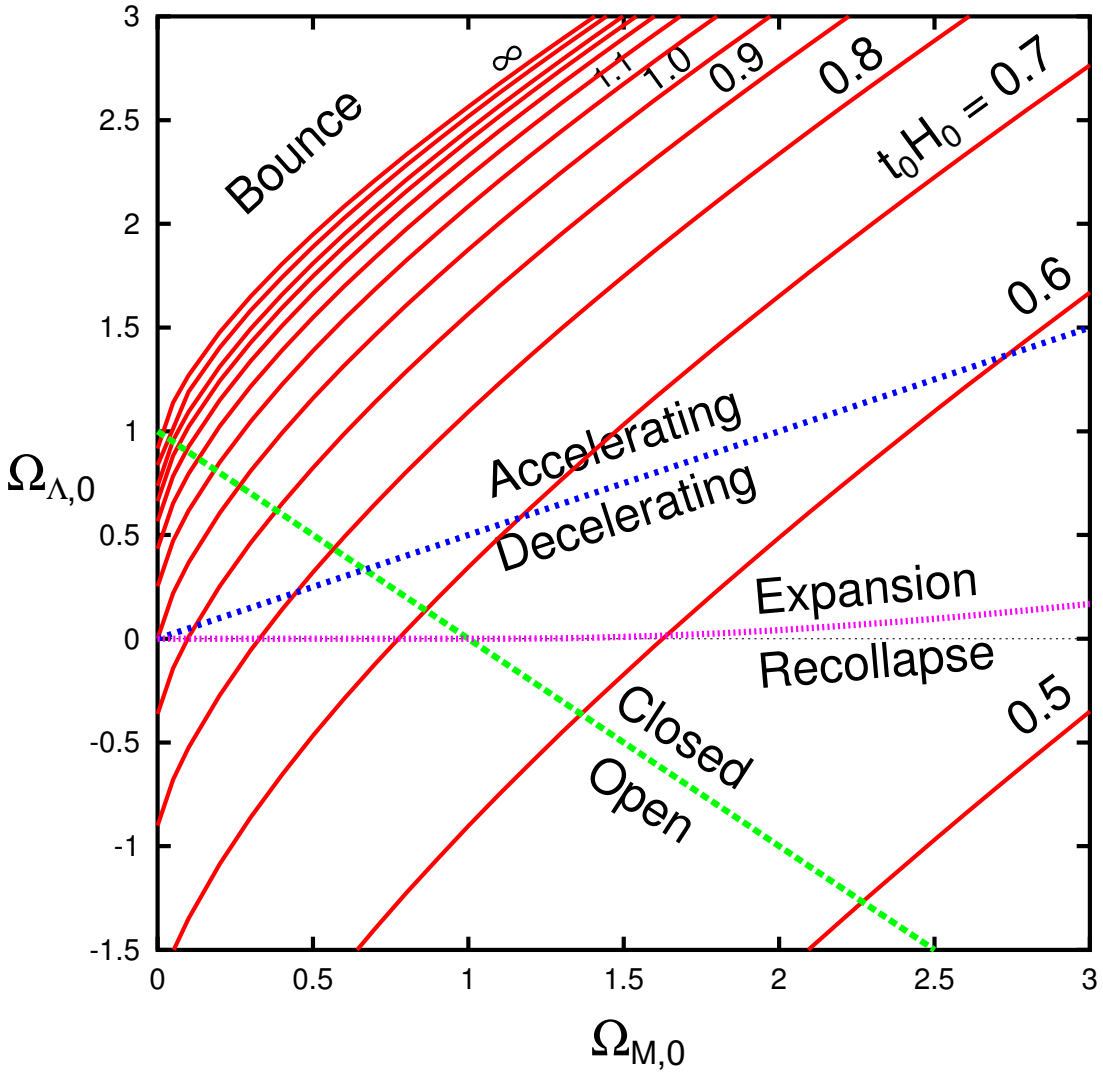
Friedmann Models

- Friedmann Equations - solutions



Friedmann Models

- Friedmann Equations - solutions



(bounce condition: $H = 0$ & $dH/dt > 0$)

