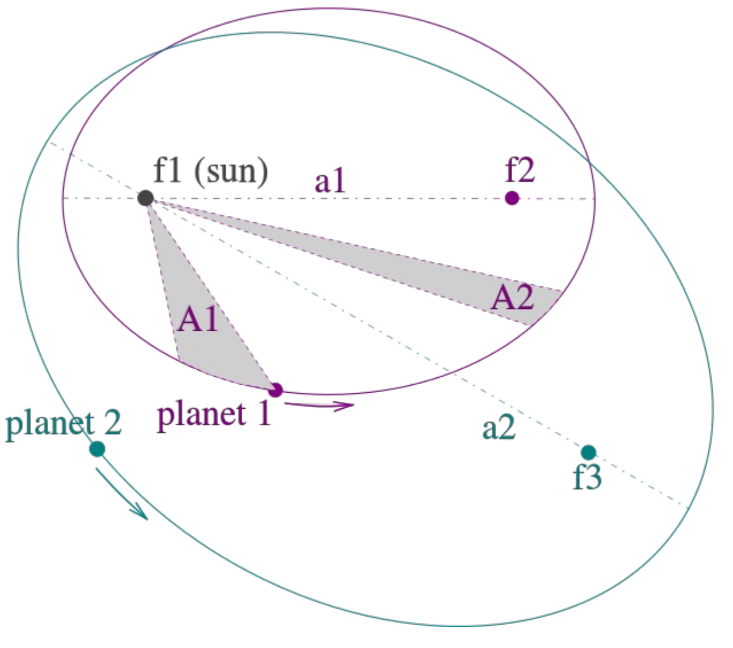
Unit 5

Differential Equations

- **example:** **Kepler's laws vs. Newton's gravity theory**

- ***Kepler's laws*** (= empirically derived description of physical phenomenon)
  - ✓ orbits of a planet is an ellipse with the Sun at one of the two focii
  - ✓ line segment joining planet & Sun sweeps out equal areas in equal times
  - ✓ $T^2 \sim a^3$ (T: orbital period, a: semi-major axis)



- ***Newton's theory*** (= mathematical explanation of physical phenomenon)

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = -G\frac{m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2)$$

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = -m_2 \frac{d^2\vec{r}_2}{dt^2}$$

differential equation:
describes how a quantity *changes*
(as opposed to simply stating some relation between quantities)

⇒ all of Kepler's law can be easily derived from Newton's theory!

Differential Equations *definitions*

- **ordinary differential equation:**

an equation of the following type

$$0 = \mathcal{G}(f^{(n)}, f^{(n-1)}, \ldots, f', f, x)$$

is called n^{th}-order ordinary differential equation (ODE) as it involves one independent variable x, a function $f(x)$, and all its derivatives up to the n^{th} order, i.e. $f'(x), f''(x), \ldots, f^{(n-1)}(x), f^{(n)}(x)$.

$\mathcal{G}(f^{(n)}, f^{(n-1)}, \ldots, f', f, x)$ is some arbitrary function of x, $f(x)$ and all derivatives of $f(x)$ up to the n^{th} order, e.g. movement in gravity field: $0 = G M / r^2 + r'' = \mathcal{G}(r'', r', r, t)$

- **Note:**
  - $f^{(n)}$ is an abbreviation for $d^n f/dx^n$
  - the order of the differential equation is determined by the highest derivative
  - the differential equation is "ordinary" because there is only one independent variable x
  - in most cases we can re-write the ODE as an ***explicit ordinary differential equation:***

$$\frac{d^n f}{dx^n} = g(f^{(n-1)}, \ldots, f', f, x)$$

- **example:**
  - the equation of motion for a particle m in a gravity field created by a constant mass M

$$m\frac{d^2 r(t)}{dt^2} = -G\frac{mM}{r^2(t)} \quad \Leftrightarrow \quad \frac{d^2 f(x)}{dx^2} = g(f', f, x) \quad \text{with} \quad g(f', f, x) = -G\frac{M}{f^2}$$

is a second order explicit ordinary differential equation.

Differential Equations *definitions*

▪ **Note:**

- a differential equation tells you how a certain quantity *changes*, i.e. the equation of motion for a particle m tells you how its position $r(t)$ changes as a function of time t
- practically every theory in physics is based upon differential equations!

▪ **Important:**

- a n^{th}-order explicit ordinary differential equation can be written as a system of n 1st-order explicit ordinary differential equations, e.g. the equation of motion for a particle in a gravity field:

2nd order differential equation		system of two 1st order differential equations
$\frac{d^2r(t)}{dt^2} = -G\frac{M}{r^2(t)}$	$\Leftrightarrow$	$\frac{dr}{dt} = v$ $\frac{dv}{dt} = -G\frac{M}{r^2}$

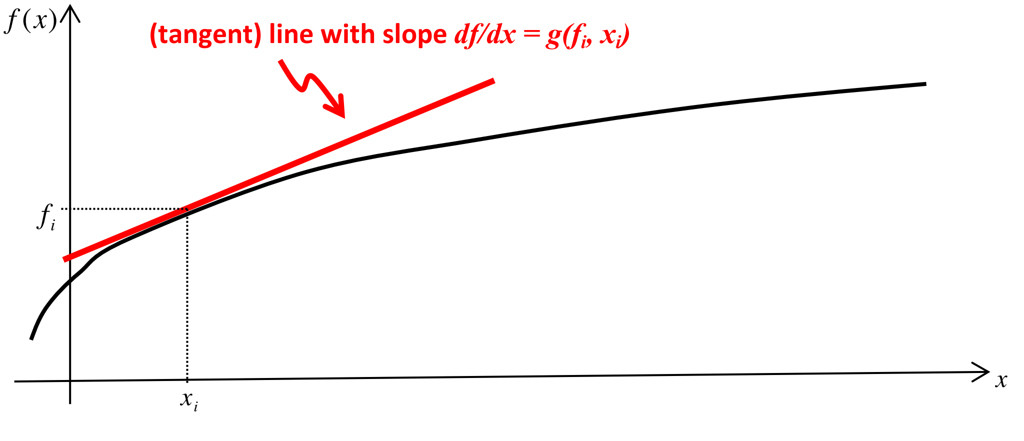
→ *we can focus on numerical techniques for solving 1st-order ordinary differential equations!*

geometrical interpretation

▪ **explicit 1st-order ordinary differential equation:**

$$\frac{df(x)}{dx} = g(f,x)$$

▪ $g(f, x)$ only tells us the slope of the tangent to $f(x)$ at a given point x:

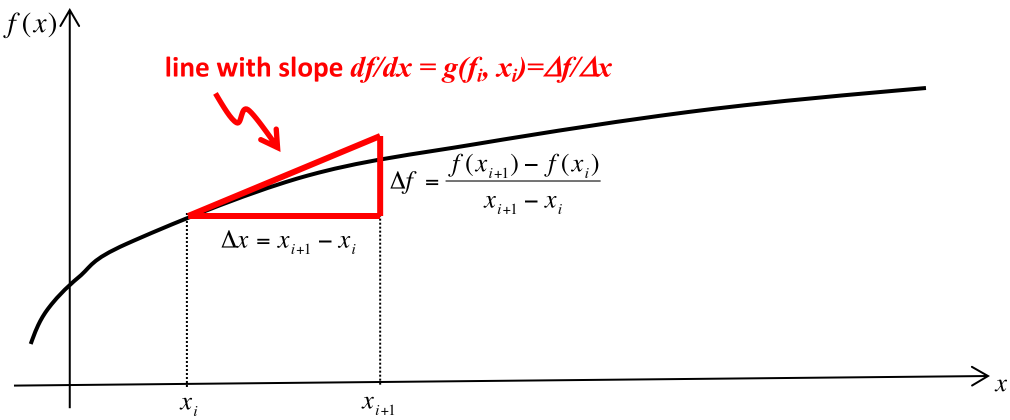


Differential Equations *Euler methods*

- **solving explicit 1st-order ordinary differential equations**

$$\frac{df(x)}{dx} = g(f, x)$$

- we construct f_{i+1} using a straight line starting in (x_i, f_i)



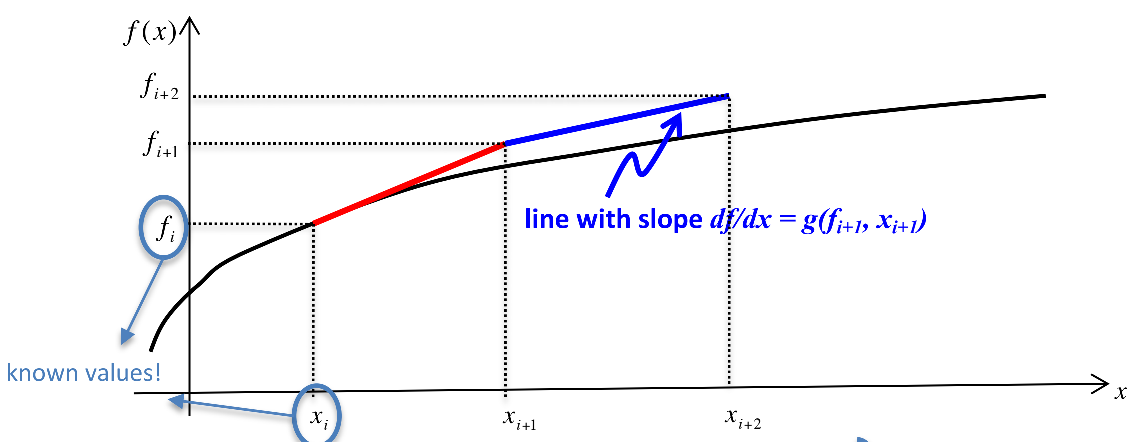
$$\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = g(f_i, x_i) \quad \Leftrightarrow \quad f(x_{i+1}) = \underbrace{f(x_i)}_{\text{needs to be known}} + \underbrace{g(f_i, x_i)}_{\text{ODE}} \underbrace{(x_{i+1} - x_i)}_{\text{personal choice}}$$

ODE

needs to be known

personal choice

- we can use that knowledge to construct the solution piece-by-piece, i.e. in small increments of Δx:



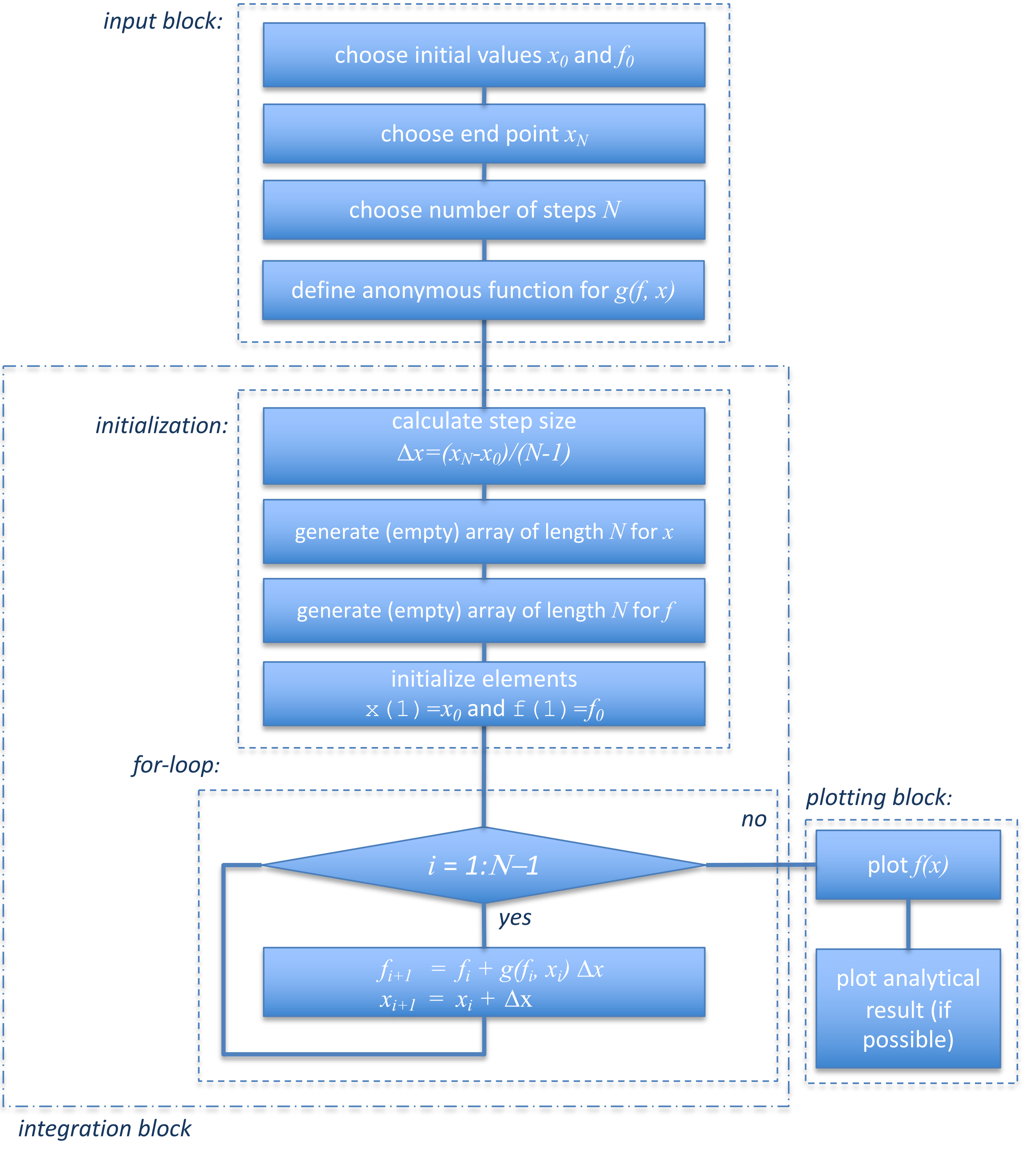
$$\left.\begin{aligned}
f(x_{i+1}) &= f(x_i) + g(f_i, x_i)(x_{i+1} - x_i) \\
f(x_{i+2}) &= f(x_{i+1}) + g(f_{i+1}, x_{i+1})(x_{i+2} - x_{i+1}) \\
&\dots \\
f(x_N) &= f(x_{N-1}) + g(f_{N-1}, x_{N-1})(x_N - x_{N-1})
\end{aligned}\right\} \text{Euler method}$$

