Friedmann Models

Alexander Knebe (Universidad Autonoma de Madrid)

IF THE UNIVERSE IS STILL EXPANDING, GET ME SOME.
Friedmann Models

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

Dark Matter + Dark Energy affect the expansion of the universe

<table>
<thead>
<tr>
<th>$\Omega_m$</th>
<th>$\Omega_\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.7</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
</tr>
<tr>
<td>5.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Billions of Years
Friedmann Models

- Robertson-Walker-Metric

- Friedmann Equations

- Friedmann Models
Friedmann Models

- Robertson-Walker-Metric

- Friedmann Equations

Friedmann derived equations for $R(t)$ in 1922, but did not use the most general metric

- Friedmann Models
Robertson-Walker-Metric

Robertson (1935) & Walker (1935): derivation of most general metric

Friedmann Equations

Friedmann derived equations for $R(t)$ in 1922, but did not use the most general metric

Friedmann Models
Friedmann Models

- **Robertson-Walker-Metric**
  - derivation
  - geodesics
  - redshift

- **Friedmann Equations**

- **Friedmann Models**
Friedmann Models

- Robertson-Walker-Metric
  - derivation
  - geodesics
  - redshift

- Friedmann Equations

- Friedmann Models
principle of general relativity

- the equations describing the laws of physics have the same form
  irrespective of the coordinate system
principle of general relativity

- the equations describing the laws of physics have the same form
  irrespective of the coordinate system

=> fulfilled by Einstein’s field equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \]
principle of general relativity

- the equations describing the laws of physics have the same form irrespective of the coordinate system

=> fulfilled by Einstein’s field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

we require a metric $g_{\mu\nu}$
- **principle of general relativity**

  • the equations describing the laws of physics have the same form *irrespective of the coordinate system*

  \[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \]

- **cosmological principle**

  • space is homogeneous and isotropic
principle of general relativity

- the equations describing the laws of physics have the same form irrespective of the coordinate system

=> fulfilled by Einstein’s field equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \]

cosmological principle

- space is homogeneous and isotropic
- cosmological principle

  • isotropy

  • homogeneity
- cosmological principle
  - isotropy $\iff$ CMB
  - homogeneity $?$

$T = 2.725K$
- cosmological principle

  - isotropy  $\iff$  CMB
  - homogeneity  $\iff$  galaxy surveys*

* for instance, WiggleZ Dark Energy Survey (arXiv:1205.6812) reported $R_{\perp} \sim 75$ Mpc/$h$
- principle of general relativity
  - the equations describing the laws of physics have the same form
    irrespective of the coordinate system
    \[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \]

- cosmological principle
  - space is *homogeneous* and *isotropic*
principle of general relativity

- the equations describing the laws of physics have the same form irrespective of the coordinate system

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \]

=> fulfilled by Einstein’s field equations

 cosmological principle   => fulfilled by Robertson-Walker-Metric

- space is homogeneous and isotropic
  ! ! !
• metric in general

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \]

with \( x^\mu = \text{vector} \)

\[ g_{\mu\nu} = \text{metric tensor} \]
- metric in general

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

with \( x^\mu = \text{vector} \)

\[ g_{\mu\nu} = \text{metric tensor} \]

example for 3D Euclidean space: \( ds^2 = dx^2 + dy^2 + dz^2 \) \( \Leftrightarrow g_{\mu\nu} = 1 \)
Friedmann Models

\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu \]

with \( x^\mu = (ct, x^i) \) = space-time vector

\[ g_{\mu \nu} = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \] = metric tensor

**Note:** Greek-index => four-vector (space-time), Roman-index => three-vector (space)
metric of curved space-time

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

with \( x^\mu = (ct, x^i) \) = space-time vector

\[ g_{\mu\nu} = \begin{pmatrix} -1 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \] = metric tensor

**Note:** Greek-index \( \Rightarrow \) four-vector (space-time),
Roman-index \( \Rightarrow \) three-vector (space)
Friedmann Models

- metric of curved space-time

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

with \( x^\mu = (ct, x^i) \) = space-time vector

\[ g_{\mu\nu} = \begin{pmatrix} -1 & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \] = metric tensor

any non-trivial \( g_{00} \) component can be absorbed be re-defining time \( dt' = \sqrt{g_{00}}dt \)

Note: Greek-index => four-vector (space-time), Roman-index => three-vector (space)
- metric of curved space-time

\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu \]

with

\[ x^\mu = (ct, x^i) \]

= space-time vector

\[
g_{\mu \nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\
0 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\
0 & \gamma_{31} & \gamma_{32} & \gamma_{33}
\end{pmatrix}
\]

= metric tensor

**Note:**
- Greek-index => four-vector (space-time),
- Roman-index => three-vector (space)
metric of curved space-time

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

with \[ x^\mu = (ct, x^i) \]

\[ g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ 0 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ 0 & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \]

\[ g_{0i} = 0 \] (if time depended on space, measurement of time could distinguish one place from another…)
Friedmann Models

- metric of curved space-time

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

with \( x^\mu = (ct, x^i) \) = space-time vector

\[ g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ 0 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ 0 & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \] = metric tensor

\( \gamma_{ij} \) should describe a homogeneous and isotropic 3D space

\[ g_{ij} \] should describe a homogeneous and isotropic 3D space

Robertson-Walker-Metric

Note: “space” is maximally symmetric, but “space-time” not
• metric of curved space-time

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \]

with \( x^\mu = (ct, x^i) \) = space-time vector

\[ g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ 0 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ 0 & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \] = metric tensor

\( \gamma_{ij} \) should describe a homogeneous and isotropic 3D space

\( \uparrow \) maximal symmetry for the 4D hypersurface described by \( x^i \)

Note: “space” is maximally symmetric, but “space-time” not…
Friedmann Models

- metric of curved space
  - 1D space ≡ hypersurface in 2D space

Robertson-Walker-Metric
• metric of curved space

• 1D space ≡ hypersurface in 2D space
• metric of curved space

• 1D space ≡ hypersurface in 2D space

\[ dl^2 = dx^2 + dy^2 \]

\[ R^2 = x^2 + y^2 \] => 1D distance on 2D hypersurface

\[ R^2 = x^2 + y^2 \] => 2D hypersurface
- metric of curved space

  - 1D space \(\equiv\) hypersurface in 2D space

\[
dl^2 = dx^2 + dy^2 \quad \Rightarrow \text{1D distance on 2D hypersurface}
\]

\[
R^2 = x^2 + y^2 \quad \Rightarrow \text{2D hypersurface}
\]

we like to get rid of that 2\textsuperscript{nd} dimension (e.g. \(y\)) as we live in 1D space (e.g. only \(x\))
• metric of curved space

• 1D space ≡ hypersurface in 2D space

\[ dl^2 = dx^2 + dy^2 \]

\[ R^2 = x^2 + y^2 \]

=> 1D distance on 2D hypersurface

=> 2D hypersurface
- metric of curved space

- 1D space $\equiv$ hypersurface in 2D space

\[ dl^2 = dx^2 + dy^2 \quad \Rightarrow \quad \text{1D distance on 2D hypersurface} \]

\[ R^2 = x^2 + y^2 \quad \Rightarrow \quad \text{2D hypersurface} \]

\[ dl^2 = dx^2 + dy^2 \]

\[ R^2 = x^2 + y^2 \]

\[ 0 = xdx + ydy \quad \rightarrow \quad dy = -xdx / y \]

\[ y^2 = R^2 - x^2 \]
Friedmann Models

- metric of curved space
  - 1D space \( \equiv \) hypersurface in 2D space

\[
dl^2 = dx^2 + dy^2 \\
R^2 = x^2 + y^2
\]

\[
dl^2 = dx^2 + dy^2 \Rightarrow 1D \text{ distance on 2D hypersurface}
\]

\[
0 = xdx + ydy \quad \Rightarrow \quad dy = -\frac{x dx}{y}
\]

\[
y^2 = R^2 - x^2
\]

\[
dl^2 = dx^2 + \frac{x^2 dx^2}{R^2 - x^2}
\]

\[
\Rightarrow 2D \text{ hypersurface}
\]
• metric of curved space

  • 1D space ≡ hypersurface in 2D space

\[
\begin{aligned}
    dl^2 &= dx^2 + dy^2 \\
    R^2 &= x^2 + y^2
\end{aligned}
\]

\[
\implies \quad 1\text{D distance on 2D hypersurface}
\]

\[
\begin{aligned}
    dl^2 &= dx^2 + dy^2 \\
    R^2 &= x^2 + y^2
\end{aligned}
\]

\[
\implies \quad 2\text{D hypersurface}
\]

\[
\begin{aligned}
    0 &= xdx + ydy \\
    dy &= -\frac{xdx}{y} \\
    y^2 &= R^2 - x^2
\end{aligned}
\]

\[
\begin{aligned}
    dl^2 &= dx^2 + \frac{x^2 dx^2}{R^2 - x^2} \\
    (R \to \infty \implies dl = dx)
\end{aligned}
\]
### Friedmann Models

- **metric of curved space**

  - 1D space $\equiv$ hypersurface in 2D space

\[
\begin{align*}
\gamma_{11} &= 1 + \frac{x^2}{R^2 - x^2} \\
R^2 &= x^2 + y^2 \\
dl^2 &= dx^2 + dy^2 \\
0 &= xdx + ydy \quad \rightarrow \quad dy = -\frac{x}{y}dx \\
R^2 &= x^2 + y^2 \\
dl^2 &= dx^2 + \frac{x^2dx^2}{R^2 - x^2} \quad (R \rightarrow \infty \Rightarrow dl = dx)
\end{align*}
\]


\[
\begin{align*}
\gamma_{11} &= 1 + \frac{x^2}{R^2 - x^2} \\
R^2 &= x^2 + y^2 \\
dl^2 &= dx^2 + dy^2 \\
0 &= xdx + ydy \quad \rightarrow \quad dy = -\frac{x}{y}dx \\
R^2 &= x^2 + y^2 \\
dl^2 &= dx^2 + \frac{x^2dx^2}{R^2 - x^2} \quad (R \rightarrow \infty \Rightarrow dl = dx)
\end{align*}
\]
- metric of curved space
  - 3D space $\equiv$ hypersurface in 4D space

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2$$
$$R^2 = x_1^2 + x_2^2 + x_3^2 + z^2$$
• metric of curved space

- 3D space ≡ hypersurface in 4D space

spherical 3D space: \( dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2 \)
\( R^2 = x_1^2 + x_2^2 + x_3^2 + z^2 \)

hyperbolic 3D space: \( dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2 \)
\( -R^2 = (x_1^2 + x_2^2 + x_3^2) + z^2 \)

Euclidean 3D space: \( dl^2 = dx_1^2 + dx_2^2 + dx_3^2 \)

careful: \( R \) is not the size of the Universe…
Friedmann Models

- metric of curved space

- 3D space ≡ hypersurface in 4D space

spherical 3D space: \[ dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2 \]
\[ R^2 = x_1^2 + x_2^2 + x_3^2 + z^2 \]

hyperbolic 3D space: \[ dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2 \]
\[ -R^2 = (x_1^2 + x_2^2 + x_3^2) + z^2 \]

Euclidean 3D space: \[ dl^2 = dx_1^2 + dx_2^2 + dx_3^2 \]

these are the only 3 possible cases for homogeneous & isotropic metrics!
Friedmann Models

- **metric of curved space**
  
  - 3D space ≡ hypersurface in 4D space

  spherical 3D space: \[ dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2 \]
  \[ R^2 = x_1^2 + x_2^2 + x_3^2 + z^2 \]

  hyperbolic 3D space: \[ dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2 \]
  \[ -R^2 = \left( x_1^2 + x_2^2 + x_3^2 \right) + z^2 \]

  Euclidean 3D space: \[ dl^2 = dx_1^2 + dx_2^2 + dx_3^2 \]

  these are the only 3 possible cases for homogeneous & isotropic metrics:

<table>
<thead>
<tr>
<th>3D space:</th>
<th>Euclidean</th>
<th>spherical</th>
<th>hyperbolic</th>
</tr>
</thead>
<tbody>
<tr>
<td>homogeneity:</td>
<td>( dl ) is invariant under translations ( x_i \rightarrow x_i + D )</td>
<td>inherited from (trivial) 4D symmetry</td>
<td>Sec 13.2 in Weinberg’s “Gravitation &amp; Cosmology”</td>
</tr>
<tr>
<td>isotropy:</td>
<td>( dl ) is invariant under rotations ( x_i \rightarrow R^i_k \ x^k )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
• metric of curved space

• 3D space ≡ hypersurface in 4D space

spherical 3D space:  
\[ dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2 \]
\[ R^2 = x_1^2 + x_2^2 + x_3^2 + z^2 \]

\[ \gamma_{ij} ? \]
- metric of curved space

- 3D space ≡ hypersurface in 4D space

spherical 3D space:

\[
 dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dz^2
\]
\[
 R^2 = x_1^2 + x_2^2 + x_3^2 + z^2
\]

\[\downarrow\text{differentiation}\]

\[0 = x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + zdz\]
Friedmann Models

- metric of curved space

  - 3D space \( \equiv \) hypersurface in 4D space

spherical 3D space:

\[
\begin{align*}
  dl^2 &= dx_1^2 + dx_2^2 + dx_3^2 + dz^2 \\
  R^2 &= x_1^2 + x_2^2 + x_3^2 + z^2
\end{align*}
\]

\[\downarrow \text{differentiation}\]

\[0 = x_1 dx_1 + x_2 dx_2 + x_3 dx_3 + zdz\]

\[\downarrow \text{combination with } dl^2 \text{ to eliminate } z\]

\[
dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - (x_1^2 + x_2^2 + x_3^2)}
\]

Robertson-Walker-Metric
• metric of curved space

  • 3D space ≡ hypersurface in 4D space

spherical 3D space: \( dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - (x_1^2 + x_2^2 + x_3^2)} \)
- metric of curved space

- 3D space \(\equiv\) hypersurface in 4D space

spherical 3D space: \(dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1dx_1 + x_2dx_2 + x_3dx_3)^2}{R^2 - (x_1^2 + x_2^2 + x_3^2)}\)

introduction of spherical coordinates will diagonalize \(\gamma_{ij}\)
• **metric of curved space**