- Robertson-Walker-Metric

- Friedmann Equations

- Friedmann Models
  - general trends
  - special solutions for flat and open world models
  - **horizons**

Friedmann Models

- horizons

<u>cosmological horizons measure distances…</u>

…from which one could possibly retrieve information from the past,

…or interact with in the future

Friedmann Models

- horizons: distances

$$ds^2 = (cdt)^2 - R^2(t)\left[\frac{dx^2}{1-kx^2} + x^2\left(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2\right)\right]$$

Friedmann Models

- horizons: distances

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- photons follow null-geodesics

$$0 = (cdt)^2 - R^2(t)\left[\frac{dx^2}{1-kx^2} + x^2\left(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2\right)\right]$$

Friedmann Models

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- no transverse motion

$$f(x_E) = \int_0^{x_E} \frac{dx}{\sqrt{1-kx^2}} = \int_{t_E}^{t_0} \frac{cdt}{R(t)}$$

Friedmann Models

- horizons: distances

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comoving distance

$$d_C = f(x_E) = \int_0^{x_E} \frac{dx}{\sqrt{1-kx^2}} = \int_{t_E}^{t_0} \frac{cdt}{R(t)}$$

$$f(x_E) = \begin{cases} x_E & \text{k=0} \\ \frac{1}{\sqrt{|k|}}\arcsin(\sqrt{|k|}\, x_E) & \text{k=1} \\ \frac{1}{\sqrt{|k|}}\operatorname{arcsinh}(\sqrt{|k|}\, x_E) & \text{k=-1} \end{cases}$$

Friedmann Models

- horizons: distances

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comoving distance

$$d_C = f(x_E) = \int_0^{x_E} \frac{dx}{\sqrt{1-kx^2}} = \int_{t_E}^{t_0} \frac{cdt}{R(t)}$$

proper distance

$$d_p = \frac{R(t)}{R_0} d_C = \frac{R(t)}{R_0} \int_{t_E}^{t_0} \frac{cdt}{R(t)}$$

$$f(x_E) = \begin{cases} x_E & \text{k=0} \\ \frac{1}{\sqrt{|k|}} \arcsin(\sqrt{|k|}\, x_E) & \text{k=1} \\ \frac{1}{\sqrt{|k|}} \operatorname{arcsinh}(\sqrt{|k|}\, x_E) & \text{k=-1} \end{cases}$$

Friedmann Models

- horizons: distances

$$d_p = \frac{R(t)}{R_0} d_C = \frac{R(t)}{R_0} \int_{t_E}^{t_0} \frac{c\,dt}{R(t)}$$