Differential Equations *Euler methods*

▪ **solving explicit 1st-order ordinary differential equations via the Euler method**

$$\frac{df(x)}{dx} = g(f,x) \quad \Rightarrow \quad f(x_{i+1}) = f(x_i) + g(f_i,x_i)(x_{i+1} - x_i)$$

▪ flowchart for the numerical integration of a differential equation via the Euler method:

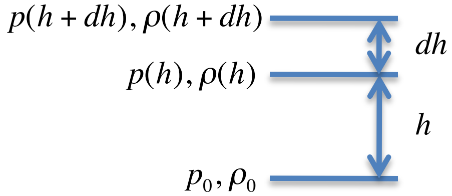


Differential Equations *Euler methods*

- **application: *barometric pressure***

  • the barometric pressure is given by the following ordinary differential equation

$$\frac{dp}{dh} = -G_E \frac{\rho_0}{p_0} p$$



…where $p(h)$ is the pressure at height h, G_E the Earth's gravitational acceleration and p_0, ρ_0 the pressure and density at sea level.

• Note:

  • the derivation of this equation is based upon $dp = -\rho\, G_E\, dh$ in combination with the Boyle-Mariotte law $p/\rho = p_0/\rho_0$ and the assumption that the Earth's gravity G_E is constant for the considered heights h.

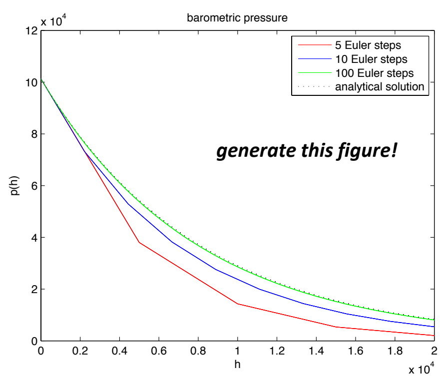
➢ **exercise:**

• use the Euler method to integrate the equation for the barometric pressure
• plot both the numerical and the analytical result in the same figure

• approach:
  • use $H_0=p_0/(\rho_0 G_E)$=8005m in your formulae
  • use p_0=101325pascals at height h_0=0m as your initial values
  • integrate up to h_N=20000m in 5, 10, and 100 integration steps, respectively
  • use an anonymous function `g=@(p,h)(-p/H0)`
  • use a function `euler()` for the integration block:

`euler.m`:

```
function [p,h] = euler(g, h0, p0, hN, N)
% initialisation block:
…
% integration block:
…
end
```