  • 3D space ≡ hypersurface in 4D space

  spherical 3D space:  
  \[ d l^2 = d x_1^2 + d x_2^2 + d x_3^2 + \frac{(x_1 d x_1 + x_2 d x_2 + x_3 d x_3)^2}{R^2 - (x_1^2 + x_2^2 + x_3^2)} \]

  \[ \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \sin \vartheta \cos \varphi \\ r \sin \vartheta \sin \varphi \\ r \cos \vartheta \end{pmatrix} \]

  \[ d x_1^2 + d x_2^2 + d x_3^2 = g_r d r^2 + g_\vartheta d \vartheta^2 + g_\varphi d \varphi^2 \]
Friedmann Models

- metric of curved space

- 3D space ≡ hypersurface in 4D space

spherical 3D space:

\[ dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - (x_1^2 + x_2^2 + x_3^2)} \]

\[ \bar{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \sin \vartheta \cos \varphi \\ r \sin \vartheta \sin \varphi \\ r \cos \vartheta \end{pmatrix} \]

\[ dx_1^2 + dx_2^2 + dx_3^2 = g_r dr^2 + g_\vartheta d\vartheta^2 + g_\varphi d\varphi^2 \]

\[ g_r = \left| \frac{\partial \bar{r}}{\partial r} \right| = 1 \]

\[ g_\vartheta = \left| \frac{\partial \bar{r}}{\partial \vartheta} \right| = r \]

\[ g_\varphi = \left| \frac{\partial \bar{r}}{\partial \varphi} \right| = r \sin \vartheta \]
- metric of curved space

- 3D space $\equiv$ hypersurface in 4D space

spherical 3D space:

\[ dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - (x_1^2 + x_2^2 + x_3^2)} \]

\[
\begin{pmatrix}
  \frac{\partial}{\partial x} \\
  \frac{\partial}{\partial y} \\
  \frac{\partial}{\partial z}
\end{pmatrix} = \begin{pmatrix}
  r \sin \vartheta \cos \varphi \\
  r \sin \vartheta \sin \varphi \\
  r \cos \vartheta
\end{pmatrix}
\]

\[
= dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 + \frac{(\vec{r} \cdot d\vec{r})^2}{R^2 - r^2}
\]

\[
= dr^2 + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 + \frac{r^2 dr^2}{R^2 - r^2}
\]

\[
= dr^2 \left[ \frac{r^2}{R^2 - r^2} + 1 \right] + r^2 \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right)
\]

\[
= dr^2 \left[ \frac{R^2}{R^2 - r^2} \right] + r^2 \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right)
\]
metric of curved space

• 3D space ≡ hypersurface in 4D space

spherical 3D space: 

\[ dl^2 = dr^2 \left[ \frac{R^2}{R^2 - r^2} \right] + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \]
Friedmann Models

- metric of curved space

  - 3D space ≡ hypersurface in 4D space

spherical 3D space:

\[
dl^2 = dr^2 \left[ \frac{R^2}{R^2 - r^2} \right] + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)
\]

introduction of rescaled (comoving) coordinates

\[
r = Rx
\]

\[
dr = Rdx
\]

\[
= R^2 \left( dx^2 \left[ \frac{1}{1 - x^2} \right] + x^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right)
\]
Friedmann Models

- metric of curved space

  - 3D space \(\equiv\) hypersurface in 4D space

spherical 3D space:

\[
dl^2 = R^2 \left( \frac{dx^2}{1 - x^2} + x^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right)
\]
Friedmann Models

- **metric of curved space**

  - 3D space ≡ hypersurface in 4D space

  \[
  dl^2 = R^2 \left( \frac{dx^2}{1-x^2} + x^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right)
  \]

  spherical 3D space: \( k = 0 \):

  \[
  \Rightarrow R^2 \left( \frac{dx^2}{1-kx^2} + x^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right)
  \]

  generalisation to hyperbolic & Euclidean spaces

  - \( k = +1 \): spherical
  - \( k = 0 \): Euclidean
  - \( k = -1 \): hyperbolic

Robertson-Walker-Metric
• metric of curved space

• 3D space ≡ hypersurface in 4D space

\[ dl^2 = R^2 \left( \frac{dx^2}{1-kx^2} + x^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right) \]

\( k = +1: \) spherical
\( k = 0: \) Euclidean
\( k = -1: \) hyperbolic
- metric of curved space

- 3D space ≡ hypersurface in 4D space

\[
dl^2 = R^2 \left( \frac{dx^2}{1-kx^2} + x^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right)\]

\[
\gamma_{ij} = \begin{pmatrix}
\frac{R^2}{1-kx^2} & 0 & 0 \\
0 & R^2 x^2 & 0 \\
0 & 0 & R^2 x^2 \sin^2 \theta
\end{pmatrix}
\]

\( k = +1: \) spherical  
\( k = 0: \) Euclidean  
\( k = -1: \) hyperbolic
• metric of curved space

- 3D space ≡ hypersurface in 4D space

\[ dl^2 = R^2 \left( \frac{dx^2}{1- kx^2} + x^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right) \]

\[ k = +1: \text{ spherical} \]
\[ k = 0: \text{ Euclidean} \]
\[ k = -1: \text{ hyperbolic} \]

\[ \gamma_{ij} = \begin{pmatrix}
\frac{R^2}{1-kx^2} & 0 & 0 \\
0 & R^2 x^2 & 0 \\
0 & 0 & R^2 x^2 \sin^2 \theta
\end{pmatrix} \quad \Rightarrow \quad g_{\mu \nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{R^2}{1-kx^2} & 0 & 0 \\
0 & 0 & R^2 x^2 & 0 \\
0 & 0 & 0 & R^2 x^2 \sin^2 \theta
\end{pmatrix} \]
Robertson-Walker-Metric

\[ ds^2 = -(cdt)^2 + R^2 \left[ \frac{dx^2}{1-kx^2} + x^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \]

or equivalently:

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2 \vartheta \end{pmatrix} \]
- **Robertson-Walker-Metric**

\[
ds^2 = -(cdt)^2 + R^2 \left[ \frac{dx^2}{1-kx^2} + x^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right]
\]

or equivalently:

\[
x^\mu = (ct, x^i)
\]

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{R^2}{1-kx^2} & 0 & 0 \\
0 & 0 & R^2x^2 & 0 \\
0 & 0 & 0 & R^2x^2 \sin^2 \theta
\end{pmatrix}
\]

**Notes:**
- curvature of 3D space: \( ^{(3)}R = 6 \frac{k}{R^2} \)
- curvature of space-time: \( ^{(4)}R = 6 \frac{\ddot{R}}{R} + 6 \left( \frac{\dot{R}}{R} \right)^2 + 6 \frac{k}{R^2} \)
Robertson-Walker-Metric

\[ ds^2 = -(c dt)^2 + R^2 \left[ \frac{dx^2}{1 - k x^2} + x^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right] \]

or equivalently:

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

with

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & \frac{R^2}{1 - k x^2} & 0 & 0 & 0 \\
0 & 0 & R^2 x^2 & 0 & 0 \\
0 & 0 & 0 & R^2 x^2 \sin^2 \vartheta & 0 
\end{pmatrix}
\]

can be used to calculate geodesics...
Friedmann Models

- Robertson-Walker-Metric
  - derivation
  - geodesics
  - redshift

- Friedmann Equations

- Friedmann Models
- geodesics

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2 \vartheta \end{pmatrix} \]
Friedmann Models

- geodesics

\[
x^\mu = (ct, x^i)
\]

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} =
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{R^2}{1-kx^2} & 0 & 0 \\
0 & 0 & R^2x^2 & 0 \\
0 & 0 & 0 & R^2x^2 \sin^2 \theta
\end{pmatrix}
\]

geodesic equation (absence of external forces):

\[
\frac{d^2 x^\mu}{ds^2} = -\Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds}
\]

\[
\text{with} \quad \Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\kappa} \left[ \frac{\partial g_{\kappa\nu}}{\partial x^\lambda} + \frac{\partial g_{\kappa\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\kappa} \right]
\]

(Christoffel symbols/affine connection)
Friedmann Models

- geodesics

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

with \[ g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & R^2 & 0 & 0 \\
0 & 0 & 1 - kx^2 & 0 \\
0 & 0 & 0 & R^2 x^2 \sin^2 \theta
\end{pmatrix} \]

geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \]

with \[ u^\mu = \frac{dx^\mu}{ds} \]

and \[ \Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\kappa} \left[ \frac{\partial g_{\kappa\nu}}{\partial x^\lambda} + \frac{\partial g_{\kappa\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\kappa} \right] \]
geodesics

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \]

with

\[ u^\mu = \frac{dx^\mu}{ds} \]

\[ \Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\kappa} \left[ \frac{\partial g_{\kappa\nu}}{\partial x^\lambda} + \frac{\partial g_{\kappa\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\kappa} \right] \]

\[ \Gamma^i_{0j} = \frac{\dot{R}}{R} \gamma_{ij} \]

\[ \Gamma^i_{ij} = \frac{\dot{R}^2 R^2}{\dot{R}} \delta_{ij} \]

\[ \Gamma^i_{jk} = \frac{1}{2} \gamma^{il} \left( \frac{\partial \gamma_{kl}}{\partial x^j} + \frac{\partial \gamma_{jl}}{\partial x^k} - \frac{\partial \gamma_{jk}}{\partial x^l} \right) \]

(all other components are zero)
Friedmann Models

- **geodesics**

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2 \theta \end{pmatrix} \]

geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds} \quad \text{and} \quad \Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\kappa} \left[ \frac{\partial g_{\kappa\nu}}{\partial x^\lambda} + \frac{\partial g_{\kappa\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\kappa} \right] \]

- \( \mu = 0 \) component:

\[ \Gamma^0_{ij} = \frac{\dot{R}}{R} \gamma_{ij} \]
Friedmann Models

- geodesics

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{R^2}{1-kx^2} & 0 & 0 \\
0 & 0 & R^2x^2 & 0 \\
0 & 0 & 0 & R^2x^2 \sin^2 \theta
\end{pmatrix} \]

geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds} \]

- \( \mu = 0 \) component:

\[ 0 = \frac{du^0}{ds} + \Gamma^0_{\nu\lambda} u^\nu u^\lambda \]

\[ \Gamma^0_{ij} = \frac{\dot{R}}{R} \gamma_{ij} \]
Friedmann Models

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

with \( g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2 \vartheta \end{pmatrix} \)

geodesic equation (absence of external forces):\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \]

\( \Gamma^0 = \frac{\dot{R}}{R} \gamma_{ij} \)

\( \mu = 0 \) component in FRW metric: \[ 0 = \frac{du^0}{ds} + \Gamma^0_{\nu\lambda} u^\nu u^\lambda = \frac{du^0}{ds} + \Gamma^0_{ij} u^i u^j \]
Friedmann Models

- geodesics

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \]

with \( g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2x^2 & 0 \\ 0 & 0 & 0 & R^2x^2\sin^2\vartheta \end{pmatrix} \)

geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds} \]

- \( \mu = 0 \) component in FRW metric:

\[ \Gamma^0_{ij} = \frac{\dot{R}}{R} \gamma_{ij} \]

\[ 0 = \frac{du^0}{ds} + \Gamma^0_{\nu\lambda} u^\nu u^\lambda = \frac{du^0}{ds} + \Gamma^0_{ij} u^i u^j = \frac{du^0}{ds} + \frac{\dot{R}}{R} \gamma_{ij} u^i u^j \]
Friedmann Models

- geodesics

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

with

\[ g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{R^2}{1-kx^2} & 0 & 0 \\
0 & 0 & R^2x^2 & 0 \\
0 & 0 & 0 & R^2x^2 \sin^2 \theta
\end{pmatrix} \]

geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds} \]

- \( \mu = 0 \) component in FRW metric:

physical 3D velocity: \( |u|^2 = \gamma_{ij} u^i u^j \)

\[ 0 = \frac{du^0}{ds} + \Gamma^0_{\nu\lambda} u^\nu u^\lambda = \frac{du^0}{ds} + \Gamma^0_{ij} u^i u^j = \frac{du^0}{ds} + \frac{\dot{R}}{R} \gamma_{ij} u^i u^j = \frac{du^0}{ds} + \frac{\dot{R}}{R} |u|^2 \]
Friedmann Models

- **geodesics**

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-\kappa x^2} & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2 \vartheta \end{pmatrix} \]

geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds} \]

- \( \mu = 0 \) component in FRW metric:

\[ 0 = \frac{du^0}{ds} + \frac{\dot{R}}{R} |u|^2 \]
geodesics

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

\[ g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{R^2}{1-kx^2} & 0 & 0 \\
0 & 0 & R^2 x^2 & 0 \\
0 & 0 & 0 & R^2 x^2 \sin^2 \vartheta
\end{pmatrix} \]

geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \]

\[ u^\mu = \frac{dx^\mu}{ds} \]

\[ \mu = 0 \] component in FRW metric:

\[ 0 = \frac{du^0}{ds} + \frac{\dot{R}}{R} |u|^2 \]
Friedmann Models

- Geodesics

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & R^2 & 0 & 0 \\
0 & 0 & \frac{R^2}{1-\kappa x^2} & 0 \\
0 & 0 & 0 & R^2 x^2 \sin^2 \theta
\end{pmatrix} \]

Geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds} \]

- \( \mu = 0 \) component in FRW metric:

\[ 0 = \frac{du^0}{ds} + \frac{\dot{R}}{R} |u|^2 \]

\[ \frac{du^0}{ds} = \frac{du^0}{dt} \frac{dt}{ds} = -\frac{du^0}{dt} \frac{u^0}{c} \]
Friedmann Models

- geodesics

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & R^2 & 0 & 0 \\ 0 & 0 & R^2 x^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2 \theta \end{pmatrix} \]

geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds} \]

- \( \mu = 0 \) component in FRW metric:

\[ 0 = \frac{du^0}{ds} + \frac{\dot{R}}{R} |u|^2 \]

\[ \frac{du^0}{ds} = \frac{du^0}{dt} \frac{dt}{ds} = -\frac{du^0}{dt} \frac{u^0}{c} = -\frac{u^0 du^0}{cdt} \]

\[ -c^2 = u_\nu u^\nu \Rightarrow u^0 du^0 = -|u| |d|u| \]

Robertson-Walker-Metric
Friedmann Models

• geodesics

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & R^2 & 0 & 0 \\ 0 & 0 & \frac{R^2 x^2}{1-kx^2} & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2 \vartheta \end{pmatrix} \]

geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds} \]