Friedmann Models

- horizons: distances

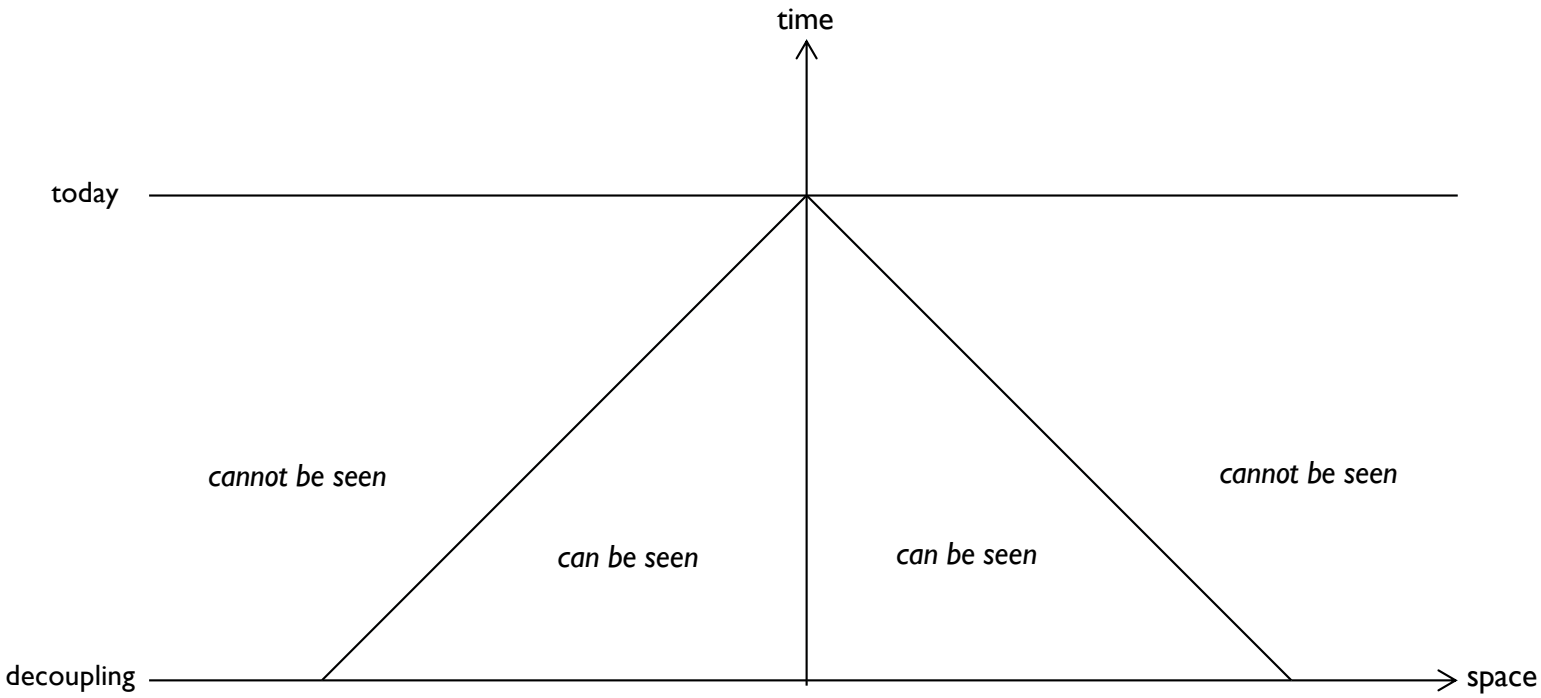
$$d_p = \frac{R(t)}{R_0} d_C = \frac{R(t)}{R_0} \int_{t_E}^{t_0} \frac{c\,dt}{R(t)}$$

different bounds define different horizons...

Friedmann Models

- horizons
  - particle horizon: max. distance particle can have travelled since decoupling