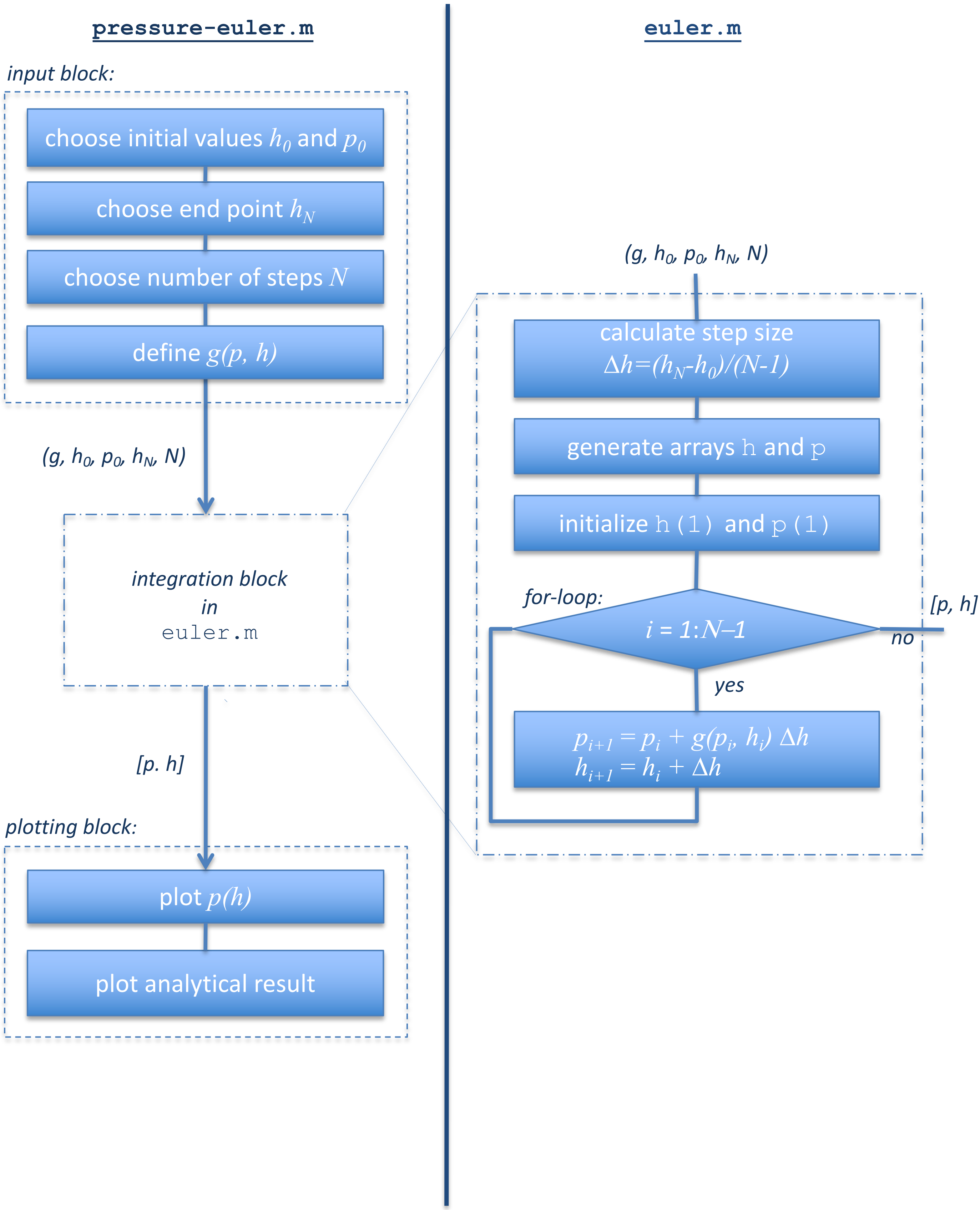


• Note:
  • `euler()` takes as arguments the anonymous function g, the initial values h_0 and p_0, the end point of the integration h_N and the number of integration steps N
  • `euler()` returns the two vectors `p` and `h` containing
    `h` = N values of h from h_0 to h_N
    `p` = the solution of the differential equation, i.e. $p(h_0)$ to $p(h_N)$
• you need to calculate the analytical solution using paper-and-pen

flowchart on the following page ➔

Differential Equations *Euler methods*

- **application:** ***barometric pressure – flowchart for exercise***



Differential Equations *Euler methods*

- the Euler method only uses $g(f,x)$ in x_i to construct a piece-by-piece solution

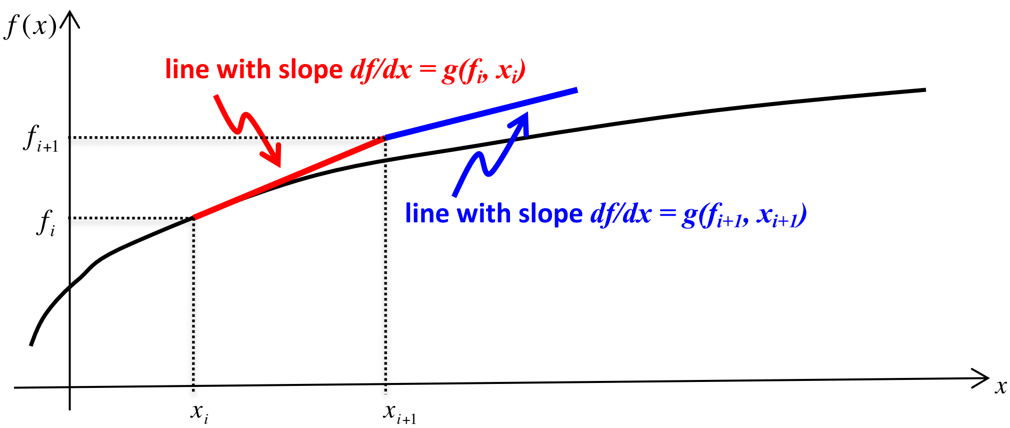
$$f(x_{i+1}) = f(x_i) + g(f_i,x_i)(x_{i+1} - x_i) \qquad \textit{Euler method}$$

- to obtain better accuracy of the solution we can perform a "trial step":

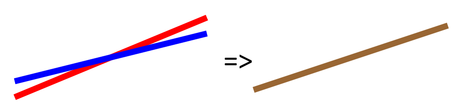
$$f(x_{i+1}) = f(x_i) + \frac{g(f_i,x_i) + g(f_{i+1},x_{i+1})}{2}(x_{i+1} - x_i) \qquad \textit{modified Euler method}$$

- Note:
  - to be able to calculate $g(f_{i+1}, x_{i+1})$ we need to perform a standard Euler step first!
  - for the actual integration step we then used the average slope $(g_i+g_{i+1})/2$

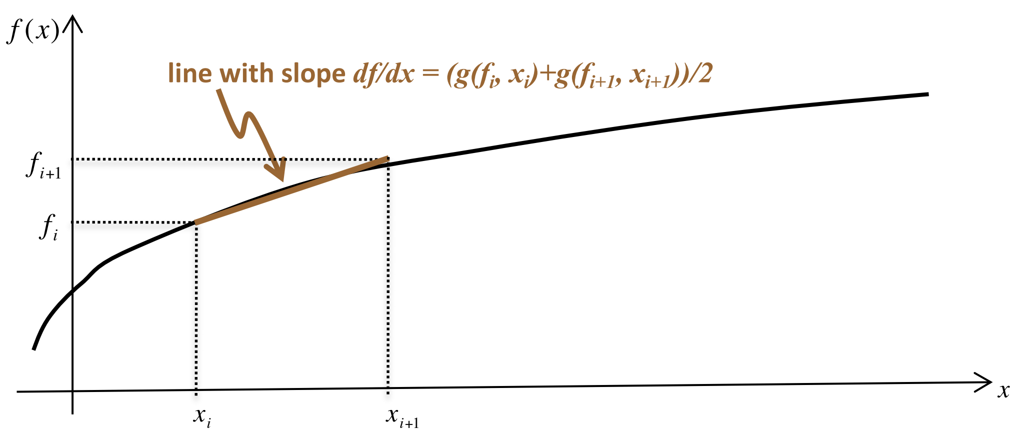
1. trial step (just to obtain g_{i+1}):



slope averaging:



2. actual integration step:



Differential Equations *Euler methods*

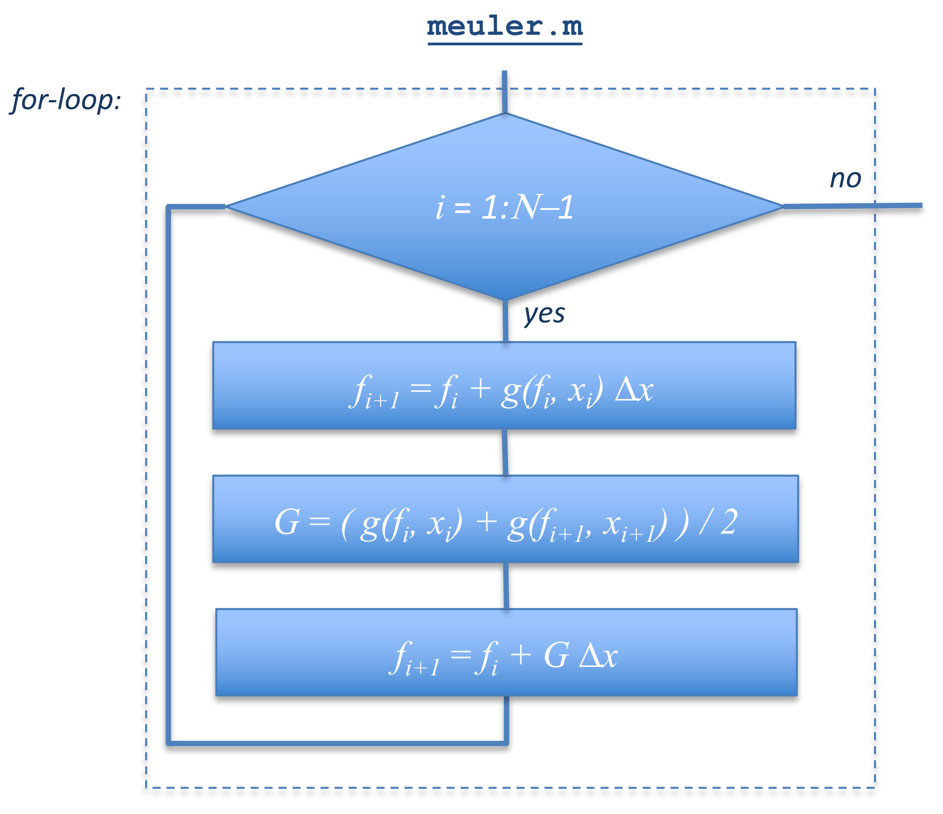
- the Euler method only uses $g(f,x)$ in x_i to construct a piece-by-piece solution

$$f(x_{i+1}) = f(x_i) + g(f_i,x_i)(x_{i+1} - x_i) \qquad \textit{Euler method}$$

- to obtain better accuracy of the solution we can perform a "trial step":

$$f(x_{i+1}) = f(x_i) + \frac{g(f_i,x_i) + g(f_{i+1},x_{i+1})}{2}(x_{i+1} - x_i) \qquad \textit{modified Euler method}$$

- flowchart for modified Euler method (showing only the for-loop!):

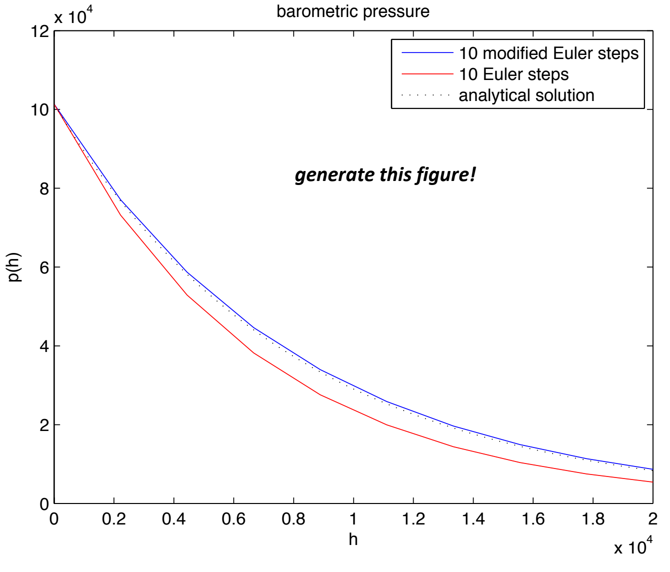


Differential Equations *Euler methods*

➢ **exercise:**

• use the modified Euler method to integrate the equation for the barometric pressure

• plot the numerical result as derived by the modified and the standard Euler method alongside the analytical result in the same figure:



• approach:

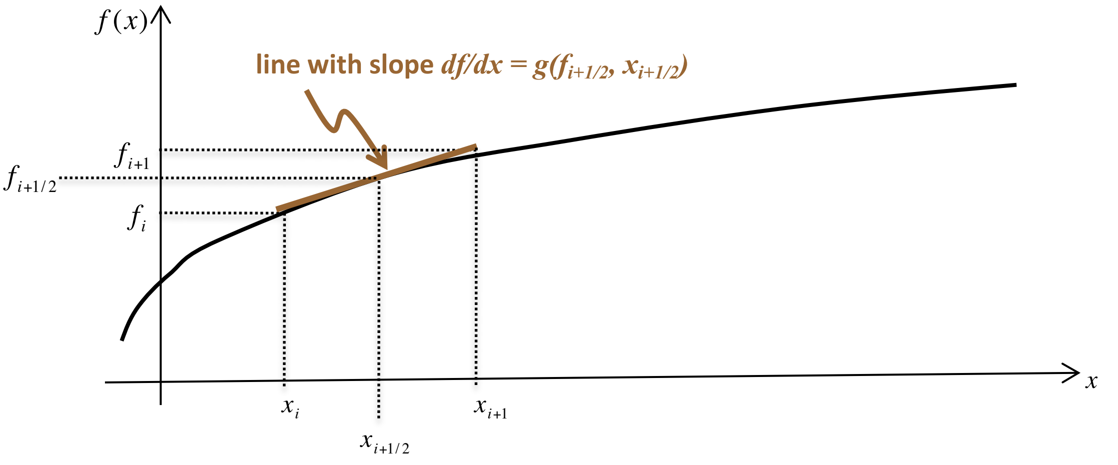
• simply generate a new script `meuler.m` containing the function `meuler()` by modifying your script `euler.m`

- 2^{nd} order Runge-Kutta scheme:

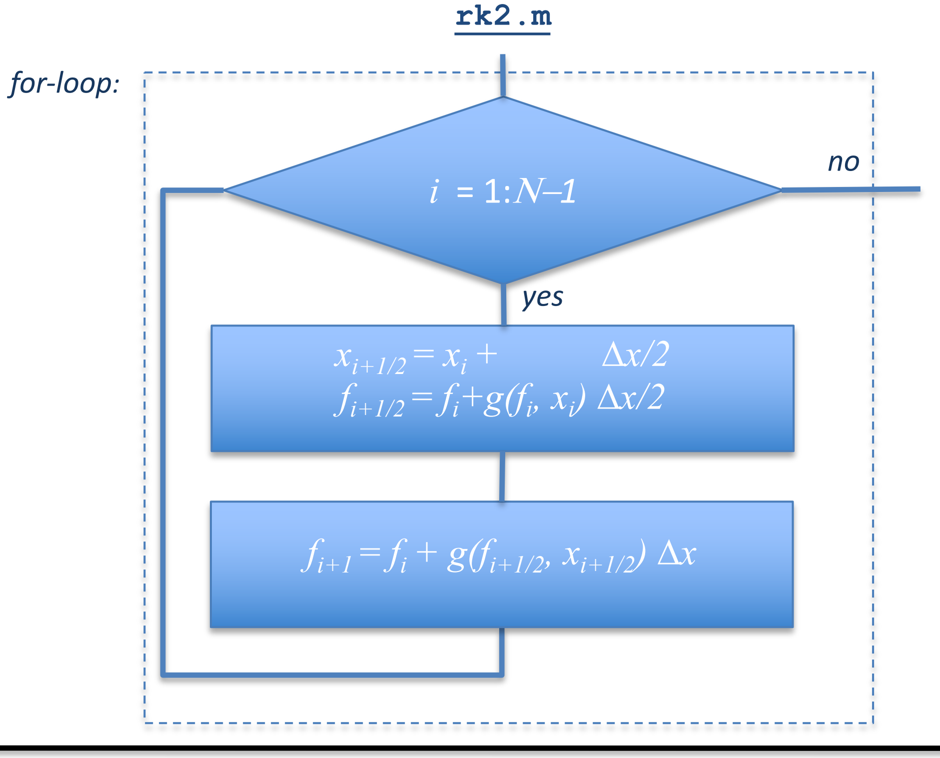
$$f(x_{i+1}) = f(x_i) + g(f_{i+1/2}, x_{i+1/2})(x_{i+1} - x_i)$$

- Note: "2^{nd} order" refers to the accuracy of the scheme

- geometrical interpretation:



- flowchart for the 2^{nd} order Runge-Kutta scheme (showing only the for-loop!):



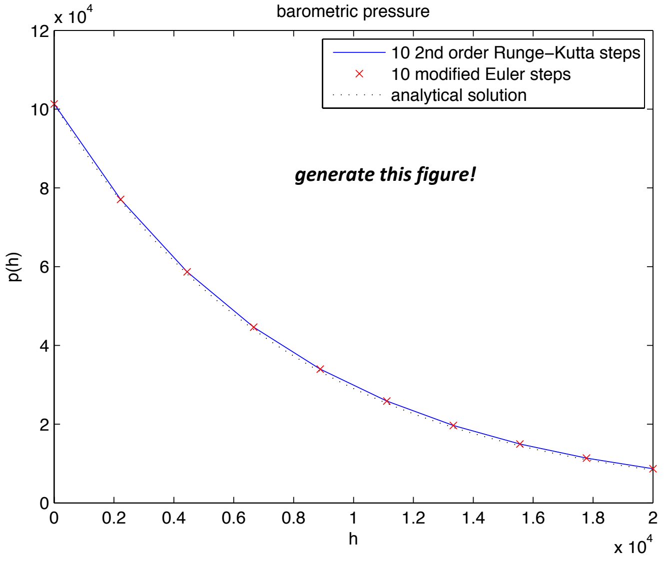
▪ 2^{nd} order Runge-Kutta scheme:

$$f(x_{i+1}) = f(x_i) + g(f_{i+1/2}, x_{i+1/2})(x_{i+1} - x_i)$$

➢ **exercise:**

• use the 2^{nd} order Runge-Kutta method to solve the equation for the barometric pressure

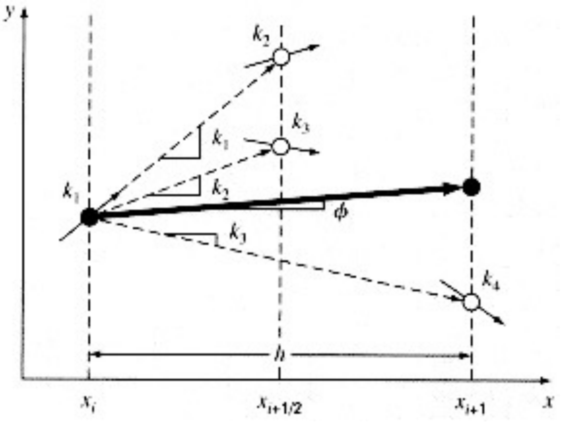
• plot the numerical result as derived by the Runge-Kutta and the modified Euler scheme alongside the analytical result in the same figure:



- 4th order Runge-Kutta scheme:

$$f(x_{i+1}) = f(x_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)(x_{i+1} - x_i)$$

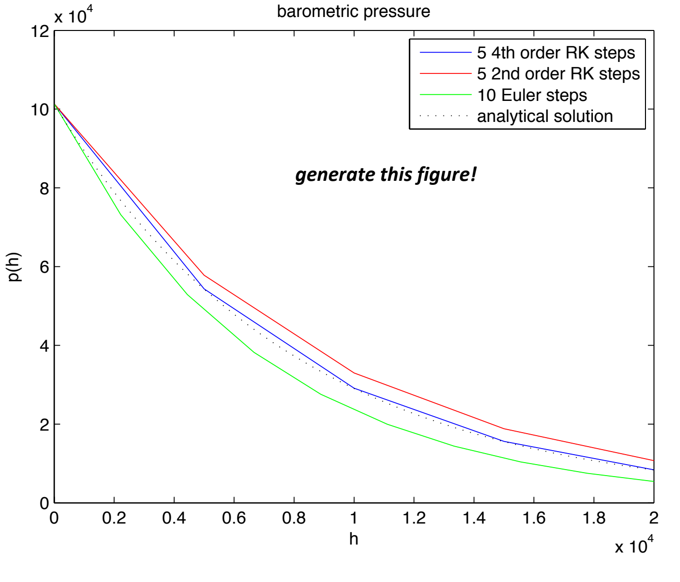
$$k_1 = g\left(f_i, x_i\right)$$
$$k_2 = g\left(f_i + k_1\frac{(x_{i+1} - x_i)}{2}, x_i + \frac{(x_{i+1} - x_i)}{2}\right)$$
$$k_3 = g\left(f_i + k_2\frac{(x_{i+1} - x_i)}{2}, x_i + \frac{(x_{i+1} - x_i)}{2}\right)$$
$$k_4 = g\left(f_i + k_3(x_{i+1} - x_i), x_{i+1}\right)$$