• \( \mu = 0 \) component in FRW metric:

\[ 0 = \frac{du^0}{ds} + \frac{\dot{R}}{R}|u|^2 \]

\[ \frac{du^0}{ds} = \frac{du^0}{dt} \frac{dt}{ds} = -\frac{du^0}{dt} \frac{u^0}{c} = -\frac{u^0 du^0}{cdt} = \frac{|u| |d| u|}{cdt} = \frac{|u|}{c} |\dot{u}| \]
Friedmann Models

- **geodesics**

\[ x'^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & R^2 & 0 & 0 \\ 0 & 0 & 1 - kx^2 & 0 \\ 0 & 0 & 0 & R^2 x^2 \sin^2 \vartheta \end{pmatrix} \]

geodesic equation (absence of external forces):

\[ 0 = \frac{d\mu}{ds} + \Gamma^\lambda_\nu_\lambda \mu \nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds} \]

- **\( \mu = 0 \)** component in FRW metric:

\[ 0 = \frac{d\mu^0}{ds} + \frac{\dot{R}}{R} |u|^2 \]

\[ \frac{du^0}{ds} = \frac{du^0}{dt} \frac{dt}{ds} = -\frac{du^0}{dt} \frac{u^0}{c} = -\frac{u^0 du^0}{cdt} = \frac{|u| du}{cdt} = \frac{|u|}{c} |\dot{u}| \]
- **geodesics**

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{R^2}{1-kx^2} & 0 & 0 \\ 0 & 0 & R^2x^2 & 0 \\ 0 & 0 & 0 & R^2x^2 \sin^2 \vartheta \end{pmatrix} \]

Geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds} \]

- \( \mu = 0 \) component in FRW metric:

\[ 0 = \frac{|u|}{c} |\dot{u}| + \frac{\dot{R}}{R} |u|^2 \]
Friedmann Models

- geodesics

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]

with

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & R^2 & 0 & 0 \\
0 & 0 & R^2 x^2 & 0 \\
0 & 0 & 0 & R^2 x^2 \sin^2 \vartheta
\end{pmatrix}
\]

geodesic equation (absence of external forces):

\[
0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds}
\]

- \( \mu = 0 \) component in FRW metric:

\[
\frac{\dot{R}}{R} \propto -\frac{\dot{R}}{R}
\]
Friedmann Models

- geodesics

\[ x^\mu = (ct, x^i) \]

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & R^2 & 0 & 0 \\
0 & 0 & 1-\kappa x^2 & 0 \\
0 & 0 & 0 & R^2 x^2 \sin^2 \vartheta
\end{pmatrix} \]

geodesic equation (absence of external forces):

\[ 0 = \frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \quad \text{with} \quad u^\mu = \frac{dx^\mu}{ds} \]

- \( \mu = 0 \) component in FRW metric:

\[ \frac{|\dot{u}|}{|u|} \propto -\frac{\dot{R}}{R} \quad \frac{d|u|}{dt} = \frac{d|u|}{dR} \frac{\dot{R}}{R} \]

\[ \frac{1}{|u|} \frac{d|u|}{dR} \propto -\frac{\dot{R}}{R} \quad \frac{d|u|}{|u|} \propto -\frac{dR}{R} \quad \rightarrow \quad |u| \propto \frac{1}{R} \]

Robertson-Walker-Metric
geodesics in FRW metric:

\[ |p| \propto \frac{1}{R} \]

the momentum of particles scale like \(1/R\)
Friedmann Models

- Robertson-Walker-Metric
  - derivation
  - geodesics
  - redshift

- Friedmann Equations

- Friedmann Models
Robertson-Walker-Metric

\[
ds^2 = -(cdt)^2 + R^2(t) \left[ \frac{dx^2}{1-kx^2} + x^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right]
\]

**Note:** $R(t)$ can be an arbitrary function of time
Friedmann Models

- Robertson-Walker-Metric

$$ds^2 = -(cdt)^2 + R^2(t) \left[ \frac{dx^2}{1-kx^2} + x^2 \left( d\theta^2 + \sin^2\theta d\varphi^2 \right) \right]$$

any connection to observables?
- Robertson-Walker-Metric

\[
ds^2 = -(cdt)^2 + R^2(t) \left[ \frac{dx^2}{1-kx^2} + x^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right]
\]

comoving coordinates: \( \bar{r}(t) = R(t) \bar{x} \)

any connection to observables?
- cosmological expansion

all length scales scale like $R(t)$

\[ \tilde{r}(t) = R(t)\tilde{x} \]
- cosmological expansion

all length scales scale like $R(t)$

$$\vec{r}(t) = R(t) \vec{x}$$

**Note:**
- space can be curved and is not necessarily Euclidean
- even though we call it ‘expanding’ we simply consider $R(t) \neq \text{const.}$
- cosmological expansion

all length scales scale like $R(t)$

$$\vec{r}(t) = R(t)\vec{x}$$
- cosmological expansion

all length scales scale like $R(t)$

$$\tilde{r}(t) = R(t)\tilde{x}$$

how to quantify this ‘stretch’?
Friedmann Models

- cosmological expansion and redshift

All length scales scale like $R(t)$

$$\ddot{r}(t) = R(t)\ddot{x}$$

how to quantify this ‘stretch’:

redshift

$$z = \frac{\lambda_0 - \lambda_E}{\lambda_E}$$

$\lambda_0$ : observed

$\lambda_E$ : emitted

expanding Universe $R(t)$
• cosmological expansion and redshift

all length scales scale like $R(t)$

$$
\ddot{r}(t) = R(t)\ddot{x}
$$

how to quantify this ‘stretch’:

redshift

$$
z = \frac{\lambda_0 - \lambda_E}{\lambda_E}
$$
- cosmological expansion and redshift

all length scales scale like $R(t)$

$$
\tilde{r}(t) = R(t)\tilde{x}
$$

relation between $z$ and $R$?!

how to quantify this ‘stretch’:

redshift

$$
z = \frac{\lambda_0 - \lambda_E}{\lambda_E}
$$

$\lambda_E$: emitted
$\lambda_0$: observed
- expansion factor and redshift

\[ z = \frac{\lambda_0 - \lambda_E}{\lambda_E} \quad \overset{?}{\longleftrightarrow} \quad \vec{r}(t) = R(t)\vec{x} \]
Friedmann Models

- expansion factor and redshift

\[ z = \frac{\lambda_0 - \lambda_E}{\lambda_E} \quad \text{(de Broglie relation)} \]

\[ \vec{r}(t) = R(t) \vec{x} \]

\[ \lambda_0 : \text{observed} \]

\[ \lambda_E : \text{emitted} \]

- de Broglie relation: \[ p = \frac{h}{\lambda} \]
- expansion factor and redshift
\[ z = \frac{\lambda_0 - \lambda_E}{\lambda_E} \quad \overset{?}{\longleftrightarrow} \quad \vec{r}(t) = R(t)\vec{x} \]

- de Broglie relation: \[ p = \frac{h}{\lambda} \]
- FRW geodesics: \[ p \propto \frac{1}{R} \]
Friedmann Models

- expansion factor and redshift

\[ z = \frac{\lambda_0 - \lambda_E}{\lambda_E} \quad \Leftrightarrow \quad \ddot{r}(t) = R(t)\ddot{x} \]

- de Broglie relation:
  \[ p = \frac{h}{\lambda} \]

- FRW geodesics:
  \[ p \propto \frac{1}{R} \]

\[ \frac{\lambda_0}{R_0} = \frac{\lambda_E}{R_E} \]
Friedmann Models

- expansion factor and redshift

$$z = \frac{\lambda_0 - \lambda_E}{\lambda_E} \quad \& \quad \frac{\lambda_0}{R_0} = \frac{\lambda_E}{R_E}$$

$$\Rightarrow \quad \frac{R_E}{R_0} = \frac{\lambda_E}{\lambda_0} = \frac{1}{\lambda_0/\lambda_E} = \frac{1}{\lambda_0/\lambda_E - 1 + 1} = \frac{1}{z + 1}$$

$$\Rightarrow \quad \frac{R(t_E)}{R(t_0)} = \frac{1}{1 + z}$$

(very important relation; does not depend on cosmology)
Friedmann Models

- expansion factor and redshift

\[ a(t) = \frac{1}{1 + z} \]

cosmic expansion factor: \[ a(t) = \frac{R(t)}{R(t_0)} \]

(very important relation; does **not** depend on cosmology)
Friedmann Models

- expansion factor and redshift

\[ a(t) = \frac{1}{1 + z} \text{ observable} \]

cosmic expansion factor: \[ a(t) = \frac{R(t)}{R(t_0)} \]

(very important relation; does **not** depend on cosmology)
- expansion factor and redshift

\[ a(t) = \frac{1}{1 + z} \]  

(very important relation; does not depend on cosmology)
Friedmann Models

- expansion factor and redshift

...but how to calculate?

$$a(t) = \frac{1}{1 + z}$$

cosmic expansion factor:  

$$a(t) = \frac{R(t)}{R(t_0)}$$

(very important relation; does not depend on cosmology)
Friedmann Models

- Robertson-Walker-Metric

- Friedmann Equations
  - derivation
  - equation of state
  - adiabatic expansion
  - density parameters

- Friedmann Models
• derivation

  • Einstein’s Field Equations

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}
\]
Friedmann Models

- derivation

  - Einstein’s Field Equations

  \[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \]

  properties of matter

  ...as described by the energy-stress tensor \( T_{\mu\nu} \)

  \[
  \begin{pmatrix}
  T^{00} & T^{01} & T^{02} & T^{03} \\
  T^{10} & T^{11} & T^{12} & T^{13} \\
  T^{20} & T^{21} & T^{22} & T^{23} \\
  T^{30} & T^{31} & T^{32} & T^{33}
  \end{pmatrix}
  \]

  c^2 \cdot (\text{energy/density})

  \text{momentum density}

  \text{momentum density}

  \text{momentum flux}

  \text{shear stress}

  \text{pressure}
Friedmann Models

- derivation

  - Einstein’s Field Equations

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}
\]

...as described by the Ricci tensor \( R_{\mu\nu} \) and Ricci scalar \( R \)

\[
R_{\mu\nu} = f\left(g_{\mu\nu}\right)
\]

\[
R = h\left(R_{\mu\nu}\right)
\]

...as described by the energy-stress tensor \( T_{\mu\nu} \)
• derivation

Einstein’s Field Equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \]

pure geometry

properties of matter

“matter tells space-time how to curve, and curved space-time tells matter how to move.”

(J.A. Wheeler)
Friedmann Models

- derivation

- Einstein's Field Equations

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}
\]

pure geometry

isotropy/homogeneity

\[
g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{R^2}{1-kx^2} & 0 & 0 \\
0 & 0 & R^2 x^2 & 0 \\
0 & 0 & 0 & R^2 x^2 \sin^2 \theta
\end{pmatrix}
\]

properties of matter

perfect fluid

\[
T_{\mu\nu} = -pg^{\mu\nu} + (\rho c^2 + p)u^\mu u^\nu
\]

Careful: \( R \) in the metric tensor \( g_{\mu\nu} \) is not the Ricci scalar \( R \).
- derivation

• Einstein’s Field Equations

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}
\]

pure geometry

isotropy/homogeneity

\[
g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{R^2}{1-kx^2} & 0 & 0 \\
0 & 0 & R^2x^2 & 0 \\
0 & 0 & 0 & R^2x^2\sin^2\vartheta
\end{pmatrix}
\]

perfect fluid

\[
T_{\mu\nu} = -pg^{\mu\nu} + (\rho c^2 + p)u^\mu u^\nu
\]

Friedmann-Equations
\( \left( \frac{\dot{R}}{R} \right)^2 = \frac{8 \pi G}{3} \rho - \frac{kc^2}{R^2} + \frac{c^2 \Lambda}{3} \)

\( \frac{\ddot{R}}{R} = -\frac{4 \pi G}{3} \rho \left( \frac{3p}{c^2} \right) + \frac{c^2 \Lambda}{3} \)
- derivation

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{R^2} + \frac{c^2 \Lambda}{3}
\]

\[
\frac{\ddot{R}}{R} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) + \frac{c^2 \Lambda}{3}
\]

- critical density

\[ k = 0, \quad \Lambda = 0 \quad \rightarrow \quad \rho \equiv \rho_{\text{crit}} = \frac{3H^2}{8\pi G}, \quad H = \frac{\dot{R}}{R} \]
Friedmann Models

- derivation

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{R^2} + \frac{c^2 \Lambda}{3}
\]

\[
\frac{\ddot{R}}{R} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) + \frac{c^2 \Lambda}{3}
\]

- critical density \( \rightarrow \) will be our favourite unit for densities later on...