$$R_p(t) = R(t)\int_{t_{dec}}^{t} \frac{c\,dt'}{R(t')}$$



Friedmann Models

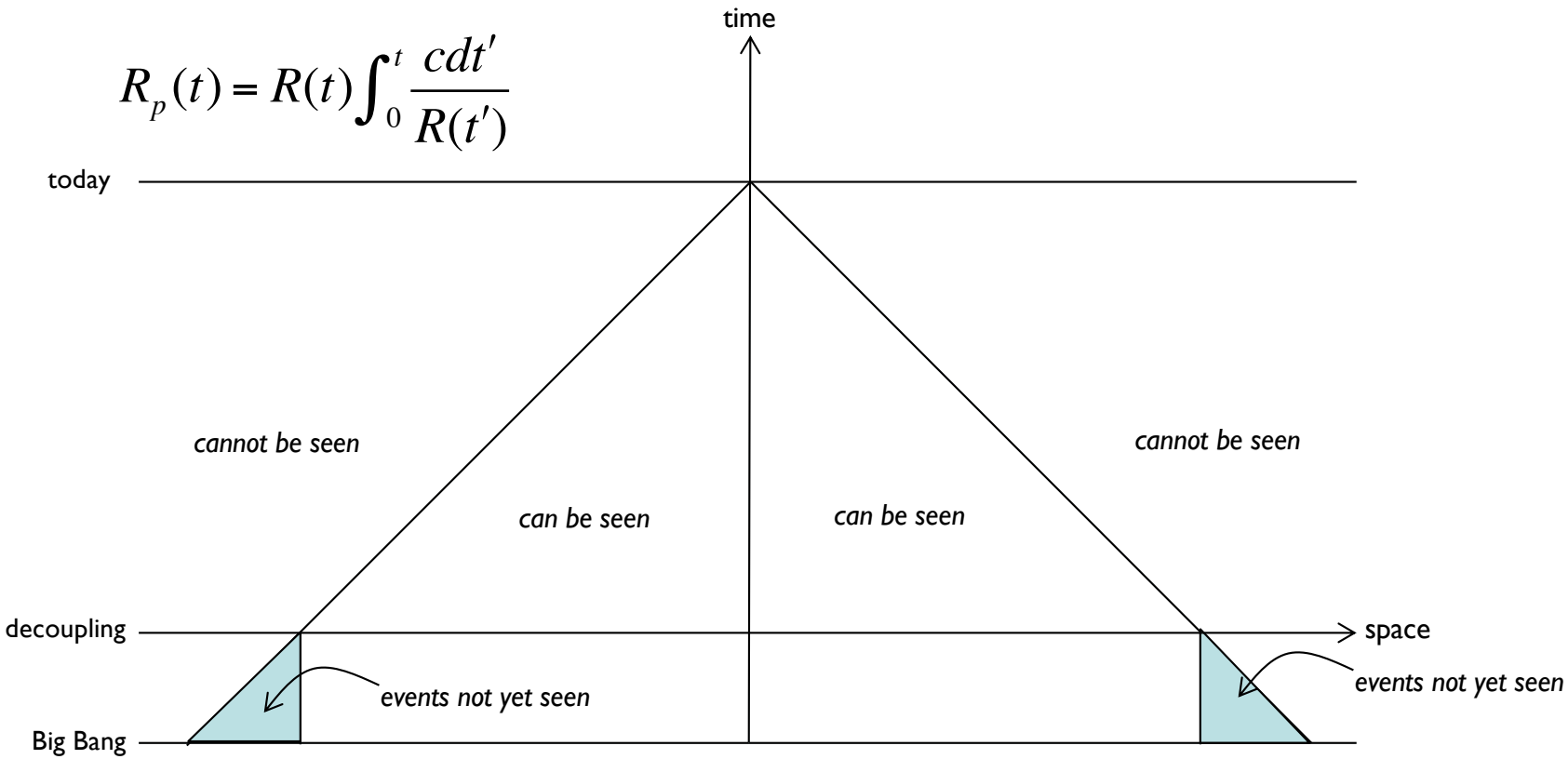
- horizons

  - particle horizon: max. distance particle can have travelled since decoupling

$$R_p(t) = R(t)\int_{t_{dec}}^{t} \frac{cdt'}{R(t')}$$

  - "particle horizon": max. distance photon can have travelled since big bang (there are events we have not have seen yet…)

$$R_p(t) = R(t)\int_{0}^{t} \frac{cdt'}{R(t')}$$



Friedmann Models

- horizons

  - particle horizon: max. distance particle can have travelled since decoupling

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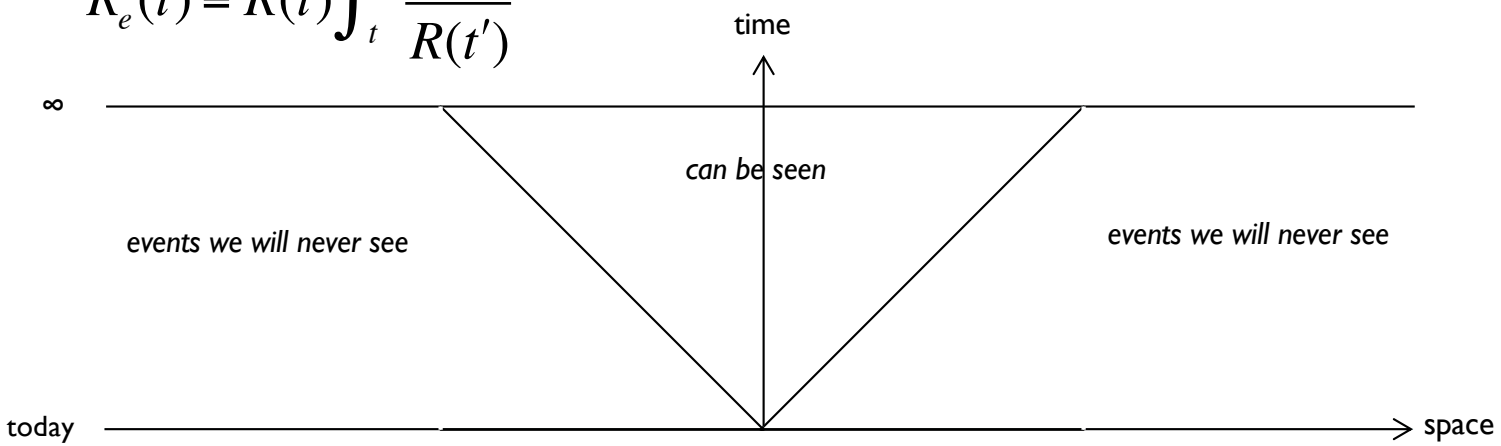
  - "particle horizon": max. distance photon can have travelled since big bang (there are events we have not have seen yet…)

$$R_p(t) = R(t)\int_{0}^{t} \frac{cdt'}{R(t')}$$

  - event horizon: max. distance particle can travel from now onwards (there may be events we will never see…)

$$R_e(t) = R(t)\int_{t}^{\infty} \frac{cdt'}{R(t')}$$



Friedmann Models

- horizons

  - particle horizon: max. distance particle can have travelled since decoupling

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  - Hubble radius: distance at which recessional velocity equals speed of light

$$R_H(t) = \frac{c}{H}$$

Friedmann Models

- horizons

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$$R_e(t) = R(t)\int_{t}^{\infty} \frac{cdt'}{R(t')}$$

  - (comoving) Hubble radius: distance at which recessional velocity equals speed of light