\( k = 0, \ \Lambda = 0 \quad \rightarrow \quad \rho \hat{=} \rho_{\text{crit}} = \frac{3H^2}{8\pi G}, \ H = \frac{\dot{R}}{R} \)
Friedmann Models

- derivation

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{R^2} + \frac{c^2 \Lambda}{3}
\]

\[
\frac{\ddot{R}}{R} = -\frac{4 \pi G}{3} \left( \rho + \frac{3 p}{c^2} \right) + \frac{c^2 \Lambda}{3}
\]

- critical density

\[ k = 0, \quad \Lambda = 0 \quad \rightarrow \quad \rho \doteq \rho_{\text{crit}} = \frac{3 H^2}{8 \pi G}, \quad H = \frac{\dot{R}}{R} \]

- equation of state

\[ p = p(\rho) \]
Friedmann Models

- Robertson-Walker-Metric

- Friedmann Equations
  - derivation
  - equation of state
  - adiabatic expansion
  - density parameters

- Friedmann Models
- equation of state
  - solving the Friedmann Equations still requires an equation-of-state
• equation of state

  - solving the Friedmann Equations still requires an equation-of-state

  - we will restrict ourselves to barotropic fluids (i.e. linear relation between $p$ and $\rho$):

  \[ p = \omega(T) \rho c^2 \]
- equation of state

  - solving the Friedmann Equations still requires an equation-of-state

  - we will restrict ourselves to barotropic fluids (i.e. linear relation between $p$ and $\rho$):

    $$ p = \omega(T) \rho c^2 $$

    $$ p = n k_B T $$
- equation of state
  - solving the Friedmann Equations still requires an equation-of-state
  - we will restrict ourselves to barotropic fluids (i.e. linear relation between $p$ and $\rho$):

$$p = \omega(T)\rho c^2$$
- **equation of state**
  
  - solving the Friedmann Equations still requires an equation-of-state
  
  - we will restrict ourselves to barotropic fluids (i.e. linear relation between $p$ and $\rho$):

\[
p = \omega(T) \rho c^2
\]
Friedmann Models

- equation of state

- solving the Friedmann Equations still requires an equation-of-state

- we will restrict ourselves to barotropic fluids (i.e. linear relation between \( p \) and \( \rho \)):

\[
p = \omega(T) \rho c^2
\]

\[
\begin{align*}
p &= nk_B T \\
&= \frac{k_B T}{m_p c^2} \rho_m c^2 \\
&= \frac{k_B T}{m_p c^2} \left( \frac{\rho c^2}{(1 + \frac{k_B T}{(\gamma - 1)m_p c^2})} \right)
\end{align*}
\]

\[
p = (\gamma - 1)\varepsilon \quad \gamma = \text{adiabatic coefficient (see next sub-section…)}
\]

mass density \( \rightarrow \) energy density:

\[
\rho c^2 = \rho_m c^2 + \varepsilon = \rho_m + \frac{p}{(\gamma - 1)} = \rho_m c^2 + \frac{nk_B T}{(\gamma - 1)} = \rho_m c^2 + \frac{\rho_m c^2 k_B T}{(\gamma - 1)m_p c^2} = \rho_m c^2 \left( 1 + \frac{k_B T}{(\gamma - 1)m_p c^2} \right)
\]
- equation of state

  - solving the Friedmann Equations still requires an equation-of-state

  - we will restrict ourselves to barotropic fluids (i.e. linear relation between $p$ and $\rho$):

    \[ p = \omega(T) \rho c^2 \]

\[ p = nk_B T \frac{k_B T}{m_p c^2} \rho_m c^2 = \frac{k_B T}{m_p c^2}\frac{\rho_c^2}{k_B T} \frac{k_B T}{(\gamma - 1) m_p c^2} = \omega(T) \rho c^2 \]
• equation of state

  • solving the Friedmann Equations still requires an equation-of-state

  • we will restrict ourselves to barotropic fluids (i.e. linear relation between $p$ and $\rho$):

  $$ p = \omega(T) \rho c^2 $$

  • the main types are:

  • radiation $\ w = \ ?$

  • collisionless matter $\ w = \ ?$
Friedmann Models

- equation of state

  - solving the Friedmann Equations still requires an equation-of-state

  - we will restrict ourselves to barotropic fluids (i.e. linear relation between $p$ and $\rho$):

    $$ p = \omega(T) \rho c^2 $$

  - the main types are:

    - radiation $w = 1/3$

    - collisionless matter $w = 0$
- equation of state

  • solving the Friedmann Equations still requires an equation-of-state

  • we will restrict ourselves to barotropic fluids (i.e. linear relation between $p$ and $\rho$):

  \[ p = \omega(T) \rho c^2 \]

  • the main types are:

    • radiation \hspace{1cm} w = 1/3

    • collisionless matter \hspace{1cm} w = 0

    • vacuum energy \hspace{1cm} w = -1

    • curvature \hspace{1cm} w = -1/3

  can be cast into energy densities, too

  \[
  \left( \frac{\dot{R}}{R} \right)^2 = \frac{8 \pi G}{3} \rho - \frac{k c^2}{R^2} + \frac{c^2 \Lambda}{3}
  \]
Friedmann Models

- Robertson-Walker-Metric

- Friedmann Equations
  - derivation
  - equation of state
  - adiabatic expansion
  - density parameters

- Friedmann Models
Friedmann Models

- adiabatic expansion

  - energy conservation:

    \[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) \]
Friedmann Models

- adiabatic expansion
  - energy conservation:
    \[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) \]
    follows directly from \( T^{\mu\nu} = 0 \)
• adiabatic expansion

  • energy conservation: \[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) \]

  follows from combination of 1. and 2. Friedmann equation:
Friedmann Models

- adiabatic expansion
  - energy conservation:

\[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) \]

follows from combination of 1. and 2. Friedmann equation:

1. Friedmann Equation

\[ \frac{\ddot{R}}{R} = \frac{8\pi G}{3} \rho R^2 - \rho c^2 \frac{k}{R^2} + \frac{c^2}{3} \Lambda R^2 \]
Friedmann Models

- adiabatic expansion

- energy conservation:
  \[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} \left( \rho + \frac{p}{c^2} \right) \]

follows from combination of 1. and 2. Friedmann equation:

1. Friedmann Equation

\[
\begin{align*}
\ddot{R} &= \frac{8\pi G}{3} \rho R^2 - kc^2 + \frac{c^2 \Lambda}{3} R^2 \\
2R\dddot{R} &= \frac{8\pi G}{3} \left( \rho \frac{\dot{R}^2}{R} + 2\rho R \ddot{R} \right) + 2R\dot{R} c^2 \Lambda \\
\dddot{R} &= \frac{4\pi G}{3} \left( \rho \frac{\dot{R}^2}{R} + 2\rho R \right) + R \frac{c^2 \Lambda}{3} \\
\frac{\dddot{R}}{R} &= \frac{4\pi G}{3} \left( \rho \frac{\dot{R}^2}{R^2} + 2\rho \right) + \frac{c^2 \Lambda}{3}
\end{align*}
\]
Friedmann Models

- adiabatic expansion

  • energy conservation:

  \[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) \]

follows from combination of 1. and 2. Friedmann equation:

1. Friedmann Equation

\[
\frac{\ddot{R}}{R} = \frac{8\pi G}{3} \rho R^2 - kc^2 + \frac{c^2 \Lambda}{3} R^2
\]

\[
2 \frac{\ddot{R}}{R} = \frac{8\pi G}{3} (\rho \frac{\dot{R}^2}{R} + 2 \rho R \dot{R}) + 2 \dot{R} \ddot{R} \frac{c^2 \Lambda}{3}
\]

\[
\frac{\dddot{R}}{R} = \frac{4\pi G}{3} (\rho \frac{\ddot{R}}{R} + 2 \rho \frac{\dot{R}}{R}) + R \frac{c^2 \Lambda}{3}
\]

\[
\frac{\dddot{R}}{R} = \frac{4\pi G}{3} (\rho \frac{\ddot{R}}{R} + 2 \rho) + \frac{c^2 \Lambda}{3}
\]

2. Friedmann Equation

\[
\frac{\dddot{R}}{R} = -\frac{4\pi G}{3} \left(\rho + \frac{3p}{c^2}\right) + \frac{c^2 \Lambda}{3}
\]
Friedmann Models

- adiabatic expansion

  - energy conservation:

\[
0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2})
\]

follows from combination of 1. and 2. Friedmann equation:

1. Friedmann Equation

\[
\frac{\ddot{R}}{R} = \frac{8\pi G}{3} \rho R^2 - k c^2 + \frac{c^2 \Lambda}{3} R^2
\]

\[
2 \dot{R} R = \frac{8\pi G}{3} (\rho R^2 + 2 \rho R \dot{R}) + 2 R \dot{R} \frac{c^2 \Lambda}{3}
\]

\[
\ddot{R} = \frac{4\pi G}{3} (\rho \frac{\dot{R}}{R} + 2 \rho R) + R \frac{c^2 \Lambda}{3}
\]

\[
\frac{\ddot{R}}{R} = \frac{4\pi G}{3} (\rho \frac{\dot{R}}{R} + 2 \rho R) + \frac{c^2 \Lambda}{3}
\]

2. Friedmann Equation

\[
\frac{\ddot{R}}{R} = - \frac{4\pi G}{3} (\rho + \frac{3p}{c^2}) + \frac{c^2 \Lambda}{3}
\]
Friedmann Models

- adiabatic expansion
  
  - energy conservation: \[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) \]
  
  - 2\textsuperscript{nd} law of thermodynamics: \[ TdS = dU + pdV \]
Friedmann Models

- adiabatic expansion
  - energy conservation: \[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) \]
  - 2nd law of thermodynamics: \[ TdS = dU + p \, dV \]
- adiabatic expansion

  - energy conservation: \( 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) \)

  - 2\textsuperscript{nd} law of thermodynamics: \( TdS = dU + pdV \)

\[ dV = d(R^3) \]
\[ dU = d(V \rho c^2) = d(R^3 \rho c^2) \]
Friedmann Models

- **adiabatic expansion**
  - energy conservation: \( 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) \)
  - 2\textsuperscript{nd} law of thermodynamics: \( TdS = dU + pdV \)

\[
\begin{align*}
  dV &= d(R^3) \\
  dU &= d(R^3 \rho c^2)
\end{align*}
\]
Friedmann Models

Friedmann Equations

- **adiabatic expansion**

  - energy conservation:
    
    \[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) \]

  - 2\textsuperscript{nd} law of thermodynamics:
    
    \[ TdS = dU + pdV \]

    \[
    dV = d(R^3) \\
    dU = d(R^3 \rho c^2)
    \]

    combine with energy conservation → requires time derivatives...
• adiabatic expansion

- energy conservation: \[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} \left( \rho + \frac{p}{c^2} \right) \]

- 2\textsuperscript{nd} law of thermodynamics: \[ TdS = dU + pdV \]

\[
\begin{align*}
    dV &= d(R^3) \\
    dU &= d(R^3 \rho c^2) \\
    \frac{dU}{dt} + p \frac{dV}{dt} &= \frac{d}{dt} \left( R^3 \rho c^2 \right) + p \frac{d(R^3)}{dt} = 3R^2 \dot{R} \rho c^2 + R^3 \dot{\rho} c^2 + 3pR^2 \dot{R}
\end{align*}
\]
Friedmann Models

- adiabatic expansion

  • energy conservation: \[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) \]

  • 2\textsuperscript{nd} law of thermodynamics: \[ TdS = dU + pdV \]

\[
\begin{align*}
  dV &= d(R^3) \\
  dU &= d(R^3 \rho c^2) \\
  \frac{dU}{dt} + p \frac{dV}{dt} &= \frac{d(R^3 \rho c^2)}{dt} + p \frac{d(R^3)}{dt} = 3R^2 \dot{\rho} c^2 + R^3 \dot{\rho} c^2 + 3pR^2 \dot{R} = 0
\end{align*}
\]
Friedmann Models

- adiabatic expansion
  - energy conservation:
    \[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} \left(\rho + \frac{p}{c^2}\right) \]
  - 2\textsuperscript{nd} law of thermodynamics:
    \[ TdS = dU + pdV \]

\[
\begin{align*}
  dV &= d(R^3) \\
  dU &= d(R^3 \rho c^2) \\
  \frac{dU}{dt} + p \frac{dV}{dt} &= \frac{d(R^3 \rho c^2)}{dt} + p \frac{d(R^3)}{dt} = 3R^2 \dot{R} \rho c^2 + R^3 \dot{\rho} c^2 + 3pR^2 \dot{R} = 0 \\
  \Rightarrow \quad T \frac{dS}{dt} = 0 &\iff TdS = 0
\end{align*}
\]
- **adiabatic expansion**
  - energy conservation:
    \[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2}) \]
  - 2\textsuperscript{nd} law of thermodynamics:
    \[ TdS = dU + pdV = 0 \]
- adiabatic expansion
  
  - energy conservation:
    
    $0 = \dot{\rho} + 3 \frac{\dot{R}}{R} (\rho + \frac{p}{c^2})$
    
  - 2\textsuperscript{nd} law of thermodynamics:
    
    $TdS = dU + pdV = 0$

  during cosmic expansion entropy is conserved
  and hence the expansion is adiabatic:

  $dU + pdV = 0$
- adiabatic expansion

  - cosmic expansion: \[ dU + pdV = 0 \]
Friedmann Models

Friedmann Equations

- adiabatic expansion

  - cosmic expansion: \( dU + pdV = 0 \)

- implication for barotropic fluids \( p = \omega \rho c^2 \):
Friedmann Models

- adiabatic expansion
  - cosmic expansion: \( dU + pdV = 0 \)

- implication for barotropic fluids \( p = \omega \rho c^2 \):

\[
\begin{align*}
  dU &= d(R^3 \rho c^2) \\
  dV &= d(R^3)
\end{align*}
\]

\[
\implies d(R^3 \rho c^2) = -pd(R^3) = -\omega \rho c^2 d(R^3)
\]

\[
0 = R^3 d\rho + \rho d(R^3) + \omega \rho d(R^3)
\]

\[
= R^3 d\rho + (1 + \omega) \rho d(R^3)
\]

\[
\frac{1}{\rho} d\rho = -(1 + \omega) \frac{d(R^3)}{R^3}
\]

\[
\rho R^{3(1+\omega)} = \text{const.}
\]
Friedmann Models

- adiabatic expansion
  - cosmic expansion: \( dU + pdV = 0 \)

- implication for barotropic fluids \( p = \omega \rho c^2 \):
  - radiation \( w = 1/3 \) \( \Rightarrow \rho_r \propto R^{-4} \)
  - collisionless matter \( w = 0 \) \( \Rightarrow \rho_m \propto R^{-3} \)
  - vacuum energy \( w = -1 \) \( \Rightarrow \rho_\Lambda = \text{const.} \)
  - curvature \( w = -1/3 \) \( \Rightarrow \rho_k \propto R^{-2} \)

\[ \rho R^{3(1+\omega)} = \text{const.} \]
Friedmann Models

\begin{itemize}
  \item adiabatic expansion
    \begin{itemize}
    \item cosmic expansion: \quad dU + pdV = 0
    \end{itemize}
\end{itemize}

\begin{itemize}
  \item implication for barotropic fluids \quad p = \omega \rho c^2:
    \begin{itemize}
    \item radiation \quad w = 1/3 \quad \Rightarrow \quad \rho_r \propto R^{-4}
    \item collisionless matter \quad w = 0 \quad \Rightarrow \quad \rho_m \propto R^{-3}
    \item vacuum energy \quad w = -1 \quad \Rightarrow \quad \rho_\Lambda = \text{const.}
    \item curvature \quad w = -1/3 \quad \Rightarrow \quad \rho_k \propto R^{-2}
    \end{itemize}
\end{itemize}

\[ \rho R^{3(1+\omega)} = \text{const.} \]

different components dominate at different times!
- adiabatic expansion

  - cosmic expansion: \( dU + pdV = 0 \)

- implication for barotropic fluids \( p = \omega \rho c^2 \):

  - radiation \( w = 1/3 \) \( \Rightarrow \rho_r \propto R^{-4} \)
  - collisionless matter \( w = 0 \) \( \Rightarrow \rho_m \propto R^{-3} \)
  - vacuum energy \( w = -1 \) \( \Rightarrow \rho_\Lambda = \text{const.} \)
  - curvature \( w = -1/3 \) \( \Rightarrow \rho_k \propto R^{-2} \)

\[
\rho R^{3(1+\omega)} = \text{const.}
\]
Friedmann Models

- adiabatic expansion
  - cosmic expansion: 
    \[ dU + pdV = 0 \]

- implication for barotropic fluids \( p = \omega \rho c^2 \):
  - radiation \( w = 1/3 \) \( \Rightarrow \rho_r \propto R^{-4} \)
  - collisionless matter \( w = 0 \) \( \Rightarrow \rho_m \propto R^{-3} \)
  - vacuum energy \( w = -1 \) \( \Rightarrow \rho_\Lambda = \text{const.} \)
  - curvature \( w = -1/3 \) \( \Rightarrow \rho_k \propto R^{-2} \)

\[ \rho R^{3(1+\omega)} = \text{const.} \]
Friedmann Models

- Robertson-Walker-Metric

- Friedmann Equations
  - derivation
  - equation of state
  - adiabatic expansion
  - density parameters

- Friedmann Models
the introduction of density parameters

leads to the most memorable form for the Friedmann equations…
- density parameters

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{R^2} + \frac{c^2 \Lambda}{3}
\]
\textbullet{} density parameters

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{R^2} + \frac{c^2 \Lambda}{3}
\]

\[
\rho = \sum_w \rho_w(R(t))
\]
• density parameters

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{R^2} + \frac{c^2 \Lambda}{3}
\]

\[
\rho = \sum \rho_w(R(t))
\]

• radiation \( w = 1/3 \) \( \Rightarrow \rho(R) \propto R^{-4} \)

• collisionless matter \( w = 0 \) \( \Rightarrow \rho(R) \propto R^{-3} \)
• density parameters

\[ \left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{R^2} + \frac{c^2 \Lambda}{3} \]

\[ \rho = \sum_w \rho_w(R(t)) \]

• radiation \( w = 1/3 \) \( \Rightarrow \rho(R) \propto R^{-4} \)

• collisionless matter \( w = 0 \) \( \Rightarrow \rho(R) \propto R^{-3} \)

**Note:**
we can also include
• the \( \Lambda \)-term \( (w=-1) \) and
• the curvature \( (w=-1/3) \)
in the total energy density….
Friedmann Models

- density parameters

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{R^2} + \frac{c^2 \Lambda}{3}
\]

\[
\rho = \sum_w \rho_w(R(t))
\]

- radiation \( w = 1/3 \) \( \Rightarrow \) \( \rho(R) \propto R^{-4} \)
- collisionless matter \( w = 0 \) \( \Rightarrow \) \( \rho(R) \propto R^{-3} \)

\[
\Rightarrow \left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \left( \rho_{m,0} \left( \frac{R}{R_0} \right)^{-3} + \rho_{r,0} \left( \frac{R}{R_0} \right)^{-4} \right) - \frac{k c^2}{R_0^2} \left( \frac{R}{R_0} \right)^{-2} + \frac{c^2 \Lambda}{3}
\]
Friedmann Models

- density parameters

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \left( \rho_{m,0} \left( \frac{R}{R_0} \right)^{-3} + \rho_{r,0} \left( \frac{R}{R_0} \right)^{-4} \right) - \frac{kc^2}{R_0^2} \left( \frac{R}{R_0} \right)^{-2} + \frac{c^2 \Lambda}{3}
\]
Friedmann Models

- Density parameters

\[
\left( \frac{\dot{R}}{R} \right) = \frac{8\pi G}{3} \left( \rho_{m,0} \left( \frac{R}{R_0} \right)^{-3} + \rho_{r,0} \left( \frac{R}{R_0} \right)^{-4} \right) - \frac{kc^2}{R_0^2} \left( \frac{R}{R_0} \right)^{-2} + \frac{c^2 \Lambda}{3}
\]

- Critical density

\[
\rho_{\text{crit,0}} = \frac{3H_0^2}{8\pi G}
\]
Friedmann Models

- density parameters

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \left( \rho_{m,0} \left( \frac{R}{R_0} \right)^{-3} + \rho_{r,0} \left( \frac{R}{R_0} \right)^{-4} \right) - \frac{k c^2}{R_0^2} \left( \frac{R}{R_0} \right)^{-2} + \frac{c^2 \Lambda}{3}
\]

the cosmological parameters

- matter density \( \Omega_{m,0} = \frac{8\pi G}{3H_0^2} \rho_{m,0} \)
- radiation density \( \Omega_{r,0} = \frac{8\pi G}{3H_0^2} \rho_{r,0} \)
- vacuum energy \( \Omega_{\Lambda,0} = \frac{c^2}{3H_0^2} \Lambda \)
- curvature \( \Omega_{k,0} = -\frac{c^2}{H_0^2 R_0^2} k \)
- expansion rate \( H_0 = \left( \frac{\dot{R}}{R} \right)_0 \)

- critical density \( \rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G} \)
Friedmann Models

- density parameters

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \left( \rho_{m,0} \left( \frac{R}{R_0} \right)^{-3} + \rho_{r,0} \left( \frac{R}{R_0} \right)^{-4} \right) - \frac{k c^2}{R_0^2} \left( \frac{R}{R_0} \right)^2 + \frac{c^2 \Lambda}{3}
\]

the cosmological parameters

- matter density \( \Omega_{m,0} = \frac{8\pi G}{3H_0^2} \rho_{m,0} \)
- radiation density \( \Omega_{r,0} = \frac{8\pi G}{3H_0^2} \rho_{r,0} \)
- vacuum energy \( \Omega_{\Lambda,0} = \frac{c^2}{3H_0^2} \Lambda \)
- curvature \( \Omega_{k,0} = -\frac{c^2}{H_0^2 R_0^2} k \)
- critical density \( \rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G} \)
- expansion rate \( H_0 = \left( \frac{\dot{R}}{R} \right)_0 \)
- **density parameters**

\[
\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3} \left( \frac{\rho_m}{R_0} - 3 + \frac{\rho_r}{R_0} - 4 \right) - \frac{k c^2}{R_0^2} \left( \frac{R}{R_0} \right)^2 + \frac{c^2 \Lambda}{3}
\]

introduction of \(\rho_{\text{crit},0}\) here leads to \(8\pi G/3\) factor, allowing for a combination with the other two energy densities...

- **the cosmological parameters**

  - matter density \(\Omega_{m,0} = \frac{8\pi G}{3H_0^2} \rho_{m,0}\)
  - radiation density \(\Omega_{r,0} = \frac{8\pi G}{3H_0^2} \rho_{r,0}\)
  - vacuum energy \(\Omega_{\Lambda,0} = \frac{c^2}{3H_0^2} \Lambda\)
  - curvature \(\Omega_{k,0} = -\frac{c^2}{H_0^2 R_0^2} k\)
  - expansion rate \(H_0 = \frac{\dot{R}}{R}_0\)

- **critical density** \(\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G}\)
• density parameters

\[
\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \left( \rho_{m,0} \left( \frac{R}{R_0} \right)^{-3} + \rho_{r,0} \left( \frac{R}{R_0} \right)^{-4} \right) - \frac{kc^2}{R_0^2} \left( \frac{R}{R_0} \right)^{-2} + \frac{c^2\Lambda}{3}
\]

the cosmological parameters

• matter density \( \Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{\text{crit},0}} \)

• radiation density \( \Omega_{r,0} = \frac{\rho_{r,0}}{\rho_{\text{crit},0}} \)

• vacuum energy \( \Omega_{\Lambda,0} = \frac{c^2}{3H_0^2} \Lambda \)

• curvature \( \Omega_{k,0} = -\frac{c^2}{H_0^2 R_0^2} k \)

• expansion rate \( H_0 = \frac{\dot{R}}{R} \)
• density parameters → Friedmann Equations in most memorizable form

\[ H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \]

The cosmological parameters:

• matter density \( \Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{\text{crit},0}} \)

• radiation density \( \Omega_{r,0} = \frac{\rho_{r,0}}{\rho_{\text{crit},0}} \)

• vacuum energy \( \Omega_{\Lambda,0} = \frac{c^2}{3H_0^2} \Lambda \)

• curvature \( \Omega_{k,0} = -\frac{c^2}{H_0^2R_0^2} k \)

• expansion rate \( H_0 = \left( \frac{\dot{R}}{R} \right)_0 \)

derivation uses \( \frac{R}{R_0} = 1/(1+z) \) and otherwise (hopefully) obvious…
Friedmann Models

• density parameters → Friedmann Equations in most memorizable form

\[ H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \]

the cosmological parameters

• matter density \( \Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{\text{crit},0}} \)

• radiation density \( \Omega_{r,0} = \frac{\rho_{r,0}}{\rho_{\text{crit},0}} \)

• vacuum energy \( \Omega_{\Lambda,0} = \frac{c^2}{3H_0^2} \Lambda \)

• curvature \( \Omega_{k,0} = -\frac{c^2}{H_0^2 R_0^2} k \)

• expansion rate \( H_0 = \left( \frac{\dot{R}}{R} \right)_0 \)

cosmic sum rule: \( z=0 \)

\[ 1 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{k,0} + \Omega_{\Lambda,0} \]
Friedmann Models

- density parameters → Friedmann Equations in most memorizable form

\[ H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \]

**cosmic sum rule:**

\[ 1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda \]

**the cosmological parameters**

- matter density \( \Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{\text{crit},0}} \)
- radiation density \( \Omega_{r,0} = \frac{\rho_{r,0}}{\rho_{\text{crit},0}} \)
- vacuum energy \( \Omega_{\Lambda,0} = \frac{c^2}{3H_0^2} \Lambda \)
- curvature \( \Omega_{k,0} = -\frac{c^2}{H_0^2 R_0^2} k \)
- expansion rate \( H_0 = \left( \frac{\dot{R}}{R} \right)_0 \)
density parameters → Friedmann Equations in most memorizable form

\[ H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \]

\[ q = -\frac{\dot{R}}{R^2} = \frac{1}{2} \Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z) \]

the cosmological parameters
- matter density \( \Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{\text{crit},0}} \)
- radiation density \( \Omega_{r,0} = \frac{\rho_{r,0}}{\rho_{\text{crit},0}} \)
- vacuum energy \( \Omega_{\Lambda,0} = \frac{c^2}{3H_0^2} \Lambda \)
- curvature \( \Omega_{k,0} = -\frac{c^2}{H_0^2 R_0^2} k \)
- expansion rate \( H_0 = \left( \frac{\dot{R}}{R} \right)_0 \)

cosmic sum rule:
\[ 1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda \]
- density parameters → Friedmann Equations in most memorizable form

\[ H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \]

\[ q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2} \Omega_m(z) + \Omega_r(z) - \Omega_{\Lambda}(z) \]

**the cosmological parameters**

- matter density \( \Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{\text{crit},0}} \)
- radiation density \( \Omega_{r,0} = \frac{\rho_{r,0}}{\rho_{\text{crit},0}} \)
- vacuum energy \( \Omega_{\Lambda,0} = \frac{c^2}{3H_0^2 \Lambda} \)
- curvature \( \Omega_{k,0} = -\frac{c^2}{H_0^2 R_0^2 k} \)
- expansion rate \( H_0 = \left( \frac{\dot{R}}{R} \right)_0 \)

**derivation:**

\[ \frac{\ddot{R}}{R} = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) + \frac{c^2 \Lambda}{3} \]

\[ = -\frac{4\pi G}{3} \left( \sum \rho_i + \frac{3}{c^2} \sum \omega_i \rho_i \right) + \frac{c^2 \Lambda}{3} \]

\[ = -\frac{4\pi G}{3} \left( \sum (1 + 3 \omega_i) \rho_i \right) + \frac{c^2 \Lambda}{3} \]

\[ = -\frac{H^2}{2} \left( \sum (1 + 3 \omega_i) \Omega_i(z) \right) + H^2 \Omega_{\Lambda}(z) \]

\[ = -\frac{H^2}{2} \left( \sum (1 + 3 \omega_i) \Omega_i(z) \right) - H^2 \Omega_{\Lambda}(z) \]

\[ = -\frac{H^2}{2} (\Omega_m(z) + 2\Omega_r(z)) - H^2 \Omega_{\Lambda}(z) \]

\[ = -\frac{H^2}{2} \Omega_m(z) + \Omega_r(z) - \Omega_{\Lambda}(z) \]

\[ q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2} \Omega_m(z) + \Omega_r(z) - \Omega_{\Lambda}(z) \]
- density parameters → Friedmann Equations in most memorizable form

\[
H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \quad \rightarrow \text{uses density parameters at present time}
\]

\[
q = -\frac{\ddot{R}}{\dot{R}^2} = \frac{1}{2} \Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)
\quad \rightarrow \text{uses density parameters at redshift } z
\]

**cosmic sum rule:**

\[
1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda
\]
• density parameters → Friedmann Equations in most memorizable form

\[ H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \] → uses density parameters at present time

\[ q = -\frac{\dddot{R}}{\dot{R}^2} = \frac{1}{2} \Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z) \] → uses density parameters at redshift \( z \)

Ω(z) ?

cosmic sum rule:

\[ 1 = \Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda \]
- density parameters → Friedmann Equations in most memorizable form

\[
H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

\[
q = -\frac{\ddot{R}}{\dot{R}^2} = \frac{1}{2} \Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)
\]

- evolution of density parameters:

\[
\Omega(z) = \frac{8\pi G}{3H^2(z)} \rho(z)
\]
- **density parameters** → Friedmann Equations in most memorizable form

\[
H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

\[
q = -\frac{\ddot{R}R}{\dot{R}^2} = \frac{1}{2} \Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)
\]