$$R_H(t) = \frac{c}{H}; \qquad R_{cH}(t) = \frac{R_0}{R}\frac{c}{H}$$

Friedmann Models

- horizons

  - particle horizon: max. distance particle can have travelled since decoupling

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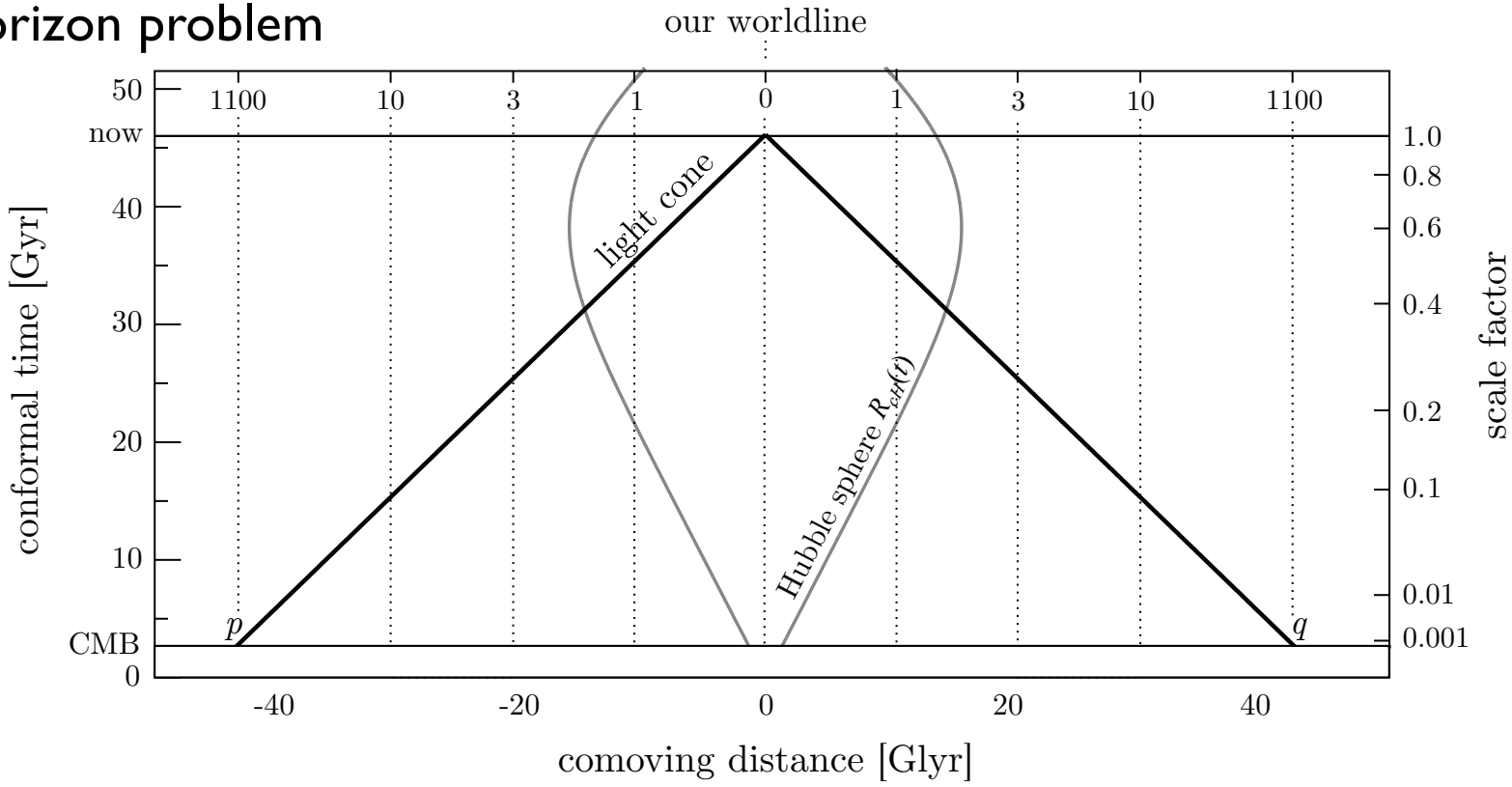
$$R_e(t) = R(t)\int_{t}^{\infty} \frac{cdt'}{R(t')}$$

  - (comoving) Hubble radius: distance at which recessional velocity equals speed of light

$$R_H(t) = \frac{c}{H}; \quad R_{cH}(t) = \frac{R_0}{R}\frac{c}{H} \xrightarrow[\dot{R}^2 \propto R^{2-3(1+\omega)}]{\text{Universe dominated by barotropic fluid } p=\omega\rho c^2} R_{cH}(t) \propto R^{\frac{1}{2}(1+3\omega)}$$