- **evolution of density parameters**:

\[
\Omega(z) = \frac{8 \pi G}{3 H^2(z)} \rho(z) = \frac{8 \pi G}{3 H^2(z)} \rho_0 (1 + z)^{3(1 + \omega)}
\]

barotropic matter
Friedmann Models

- density parameters → Friedmann Equations in most memorizable form

\[ H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \]

\[ q = -\frac{\dot{R}R}{\dot{R}^2} = \frac{1}{2} \Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z) \]

- evolution of density parameters:

\[ \Omega(z) = \frac{8\pi G}{3H^2(z)} \rho(z) = \frac{8\pi G}{3H_0^2} \frac{\rho_0 (1 + z)^{3(1+\omega)}}{(\Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0})} \]

1st Friedmann equation
Friedmann Models

- density parameters → Friedmann Equations in most memorizable form

\[ H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \]

\[ q = -\frac{\dot{R}R}{\dot{R}^2} = \frac{1}{2} \Omega_m(z) + \Omega_r(z) - \Omega_{\Lambda}(z) \]

- evolution of density parameters:

\[ \Omega(z) = \frac{8\pi G}{3H^2(z)} \rho(z) = \frac{\Omega_0 (1 + z)^{3(1+\omega)}}{(\Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0})} \]

\[ \rho_0 \rightarrow \Omega_0 \]
Friedmann Models

- density parameters → Friedmann Equations in most memorizable form

\[ H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \]

\[ q = -\frac{\ddot{R}}{\dot{R}^2} = \frac{1}{2} \frac{\dot{\Omega}_m}{\Omega_m} (z) + \Omega_r (z) - \Omega_{\Lambda} (z) \]

- evolution of density parameters:

\[ \Omega(z) = \frac{\Omega_0}{(\Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0})} \left(1 + z\right)^{3(1+\omega)} \]

- radiation \( w = 1/3 \)
- matter \( w = 0 \)
Friedmann Models

- density parameters → Friedmann Equations in most memorizable form

\[
H^2 = H_0^2 \left( \Omega_{r,0} (1+z)^4 + \Omega_{m,0} (1+z)^3 + \Omega_{k,0} (1+z)^2 + \Omega_{\Lambda,0} \right)
\]

\[
q = -\frac{\dddot{R}R}{\dot{R}^2} = \frac{1}{2} \Omega_m (z) + \Omega_r (z) - \Omega_\Lambda (z)
\]

- evolution of density parameters:

\[
\Omega(z) = \frac{\Omega_0}{\left( \Omega_{r,0} (1+z)^4 + \Omega_{m,0} (1+z)^3 + \Omega_{k,0} (1+z)^2 + \Omega_{\Lambda,0} \right)} \left( 1+z \right)^{3(1+\omega)}
\]

- radiation \( w = 1/3 \)
- matter \( w = 0 \)
- vacuum energy \( w = -1 \)

\[
\Omega(z) = \frac{c^2 \Omega_\Lambda}{3H^2(z)} \left( \Omega_m (1+z)^3 + \Omega_{r,0} (1+z)^2 + \Omega_{\Lambda,0} \right)
\]; same results as for \( \omega = -1 \)
Friedmann Models

- density parameters → Friedmann Equations in most memorizable form

\[
H^2 = H_0^2 \left( \Omega_{r,0} (1+z)^4 + \Omega_{m,0} (1+z)^3 + \Omega_{k,0} (1+z)^2 + \Omega_{\Lambda,0} \right)
\]

\[
q = -\frac{\dddot{R}}{R} = \frac{1}{2} \Omega_m(z) + \Omega_r(z) - \Omega_{\Lambda}(z)
\]

- evolution of density parameters:

\[
\Omega(z) = \frac{\Omega_0}{\left( \Omega_{r,0} (1+z)^4 + \Omega_{m,0} (1+z)^3 + \Omega_{k,0} (1+z)^2 + \Omega_{\Lambda,0} \right) (1+z)^{3(1+\omega)}}
\]

- radiation \( w = 1/3 \)
- matter \( w = 0 \)
- vacuum energy \( w = -1 \)
Friedmann Models

- density parameters → Friedmann Equations in most memorizable form

\[
H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

\[
q = -\frac{\dddot{R}}{\dot{R}^2} = \frac{1}{2} \Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)
\]

- evolution of density parameters:

\[
\Omega(z) = \frac{\Omega_0}{(\Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0})} \left(1 + z\right)^{3(1+\omega)}
\]

Further useful:

\[
\frac{\Omega_i}{\Omega_j} = \frac{\Omega_{i,0}}{\Omega_{j,0}} (1 + z)^{(n_i - n_j)} \text{ with } n = 3(1 + \omega)
\]

- radiation \( w = 1/3 \)
- matter \( w = 0 \)
- vacuum energy \( w = -1 \)
- density parameters → Friedmann Equations in most memorizable form

\[ H^2 = H_0^2 \left( \Omega_{r,0} (1+z)^4 + \Omega_{m,0} (1+z)^3 + \Omega_{k,0} (1+z)^2 + \Omega_{\Lambda,0} \right) \]

\[ q = -\frac{\ddot{R}}{\dot{R}^2} = \frac{1}{2} \frac{\Omega_m(z) + \Omega_r(z) - \Omega_\Lambda(z)}{\Omega_0} \]

- evolution of density parameters:

\[ \Omega(z) = \frac{\Omega_0}{\Omega_{r,0} (1+z)^4 + \Omega_{m,0} (1+z)^3 + \Omega_{k,0} (1+z)^2 + \Omega_{\Lambda,0}} \left( 1 + z \right)^{3(1+\omega)} \]

- cosmic sum rule:

\[ 1 = \Omega_r(z) + \Omega_m(z) + \Omega_k(z) + \Omega_\Lambda(z) \]

Further useful:

- radiation \( w = 1/3 \)
- matter \( w = 0 \)
- vacuum energy \( w = -1 \)

Further useful:

\[ \frac{\Omega_j}{\Omega_j} = \frac{z}{\Omega_{j,0}} (1+z)^{(n_i-n_j)} \]
Friedmann Models

- density parameters → Friedmann Equations in most **memorizable** form

\[ H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \]

\[ q = -\frac{\ddot{R}}{\dot{R}^2} = \frac{1}{2} \Omega_m(z) + \Omega_r(z) - \Omega_{\Lambda}(z) \]

- evolution of density parameters:

\[ \Omega(z) = \frac{\Omega_0}{(\Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0})} \left(1 + z\right)^{3(1+\omega)} \]

- cosmic sum rule:

\[ 1 = \Omega_r(z) + \Omega_m(z) + \Omega_k(z) + \Omega_{\Lambda}(z) \]

- cosmic sum rule:

\[ \frac{\Omega_i}{\Omega_j} = \frac{\Omega_{i,0}}{\Omega_{j,0}} \left(1 + z\right)^{(\omega_i - \omega_j)} \]

  - radiation \( w = 1/3 \)
  - matter \( w = 0 \)
  - vacuum energy \( w = -1 \)

Further useful:
Friedmann Models

- interplay  scale factor – redshift – time

\[
H^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

⇒ solution is \( a(t) \)

⇒ relation to redshift: \( a(t) = \frac{1}{1 + z} \)

for standard cosmological models we have a unique mapping between

\[ a \leftrightarrow t \leftrightarrow z \]

(further, the constancy of the speed of light then also relates those to distance)
Friedmann Models

- Robertson-Walker-Metric

- Friedmann Equations

- Friedmann Models
  - general trends
    - special solutions for flat and open world models
    - horizons
- Friedmann Equations

\[
\begin{align*}
\dot{R}^2 &= \frac{8\pi G}{3} \rho R^2 - kc^2 + \frac{c^2 \Lambda}{3} R^2 \\
R" &= -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right) + \frac{c^2 \Lambda}{3} R
\end{align*}
\]
- **Friedmann Equations - general behaviour**

\[
\frac{a^2}{R} = \frac{8\pi G}{3} \rho R^2 - kc^2
\]

\[
\frac{a^3}{R} = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right)
\]

- \( \Lambda = 0, \ p=\omega \rho c^2 \)
Friedmann Models

Friedmann Equations - general behaviour

\[
\begin{align*}
\dot{R}^2 &= \frac{8\pi G}{3} \rho R^2 - kc^2 \\
\ddot{R} &= -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right)
\end{align*}
\]

- $\Lambda = 0, \ p = \omega \rho c^2$ \quad $\dot{R}^2 \propto R^{2-3(1+\omega)} - kc^2$ \quad as $\rho \propto R^{-3(1+\omega)}$

Note: $2 - 3(1 + \omega) < 0$
Friedmann Equations - general behaviour

\[ \dot{R}^2 = \frac{8\pi G}{3} \rho R^2 - kc^2 \]

\[ \ddot{R} = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right) \]

- \( \Lambda = 0, p=\omega \rho c^2 \)

\[ \dot{R}^2 \propto R^{2-3(1+\omega)} - kc^2 \]

as \( \rho \propto R^{-3(1+\omega)} \)

\[ \ddot{R} < 0 \]

as \( \rho + \frac{3p}{c^2} > 0 \) \( (p = \omega \rho c^2, \omega > 0) \)

Note: \( 2 - 3(1 + \omega) < 0 \)
• Friedmann Equations - general behaviour

\[
\begin{align*}
\ddot{R} & = \frac{8\pi G}{3} \rho R^2 - kc^2 \\
\dddot{R} & = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right)
\end{align*}
\]

• \( \Lambda = 0, \ p = \omega \rho c^2 \)

\[
\dot{R}^2 \propto R^{2-3(1+\omega)} - kc^2 \\
\ddot{R} < 0 \\
\] as \( \rho \propto R^{-3(1+\omega)} \)

as \( \rho + \frac{3p}{c^2} > 0 \) \( (p = \omega \rho c^2, \ \omega > 0) \)

we observe: \( \left( \frac{\dot{R}}{R} \right)_0 > 0 \)

Note: \( 2 - 3(1 + \omega) < 0 \)
- Friedmann Equations - general behaviour

\[ \dot{R}^2 = \frac{8\pi G}{3} \rho R^2 - kc^2 \]

\[ \ddot{R} = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right) \]

• \( \Lambda = 0, p = \omega \rho c^2 \)

\[ \dot{R}^2 \propto R^{2-3(1+\omega)} - kc^2 \]

\[ \ddot{R} < 0 \]

as \( \rho \propto R^{-3(1+\omega)} \)

as \( \rho + \frac{3p}{c^2} > 0 \) \( (p = \omega \rho c^2, \omega > 0) \)

we observe: \( \left( \frac{\dot{R}}{R} \right)_0 > 0 \quad \Rightarrow \quad \left( \dot{R}^2 \right)_0 > 0 \)

\( R \neq 0 \)

Note: \( 2 - 3(1 + \omega) < 0 \)
**Friedmann Models**

- **Friedmann Equations - general behaviour**

\[
\begin{align*}
\dot{R}^2 &= \frac{8\pi G}{3} \rho R^2 - kc^2 \\
\ddot{R} &= -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right)
\end{align*}
\]

- \( \Lambda = 0, \ p = \omega \rho c^2 \)

\[
\begin{align*}
\dot{R}^2 &\propto R^{2-3(1+\omega)} - kc^2 \\
\ddot{R} &< 0
\end{align*}
\]

\[
\text{as } \rho \propto R^{-3(1+\omega)} \\
\text{as } \rho + \frac{3p}{c^2} > 0 \ (p = \omega \rho c^2, \ \omega > 0)
\]

we observe: \( \left( \frac{\dot{R}}{R} \right)_0 > 0 \Rightarrow \left( \dot{R}^2 \right)_0 > 0 \)

\( \dot{R} > 0, \ \ddot{R} < 0 \)

\( \Rightarrow R(t) \) has been monotonically increasing! \( \Rightarrow \exists t_c \text{ with } R(t_c) = 0 \Rightarrow \text{singularity!} \)

Note: \( 2 - 3(1 + \omega) < 0 \)
- **Friedmann Equations - general behaviour**

\[
\begin{align*}
\dot{R}^2 &= \frac{8\pi G}{3} \rho R^2 - kc^2 \\
\ddot{R} &= -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right)
\end{align*}
\]

- \( \Lambda = 0, \ p = \omega \rho c^2 \)

\[
\begin{align*}
\dot{R}^2 &\propto R^{2-3(1+\omega)} - kc^2 \\
\ddot{R} &< 0
\end{align*}
\]

**future:**

- \( k = 0: \ \dot{R}^2 \propto R^{2-3(1+\omega)} \rightarrow R \propto t^{2(1+\omega)/3} \)

\[
R^{3(1+\omega)/2-1} dR = dt
\]

\( \Rightarrow \) indefinite expansion!

Note: \( 2 - 3(1 + \omega) < 0 \)
Friedmann Models

- Friedmann Equations - general behaviour

\[
\begin{align*}
\dot{R}^2 &= \frac{8\pi G}{3} \rho R^2 - kc^2 \\
\ddot{R} &= -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right)
\end{align*}
\]

- \( \Lambda = 0, p=\omega \rho c^2 \)

\[
\dot{R}^2 \propto R^{2-3(1+\omega)} - kc^2
\]

as \( \rho \propto R^{-3(1+\omega)} \)

as \( \rho + \frac{3p}{c^2} > 0 \) \( (p = \omega \rho c^2, \omega > 0) \)

future:

\( k = 0: \quad R \propto R^{2-3(1+\omega)} \quad \longrightarrow \quad R \propto t^{2/3(1+\omega)} \quad \Rightarrow \text{indefinite expansion!} \)

\( k > 0: \quad R \propto R^{2-3(1+\omega)} - |k|c^2 \quad \longrightarrow \quad \exists \dot{R}(t_s) = 0 \quad \Rightarrow \text{recollapse!} \)

Note: \( 2 - 3(1 + \omega) < 0 \)

\[ \begin{align*}
\dot{R}(0) &= +\infty \\
\dot{R}(\infty) &= -|k|c^2 \\
\end{align*} \]

\( \dot{R} \) changes sign (sometime)!
Friedmann Models

- Friedmann Equations - general behaviour

\[
\begin{align*}
\dot{R}^2 &= \frac{8\pi G}{3} \rho R^2 - kc^2 \\
\ddot{R} &= -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right)
\end{align*}
\]

- \( \Lambda = 0, \ p = \omega \rho c^2 \)

\[
\dot{R}^2 \propto R^{2-3(1+\omega)} - kc^2 \quad \text{as } \rho \propto R^{-3(1+\omega)}
\]

\[
\dot{R} < 0 \quad \text{as } \rho + \frac{3p}{c^2} > 0 \quad (p = \omega \rho c^2, \ \omega > 0)
\]

future:

- \( k = 0 \):

\[
R \propto R^{2-3(1+\omega)} \quad \longrightarrow \quad R \propto t^{2/3(1+\omega)} \quad \Rightarrow \text{indefinite expansion!}
\]

- \( k > 0 \):

\[
R \propto R^{2-3(1+\omega)} - |k|c^2 \quad \longrightarrow \quad \exists \ R(t_0) = 0
\]

\( \Rightarrow \text{recollapse!} \)

- \( k < 0 \):

\[
R \propto R^{2-3(1+\omega)} + |k|c^2 \quad \longrightarrow \quad \dot{R}^2 > 0 \quad \Rightarrow \quad R(t) \uparrow
\]

\( \Rightarrow \text{increased indefinite expansion!} \)

Note: \( 2 - 3(1 + \omega) < 0 \)
- Friedmann Equations - general behaviour

\[ \ddot{R} = \frac{8\pi G}{3} \rho R^2 - kc^2 + \frac{c^2 \Lambda}{3} R^2 \]

\[ \dddot{R} = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right) + \frac{c^2 \Lambda}{3} R \]

- \( \Lambda < 0 \)
Friedmann Equations - general behaviour

\[
\frac{\ddot{R}}{R} = \frac{8\pi G}{3} \rho R^2 - kc^2 + \frac{c^2 \Lambda}{3} R^2
\]

\[
\dddot{R} = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right) + \frac{c^2 \Lambda}{3} R
\]

\[ \Lambda < 0 \]

\[
\frac{\ddot{R}}{R} = \frac{8\pi G}{3} \rho R^2 - kc^2 - \frac{c^2 |\Lambda|}{3} R^2 = \frac{8\pi G}{3} \rho_m + \frac{c^2 |\Lambda|}{3} R^2
\]

\[
\dddot{R} = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right) - \frac{c^2 |\Lambda|}{3} R = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} + \frac{c^2 |\Lambda|}{12\pi G} \right) < 0
\]
• Friedmann Models

Friedmann Equations - general behaviour

\[
\dot{R}^2 = \frac{8\pi G}{3} \rho R^2 - kc^2 + \frac{c^2 \Lambda}{3} R^2
\]

\[
\ddot{R} = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right) + \frac{c^2 \Lambda}{3} R
\]

• \(\Lambda < 0\)

\[
\dot{R}^2 = \frac{8\pi G}{3} \rho R^2 - kc^2 - \frac{c^2 |\Lambda|}{3} R^2 = \frac{8\pi G}{3} \rho_m \Lambda R^2 - kc^2
\]

\[
\ddot{R} = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right) - \frac{c^2 |\Lambda|}{3} R = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} + \frac{c^2 |\Lambda|}{2\pi G} \right) < 0
\]

“+” \(\Rightarrow\) attractive force!
Friedmann Models

- Friedmann Equations - general behaviour

\[ \ddot{R} = \frac{8\pi G}{3} \rho R^2 - kc^2 + \frac{c^2 \Lambda}{3} R^2 \]

\[ \dddot{R} = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right) + \frac{c^2 \Lambda}{3} R \]

- \( \Lambda < 0 \)

\[ \ddot{R} = \frac{8\pi G}{3} \rho R^2 - kc^2 - \frac{c^2 |\Lambda|}{3} R^2 = \frac{8\pi G}{3} \rho_{m+\Lambda} R^2 - kc^2 \]

\[ \dddot{R} = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right) - \frac{c^2 |\Lambda|}{3} R = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} + \frac{c^2 |\Lambda|}{2\pi G} \right) < 0 \]

"+" \( \Rightarrow \) attractive force!

\[ \Rightarrow \ R(t) \] starts monotonically increasing \& \( \exists \ R(t_s) = 0 \)

\[ \Rightarrow \] recollapse happens even earlier due to “attractive force”!
Friedmann Equations - general behaviour

\[
\frac{\ddot{R}}{R} = \frac{8\pi G}{3} \rho R^2 - kc^2 + \frac{c^2 \Lambda}{3} R^2
\]

\[
\dddot{R} = -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right) + \frac{c^2 \Lambda}{3} R
\]

• \( \Lambda > 0 \)
Friedmann Models

- Friedmann Equations - general behaviour

\[
\begin{align*}
\dot{R}^2 &= \frac{8\pi G}{3} \rho R^2 - kc^2 + \frac{c^2 \Lambda}{3} R^2 \\
\ddot{R} &= -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} \right) + \frac{c^2 \Lambda}{3} R
\end{align*}
\]

• \(\Lambda > 0\)

\[
\begin{align*}
\dot{R}^2 &= \frac{8\pi G}{3} \rho R^2 - kc^2 - \frac{c^2 |\Lambda|}{3} R^2 = \frac{8\pi G}{3} \rho_{m+\Lambda} R^2 - kc^2 \\
\ddot{R} &= -\frac{4\pi G}{3} R \left( \rho + \frac{3p}{c^2} + \frac{c^2 |\Lambda|}{12\pi G} \right) < 0
\end{align*}
\]

“−” \(\Rightarrow\) repulsive force!

\[\Rightarrow\] forever expanding!*  

(Note: \(\rho\) drops while \(\Lambda\) remains constant...)

*careful: there are \((\rho, \Lambda)\) combinations possible where this is not necessarily true...
Friedmann Models

- Robertson-Walker-Metric

- Friedmann Equations

- Friedmann Models
  - general trends
  - special solutions for flat and open world models
  - horizons
- **Friedmann Equations - solutions**

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]
Friedmann Equations - solutions

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]
Friedmann Equations - solutions

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]
- Friedmann Equations - solutions

\[ \left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \]
Friedmann Models

- Friedmann Equations - solutions

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

- flat model with \( \Omega_m = 1 \)

**Note:** this solution is important right after matter-radiation equality!
Friedmann Models

- Friedmann Equations - solutions

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

- flat model with \( \Omega_m = 1 \)

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \frac{R_0}{R} \right)^3
\]

\[
\frac{\dot{R}}{R} = H_0 \left( \frac{R_0}{R} \right)^{3/2}
\]

\[
R^{3/2} \frac{\dot{R}}{R} = H_0 R_0^{3/2}
\]

\[
R^{1/2} dR = H_0 R_0^{3/2} dt
\]

\[
\Rightarrow \frac{R(t)}{R_0} = \left( \frac{t}{t_0} \right)^{2/3}, \quad t_0 = \frac{2}{3H_0}
\]
Friedmann Models

- **Friedmann Equations - solutions**

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

- flat model with \( \Omega_r = 1 \)

\[
\frac{\dot{R}}{R} = H_0 \left( \frac{R_0}{R} \right)^2
\]

\[
\frac{\dot{R}}{R} = H_0 \left( \frac{R_0}{R} \right)^2
\]

\[
R^2 \frac{\ddot{R}}{R} = H_0 R_0^2
\]

\[
RdR = H_0 R_0^2 dt \quad \Rightarrow \quad \frac{R(t)}{R_0} = \left( \frac{t}{t_0} \right)^{1/2} \quad t_0 = \frac{1}{2H_0}
\]
Friedmann Models

- Friedmann Equations - solutions

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

- flat model with \(\Omega_\Lambda = 1\)

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2
\]

\[
\frac{\dot{R}}{R} = H_0
\]

\[
\frac{dR}{R} = H_0 \, dt \quad \Rightarrow \quad R(t) = e^{H_0 t} \quad \text{inflationary Universe!}
\]

\[\Omega_\Lambda(z) \to 1\]
Friedmann Models

- Friedmann Equations - solutions

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

- general model with \( \Omega_m \neq 0, \Omega_{\Lambda} = 0, \Omega_r = 0 \)

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{m,0} \left( \frac{R_0}{R} \right)^3 + \Omega_{k,0} \left( \frac{R_0}{R} \right)^2 \right)
\]

\[
\frac{\dot{R}}{R} = R^2 H_0^2 \left( \Omega_{m,0} \left( \frac{R_0}{R} \right)^3 \Omega_{k,0} \left( \frac{R_0}{R} \right) + \Omega_{k,0} \right)
\]

\[
R = H_0 R_0 \sqrt{\Omega_{m,0} \left( \frac{R_0}{R} \right) + \left( 1 - \Omega_{m,0} \right)}
\]

\[
R = H_0 R_0 \sqrt{\Omega_{m,0} \left( \frac{R_0}{R} - 1 \right) + 1}
\]
Friedmann Models

- Friedmann Equations - solutions

\[
\left( \frac{R'}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

*general model with \( \Omega_m \neq 0, \) \( \Omega_\Lambda = 0, \) \( \Omega_r = 0 \)

\[
\left( \frac{R'}{R} \right)^2 = R^2 H_0^2 \left( \Omega_{m,0} \left( \frac{R_0}{R} \right)^3 + \Omega_{k,0} \left( \frac{R_0}{R} \right)^2 \right)
\]

\[
\dot{R} = R^2 H_0 \left( \Omega_{m,0} \left( \frac{R_0}{R} \right)^2 + \Omega_{k,0} \right) \quad \Rightarrow \quad \Omega_{m,0} > 1
\]

\[
\dot{R} = H_0 R_0 \sqrt{\Omega_{m,0} \left( \frac{R_0}{R} \right) + (1 - \Omega_{m,0})} \quad \Rightarrow \quad \Omega_{m,0} < 1
\]

\[
R = H_0 R_0 \sqrt{\Omega_{m,0} \left( \frac{R_0}{R} \right) + (1 - \Omega_{m,0})}
\]

\[
\dot{R} = H_0 R_0 \sqrt{\Omega_{m,0} \left( \frac{R_0}{R} \right) - 1} + 1 \quad \Rightarrow \quad \Omega_{m,0} < 1
\]

\[
R' = H_0 \frac{R_0}{R} \left( \Omega_{m,0} \left( \frac{R_0}{R} \right) - 1 \right) + 1
\]

\[
R = H_0 R_0 \sqrt{\Omega_{m,0} \left( \frac{R_0}{R} \right) - 1} + 1
\]

\[
R = a(1 - \cos \theta) \quad a = \frac{\Omega_{m,0}}{2(\Omega_{m,0} - 1)}
\]

\[
t = b(\theta - \sin \theta) \quad b = \frac{\Omega_{m,0}}{2 H_0 (\Omega_{m,0} - 1)^{3/2}}
\]

\[
R = A(\cosh \theta - 1) \quad A = \frac{\Omega_{m,0}}{2(1 - \Omega_{m,0})}
\]

\[
t = B(\sinh \theta - \theta) \quad B = \frac{\Omega_{m,0}}{2 H_0 (1 - \Omega_{m,0})^{3/2}}
\]
• Friedmann Equations - solutions

\[
\left(\frac{\dot{R}}{R}\right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

- general model with \( \Omega_m \neq 0, \Omega_{\Lambda} = 0, \Omega_r = 0 \)

\[
\left(\frac{\dot{R}}{R}\right)^2 = H_0^2 \left( \Omega_{m,0} \left(\frac{R_0}{R}\right)^3 + \Omega_{k,0} \left(\frac{R_0}{R}\right)^2 \right)
\]

\[
\dot{R} = R^2 H_0^2 \left( \Omega_{m,0} \left(\frac{R_0}{R}\right) + \Omega_{k,0} \right) \Rightarrow \dot{R} = H_0 R_0 \sqrt{\Omega_{m,0} \left(\frac{R_0}{R}\right) + (1 - \Omega_{m,0})}
\]

- \( \Omega_{m,0} > 1 \)

\[
R = a(1 - \cos \vartheta) \quad a = \frac{\Omega_{m,0}}{2(\Omega_{m,0} - 1)}
\]

\[
t = b(\vartheta - \sin \vartheta) \quad b = \frac{\Omega_{m,0}}{2H_0(\Omega_{m,0} - 1)^{3/2}}
\]

- \( \Omega_{m,0} < 1 \)

\[
R = A(\cosh \theta - 1) \quad A = \frac{\Omega_{m,0}}{2(1 - \Omega_{m,0})}
\]

\[
t = B(\sinh \theta - \theta) \quad B = \frac{\Omega_{m,0}}{2H_0(1 - \Omega_{m,0})^{3/2}}
\]
Friedmann Equations – structure formation

- Initial Seed (i.e. Overdensity)
- Expansion (Hubble Flow)
- Collapsed Region (a galaxy)
Friedmann Equations – structure formation

- Initial Seed (i.e. Overdensity)
- Expansion (Hubble Flow)
- Decoupling of perturbation
- Collapsed Region (a galaxy)
Friedmann Models

- Friedmann Equations – structure formation

Overdensity: \( \Omega_m > 0, \Omega_\Lambda = 0, \Omega_r = 0 \)

\[
R = a(1 - \cos \vartheta) \quad a = \frac{\Omega_{m,0}}{2(\Omega_{m,0} - 1)}
\]

\[
t = b(\vartheta - \sin \vartheta) \quad b = \frac{\Omega_{m,0}}{2H_0(\Omega_{m,0} - 1)^{3/2}}
\]

Initial Seed (i.e. Overdensity)

decoupling of perturbation

Collapsed Region (a galaxy)

Expansion (Hubble Flow)

time
Friedmann Models

- Friedmann Equations - solutions

\[ \left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right) \]

- flat model with \( \Omega_m + \Omega_\Lambda = 1, \ \Omega_r = 0 \)

\[ t(z) = \frac{2}{3H_0} \frac{1}{\sqrt{\Omega_{\Lambda,0}}} \ln \left( \sqrt{\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}} \left( \frac{1}{1 + z} \right)^3 + \sqrt{\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}} \left( \frac{1}{1 + z} \right)^3 + 1 \right) \]
Friedmann Models

- Friedmann Equations - solutions

\[
\left( \frac{\dot{R}}{R} \right)^2 = H_0^2 \left( \Omega_{r,0} (1 + z)^4 + \Omega_{m,0} (1 + z)^3 + \Omega_{k,0} (1 + z)^2 + \Omega_{\Lambda,0} \right)
\]

• general model with \( \Omega_m \neq 0, \Omega_\Lambda \neq 0, \Omega_r = 0 \)

no analytical solution for the general case!
Friedmann Models

- Friedmann Equations - solutions

all models expand (or collapse)!
\[ \exists \text{ no stationary Universe...} \]
- Friedmann Equations - solutions