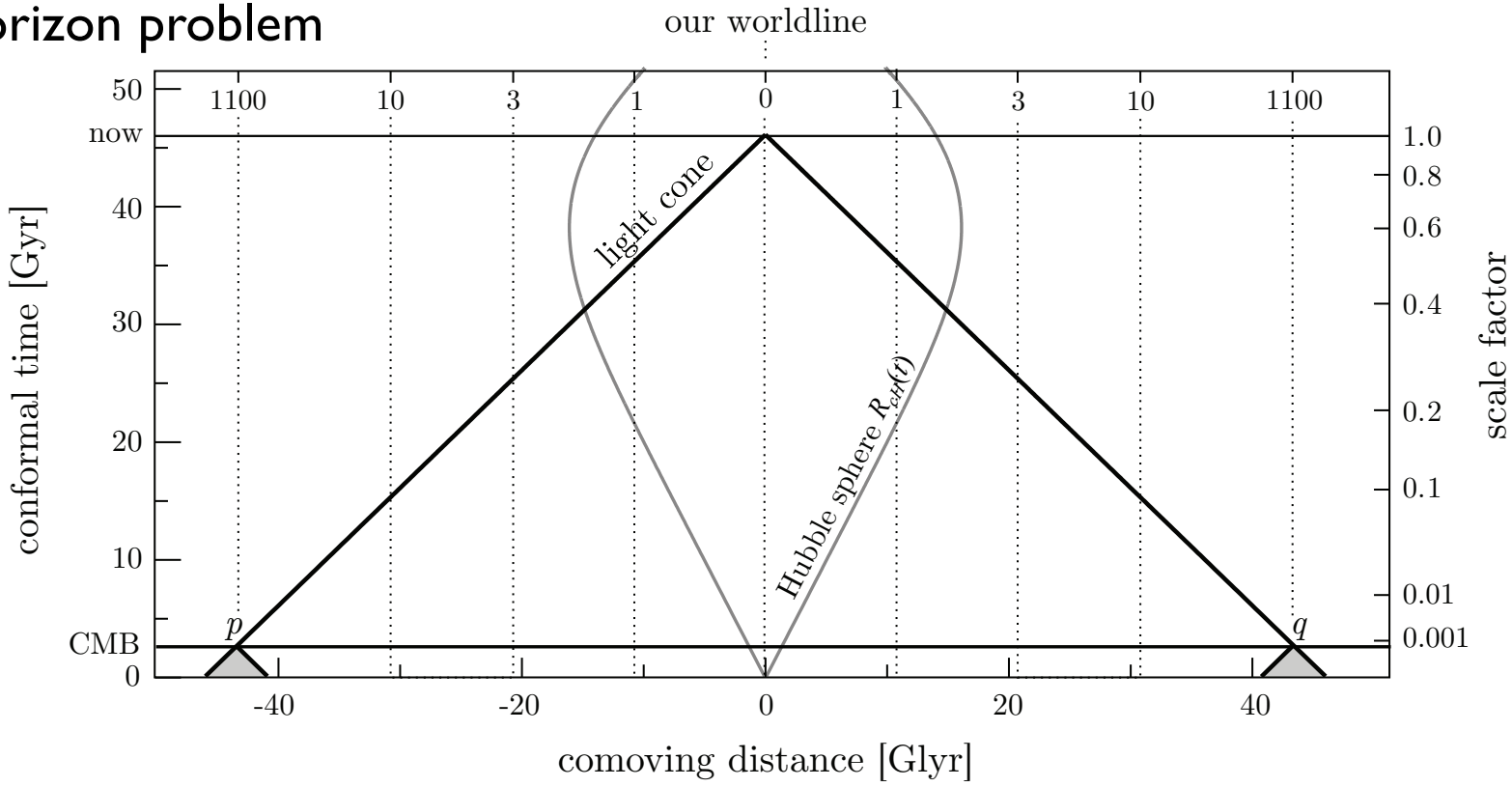
Friedmann Models

- horizon problem



Friedmann Models

- horizon problem

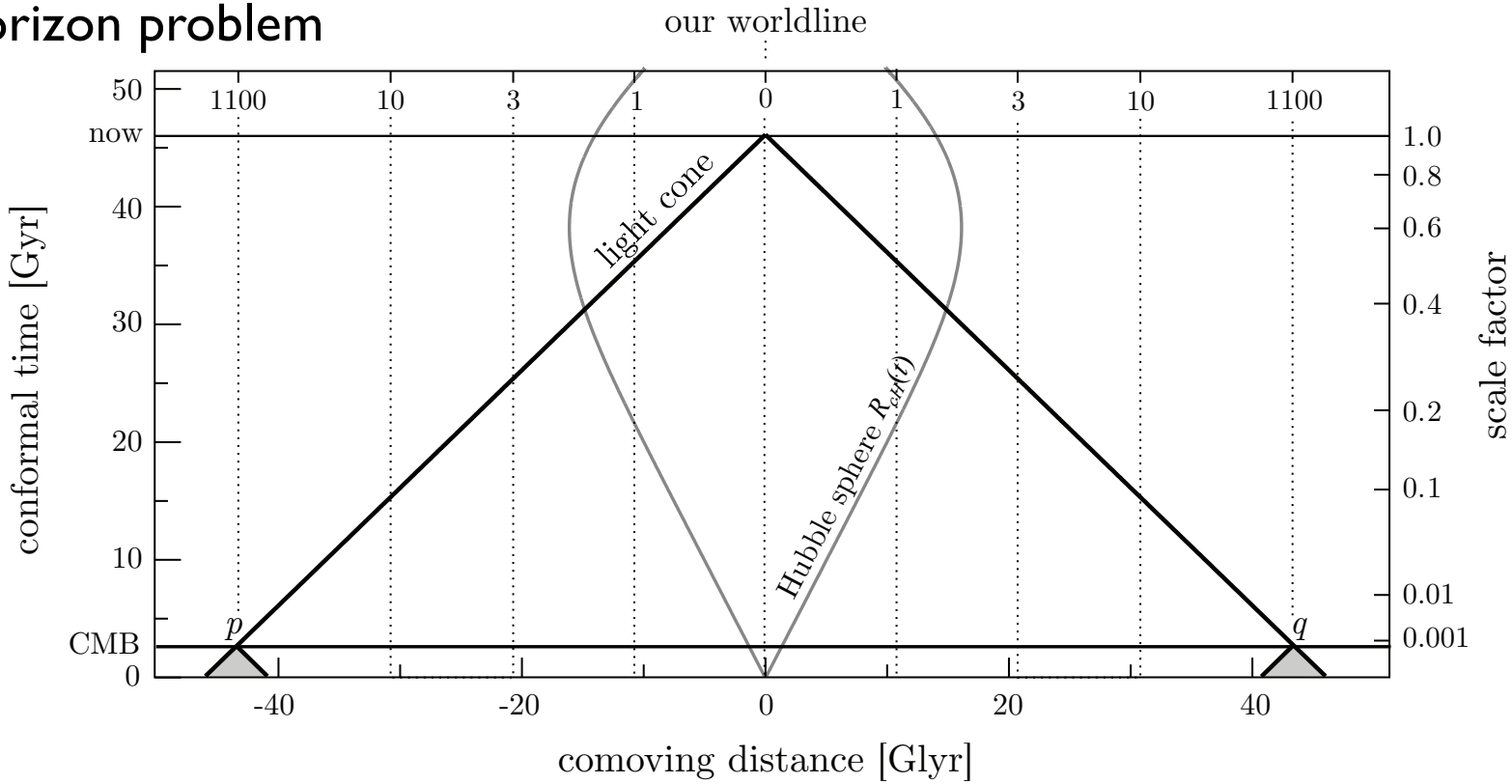


p and q are causally disconnected

(applies to any two points separated by more than 1° on the sky!)

Friedmann Models

- horizon problem



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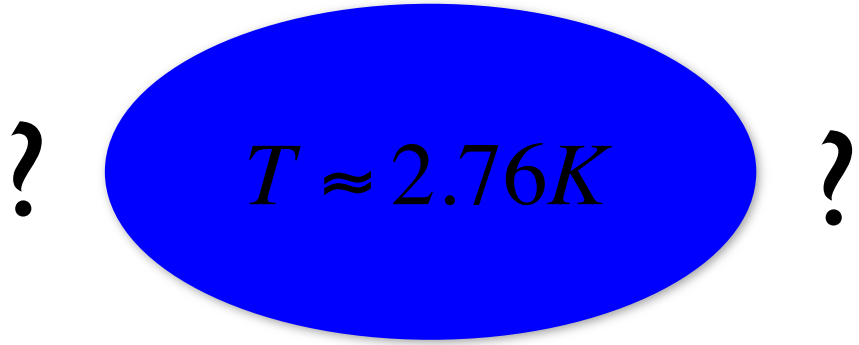

Friedmann Models

- horizon problem: solution via inflation