(bounce condition: $H = 0 \& \frac{dH}{dt} > 0$)
Friedmann Models

- Robertson-Walker-Metric

- Friedmann Equations

- Friedmann Models
  - general trends
  - special solutions for flat and open world models
  - horizons
- horizons

**cosmological horizons** measure distances…

…from which one could possibly retrieve information from the past,

…or interact with in the future
• horizons: distances

\[ ds^2 = (cdt)^2 - R^2(t) \left[ \frac{dx^2}{1-\kappa x^2} + x^2 (d\theta^2 + \sin^2(\theta)d\phi^2) \right] \]
• horizons: distances

\[ ds^2 = (c dt)^2 - R^2(t) \left[ \frac{dx^2}{1-kx^2} + x^2 \left( d\vartheta^2 + \sin^2(\vartheta) d\varphi^2 \right) \right] \]

• photons follow null-geodesics

\[ 0 = (c dt)^2 - R^2(t) \left[ \frac{dx^2}{1-kx^2} + x^2 \left( d\vartheta^2 + \sin^2(\vartheta) d\varphi^2 \right) \right] \]
- horizons: distances

\[ ds^2 = (cdt)^2 - R^2(t) \left[ \frac{dx^2}{1-kx^2} + x^2 \left( d\theta^2 + \sin^2(\theta)d\varphi^2 \right) \right] \]

- photons follow null-geodesics

\[ 0 = (cdt)^2 - R^2(t) \left[ \frac{dx^2}{1-kx^2} + x^2 \left( d\theta^2 + \sin^2(\theta)d\varphi^2 \right) \right] \]

- no transverse motion

\[ f(x_E) = \int_0^{x_E} \frac{dx}{\sqrt{1-kx^2}} = \int_{t_0}^{t_E} \frac{cdt}{R(t)} \]
Friedmann Models

- horizons: distances

\[ ds^2 = (cdt)^2 - R^2(t) \left[ \frac{dx^2}{1-kx^2} + x^2 (d\theta^2 + \sin^2(\theta)d\varphi^2) \right] \]

- photons follow null-geodesics

\[ 0 = (cdt)^2 - R^2(t) \left[ \frac{dx^2}{1-kx^2} + x^2 (d\theta^2 + \sin^2(\theta)d\varphi^2) \right] \]

- no transverse motion

Comoving distance

\[ d_C = f(x_E) = \int_0^{x_E} \frac{dx}{\sqrt{1-kx^2}} = \int_{t_E}^{t} \frac{cdt}{R(t)} \]

\[ f(x_E) = \begin{cases} 
  x_E & k=0 \\
  \frac{1}{\sqrt{|k|}} \arcsin(\sqrt{|k|} x_E) & k=1 \\
  \frac{1}{\sqrt{|k|}} \arcsinh(\sqrt{|k|} x_E) & k=-1 
\end{cases} \]
Friedmann Models

- **horizons: distances**

\[ ds^2 = (cdt)^2 - R^2(t) \left[ \frac{dx^2}{1-kx^2} + x^2 \left( d\theta^2 + \sin^2(\theta)d\phi^2 \right) \right] \]

- **photons follow null-geodesics**

\[ 0 = (cdt)^2 - R^2(t) \left[ \frac{dx^2}{1-kx^2} + x^2 \left( d\theta^2 + \sin^2(\theta)d\phi^2 \right) \right] \]

- **no transverse motion**

  **comoving distance**
  \[ d_C = f(x_E) = \int_0^{x_E} \frac{dx}{\sqrt{1-kx^2}} = \int_{t_0}^{t_E} \frac{cdt}{R(t)} \]

  \[ f(x_E) = \begin{cases} 
  x_E & k=0 \\
  \frac{1}{\sqrt{|k|}} \arcsin(\sqrt{|k|} x_E) & k=1 \\
  \frac{1}{\sqrt{|k|}} \arcsinh(\sqrt{|k|} x_E) & k=-1 
\end{cases} \]

  **proper distance**
  \[ d_p = \frac{R(t)}{R_0} d_C = \frac{R(t)}{R_0} \int_{t_0}^{t_E} \frac{cdt}{R(t)} \]
horizons: distances

\[ d_p = \frac{R(t)}{R_0} d_c = \frac{R(t)}{R_0} \int_{t_E}^{t_0} \frac{c dt}{R(t)} \]
- horizons: distances

\[
d_p = \frac{R(t)}{R_0} \quad d_C = \frac{R(t)}{R_0} \int_{t_0}^{t_F} c dt \quad \frac{R(t)}{R(t)}
\]

different bounds define different horizons...
- horizons

  - particle horizon: max. distance particle can have travelled since decoupling

\[ R_p(t) = R(t) \int_{t_{dec}}^{t} \frac{cdt'}{R(t')} \]
• particle horizon: max. distance particle can have travelled since decoupling

\[ R_p(t) = R(t) \int_{t_{\text{dec}}}^{t} \frac{c dt'}{R(t')} \]

• “particle horizon”: max. distance photon can have travelled since big bang (there are events we have not have seen yet…)

\[ R_p(t) = R(t) \int_{0}^{t} \frac{c dt'}{R(t')} \]

development

can be seen
cannot be seen
can be seen
cannot be seen

events not yet seen
events not yet seen

today
• **horizons**

  - particle horizon: max. distance particle can have travelled since decoupling

    \[ R_p(t) = R(t) \int_{t_{dec}}^{t} \frac{cdt'}{R(t')} \]

  - “particle horizon”: max. distance photon can have travelled since big bang (there are events we have not have seen yet…)

    \[ R_p(t) = R(t) \int_{0}^{t} \frac{cdt'}{R(t')} \]

  - event horizon: max. distance particle can travel from now onwards (there may be events we will never see…)

    \[ R_e(t) = R(t) \int_{t}^{\infty} \frac{cdt'}{R(t')} \]
Friedmann Models

- horizons
  - particle horizon: max. distance particle can have travelled since decoupling
    \[ R_p(t) = R(t) \int_{t_{dec}}^{t} \frac{c dt'}{R(t')} \]
  - “particle horizon”: max. distance photon can have travelled since big bang (there are events we have not have seen yet…)
    \[ R_p(t) = R(t) \int_{0}^{t} \frac{c dt'}{R(t')} \]
  - event horizon: max. distance particle can travel from now onwards (there may be events we will never see…)
    \[ R_e(t) = R(t) \int_{t}^{\infty} \frac{c dt'}{R(t')} \]
  - Hubble radius: distance at which recessional velocity equals speed of light
    \[ R_H(t) = \frac{c}{H} \]
horrizons

- particle horizon: max. distance particle can have travelled since decoupling

\[ R_p(t) = R(t) \int_{t_{dec}}^{t} \frac{cdt'}{R(t')} \]

- “particle horizon”: max. distance photon can have travelled since big bang (there are events we have not have seen yet…)

\[ R_p(t) = R(t) \int_{0}^{t} \frac{cdt'}{R(t')} \]

- event horizon: max. distance particle can travel from now onwards (there may be events we will never see…)

\[ R_e(t) = R(t) \int_{t}^\infty \frac{cdt'}{R(t')} \]

- (comoving) Hubble radius: distance at which recessional velocity equals speed of light

\[ R_H(t) = \frac{c}{H}; \quad R_{cH}(t) = \frac{R_0}{R} \frac{c}{H} \]
• **horizons**

  - particle horizon: max. distance particle can have travelled since decoupling

\[
R_p(t) = R(t) \int_{t_{dec}}^{t} \frac{c dt'}{R(t')}
\]

  - “particle horizon”: max. distance photon can have travelled since big bang (there are events we have not have seen yet…)

\[
R_p(t) = R(t) \int_{0}^{t} \frac{c dt'}{R(t')}
\]

  - event horizon: max. distance particle can travel from now onwards (there may be events we will never see…)

\[
R_e(t) = R(t) \int_{t}^{\infty} \frac{c dt'}{R(t')}
\]

  - (comoving) Hubble radius: distance at which recessional velocity equals speed of light

\[
R_H(t) = \frac{c}{H} \quad ; \quad R_{cH}(t) = \frac{R_0}{H} \frac{c}{R} \quad ; \quad \text{Universe dominated by barotropic fluid } p = \omega \rho c^2 \Rightarrow R_{cH}(t) \propto R^2^{\frac{1}{(1+3\omega)}}
\]
2. Inflation coming from $p$ and $q$ "know" that they should be at almost exactly the same temperature? The same question applies to any two points in the CMB that are separated by more than 1 degree in the sky. The homogeneity of the CMB spans scales that are much larger than the particle horizon at the time when the CMB was formed. In fact, in the standard cosmology the CMB is made of about $10^4$ disconnected patches of space. If there wasn't enough time for these regions to communicate, why do they look so similar? This is the horizon problem.

![horizon problem](image)

Figure 2.2: The horizon problem in the conventional Big Bang model. All events that we currently observe are on our past light cone. The intersection of our past light cone with the spacelike slice labelled CMB corresponds to two opposite points in the observed CMB. Their past light cones don't overlap before they hit the singularity, $a=0$, so these points appear to have been in causal contact. The same applies to any two points in the CMB that are separated by more than 1 degree on the sky.

2.2 A Shrinking Hubble Sphere

Our description of the horizon problem has highlighted the fundamental role played by the growing Hubble sphere of the standard Big Bang cosmology. A simple solution to the horizon problem therefore suggests itself: let us conjecture a phase of decreasing Hubble radius in the early universe, $\frac{dt}{dH}(<0)$

If this lasts long enough, the horizon problem can be avoided. Physically, the shrinking Hubble sphere requires a SEC-violating fluid, $1 + 3w < 0$.

2.2.1 Solution of the Horizon Problem

For a shrinking Hubble sphere, the integral in (2.1.5) is dominated by the lower limit. The Big Bang singularity is now pushed to negative conformal time, $\tau_{\text{in}} = 2H^{-1}0(1 + 3w)a^{1/2}(1+3w)^{-1/2}a^{1/2}(1+3w)! 1^{-1/2}$

This implies that there was "much more conformal time between the singularity and decoupling than we had thought"! Fig. 2.3 shows the new spacetime diagram. The past light cones of $R_{cH}(t)$ now intersect at $p$ and $q$.
The horizon problem arises because the CMB is made of about $10^4$ disconnected patches of space. If there wasn't enough time for these regions to communicate, why do they look so similar? This is the horizon problem.

Figure 2.2: The horizon problem in the conventional Big Bang model. All events that we currently observe are on our past light cone. The intersection of our past light cone with the spacelike slice labelled CMB corresponds to two opposite points in the observed CMB. Their past light cones don't overlap before they hit the singularity, $a=0$, so these points appear to have been in causal contact. This applies to any two points separated by more than $1^\circ$ on the sky.

2.2 A Shrinking Hubble Sphere

Our description of the horizon problem has highlighted the fundamental role played by the growing Hubble sphere of the standard Big Bang cosmology. A simple solution to the horizon problem therefore suggests itself: let us conjecture a phase of decreasing Hubble radius in the early universe.

If this lasts long enough, the horizon problem can be avoided. Physically, the shrinking Hubble sphere requires a SEC-violating fluid, $1 + 3w < 0$.

2.2.1 Solution of the Horizon Problem

For a shrinking Hubble sphere, the integral in (2.1.5) is dominated by the lower limit. The Big Bang singularity is now pushed to negative conformal time, $\tau < -1/(1 + 3w)$.

This implies that there was "much more conformal time between the singularity and decoupling than we had thought"! Fig. 2.3 shows the new spacetime diagram. The past light cones of $p$ and $q$ are causally disconnected (applies to any two points separated by more than $1^\circ$ on the sky!).
Inflation coming from $p$ and $q$ "know" that they should be at almost exactly the same temperature? The same question applies to any two points in the CMB that are separated by more than 1 degree in the sky. The homogeneity of the CMB spans scales that are much larger than the particle horizon at the time when the CMB was formed. In fact, in the standard cosmology the CMB is made of about $10^4$ disconnected patches of space. If there wasn't enough time for these regions to communicate, why do they look so similar? This is the horizon problem.

For a shrinking Hubble sphere, the integral in (2.1.5) is dominated by the lower limit. The Big Bang singularity is now pushed to negative conformal time, $\tau_i = \frac{2}{H_0} (1 + 3w) a_{i2}^2 (1 + 3w)$.

This implies that there was "much more conformal time between the singularity and decoupling than we had thought"! Fig. 2.3 shows the new spacetime diagram. The past light cones of $p$ and $q$ are causally disconnected (applies to any two points separated by more than 1° on the sky!)

$T \approx 2.76 K$
- horizon problem: solution via inflation

Figure 2.3: Inflationary solution to the horizon problem. The comoving Hubble sphere shrinks during inflation and expands during the conventional Big Bang evolution (at least until dark energy takes over at $\pi/2$). Conformal time during inflation is negative. The spacelike singularity of the standard Big Bang is replaced by the reheating surface, i.e. rather than marking the beginning of time it now corresponds simply to the transition from inflation to the standard Big Bang evolution. All points in the CMB have overlapping past light cones and therefore originated from a causally connected region of space.

The uniformity of the CMB is not a mystery anymore. In inflationary cosmology, $\pi=0$ is not the initial singularity, but instead becomes only a transition point between inflation and the standard Big Bang evolution. There is time both before and after $\pi=0$.}

2.2.2 Hubble Radius vs. Particle Horizon

A quick word of warning about bad (but unfortunately standard) language in the inflationary literature: Both the particle horizon $ph$ and the Hubble radius ($aH$) are often referred to simply as the “horizon”. In the standard FRW evolution (with ordinary matter) the two are roughly the same—cf. eq. (2.1.9)—so giving them the same name isn’t an issue. However, the whole point of inflation is to make the particle horizon much larger than the Hubble radius. The Hubble radius ($aH$) is the (comoving) distance over which particles can travel in the course of one expansion time. It is therefore another way of measuring whether particles are causally connected with each other: comparing the comoving separation of two particles with ($aH$) determines whether the particles can communicate with each other at a given moment (i.e. within the next Hubble time). This makes it clear that $ph$ and ($aH$) are conceptually very different: the expansion time, $t_H=\frac{dt}{d\ln a}$, is roughly the time in which the scale factor doubles.