➢ **exercise:**

- use the 4th order Runge-Kutta method to solve the equation for the barometric pressure

- plot the numerical result as derived by the 4th and 2nd order Runge-Kutta scheme and the standard Euler method alongside the analytical result in the same figure:



Differential Equations *summary*

▪ **solving an explicit 1st order explicit ordinary differential equation:**

$$\frac{df}{dx} = g(f,x)$$

- Euler method

$$f(x_{i+1}) = f(x_i) + g(f_i,x_i)(x_{i+1} - x_i)$$

- modified Euler method

$$f(x_{i+1}) = f(x_i) + \frac{g(f_i,x_i) + g(f_{i+1},x_{i+1})}{2}(x_{i+1} - x_i)$$

- 2nd order Runge-Kutta method

$$f(x_{i+1}) = f(x_i) + g(f_{i+1/2},x_{i+1/2})(x_{i+1} - x_i)$$

- 4th order Runge-Kutta scheme

$$f(x_{i+1}) = f(x_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)(x_{i+1} - x_i)$$

$$k_1 = g\left(f_i,x_i\right)$$

$$k_2 = g\left(f_i + k_1\frac{(x_{i+1} - x_i)}{2}, x_i + \frac{(x_{i+1} - x_i)}{2}\right)$$

$$k_3 = g\left(f_i + k_2\frac{(x_{i+1} - x_i)}{2}, x_i + \frac{(x_{i+1} - x_i)}{2}\right)$$

$$k_4 = g\left(f_i + k_3(x_{i+1} - x_i), x_{i+1}\right)$$

Systems of Differential Equations

- without loss of generality we will restrict ourselves to an explicit 2nd order ODE

$$\frac{d^2 f}{dx^2} = h\left(f, \frac{df}{dx}, x\right)$$

- we can split this 2nd order ODE into a coupled system of two 1st order ODEs:

$$\frac{df}{dx} = g(f,\phi,x) = \phi$$
$$\frac{d\phi}{dx} = h(f,\phi,x)$$

- solving this system only requires minimal modifications to our previous scheme(s):

$$f(x_{i+1}) = f(x_i) + G\,(x_{i+1} - x_i)$$
$$\phi(x_{i+1}) = \phi(x_i) + H\,(x_{i+1} - x_i)$$

• Euler method

$$G = g(f_i,\phi_i,x_i)$$
$$H = h(f_i,\phi_i,x_i)$$

• modified Euler method

$$G = \frac{g(f_i,\phi_i,x_i) + g(f_{i+1},\phi_{i+1},x_{i+1})}{2}$$
$$H = \frac{h(f_i,\phi_i,x_i) + h(f_{i+1},\phi_{i+1},x_{i+1})}{2}$$

• 2nd order Runge-Kutta method

$$G = g(f_{i+1/2},\phi_{i+1/2},x_{i+1/2})$$
$$H = h(f_{i+1/2},\phi_{i+1/2},x_{i+1/2})$$

- ***Note***:
  - x_{i+1}, f_{i+1} and ϕ_{i+1} in the formulae for *G()* and *H()* are (again) only temporary values and must be calculated **before** being used in *G()* and *H()*!
  - the same is obviously true for $x_{i+1/2}$, $f_{i+1/2}$ and $\phi_{i+1/2}$

Systems of Differential Equations *harmonic oscillator*

- the differential equation for a harmonic oscillator (e.g. a spring) reads as follows

$$\ddot{x} + \frac{k}{m} x = 0$$

...where k is the "spring constant" and m the mass attached to the spring.

- split this 2nd order ODE into a coupled system of two 1st order ODEs:

$$\ddot{x} + \frac{k}{m} x = 0 \quad \Rightarrow \quad \begin{aligned} \dot{x} &= g(x,v,t) = v \\ \dot{v} &= h(x,v,t) = -\frac{k}{m} x \end{aligned}$$

- ***Note***: the analytical solution for this particular equation is:

$$x(t) = A \sin(\omega t + \varphi)$$

with:

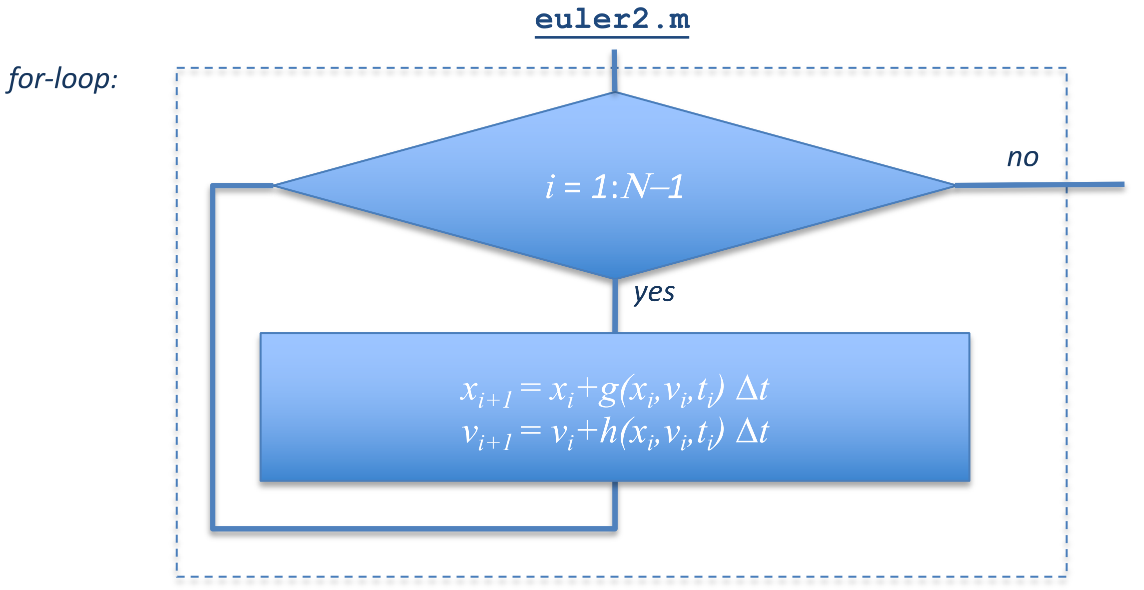
$$\omega = \sqrt{k/m}$$

$$A = \sqrt{x_0^2 + (v_0/\omega)^2}$$

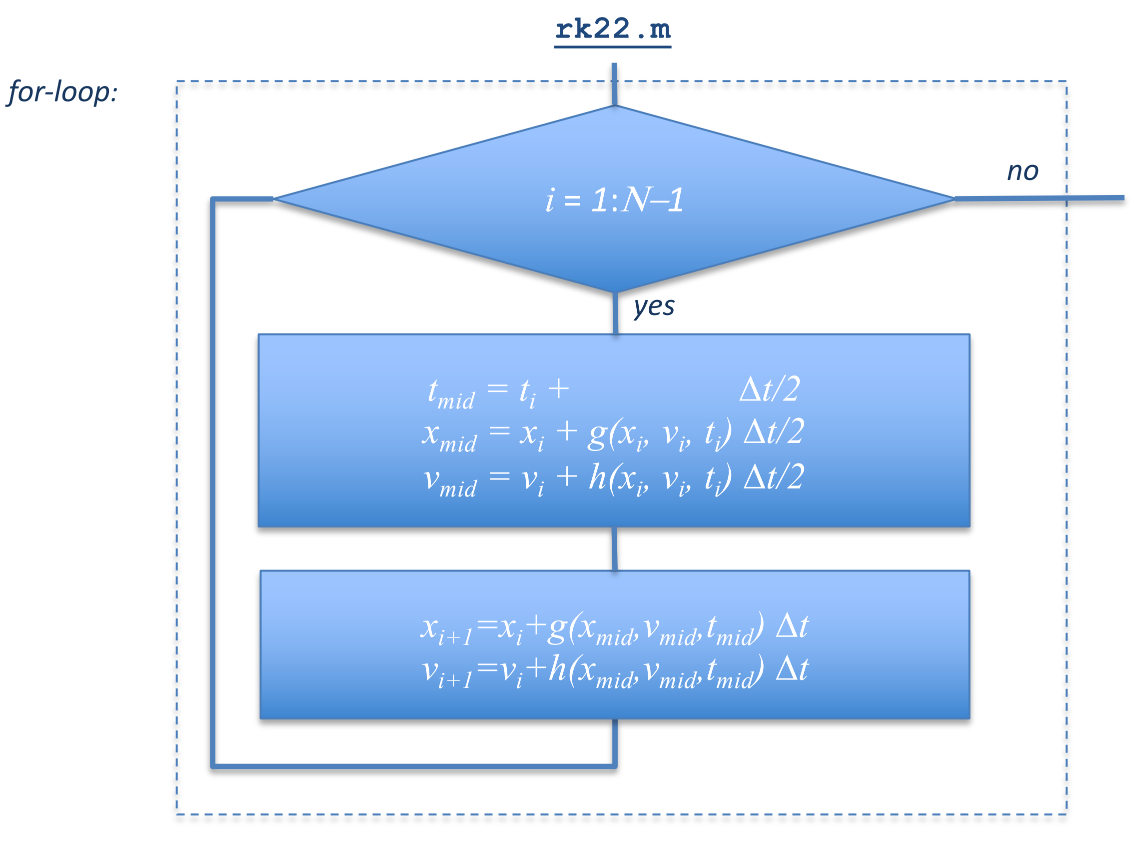
$$\varphi = \mathrm{atan}(\omega\, x_0 / v_0)$$

Systems of Differential Equations *harmonic oscillator*

- standard Euler method:



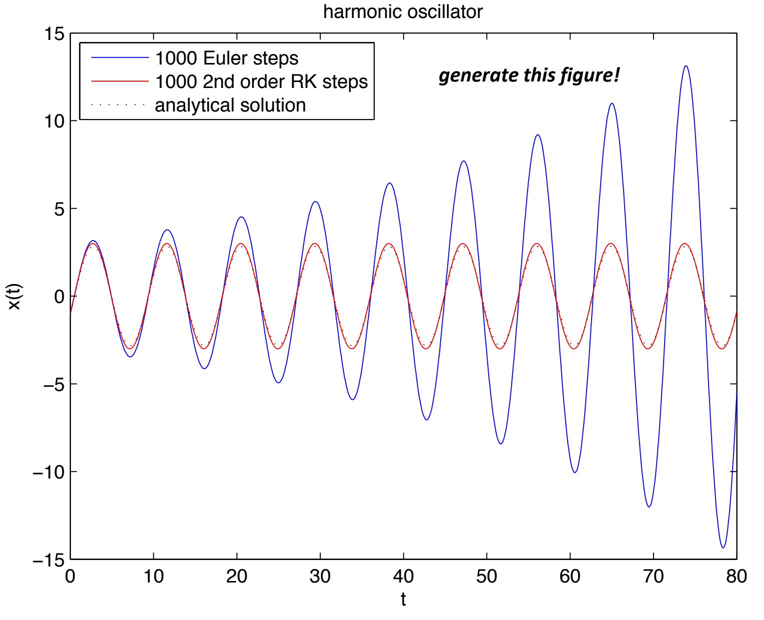
- 2nd order Runge-Kutta method:



Systems of Differential Equations *harmonic oscillator*

➢ **exercise:**

• consider a spring with the following specifications: k=1kg/s^2 and m=2kg
• use the initial conditions t_0=0s, x_0=-1m, v_0=2m/s
• solve the differential equation for the harmonic oscillator using the standard Euler method and the 2nd order Runge-Kutta scheme
• plot the numerical solutions alongside the analytical solution up to the end point t_N=80s:



• approach:

• you now have to define two anonymous functions

```
g = @(x,v,t)(v);
h = @(x,v,t)(-k/m * x);
```

• both these functions must be passed to your integration function, e.g. `euler2()`

`euler2.m`:

```
function [x,v,t] = euler2(g, h, t0, x0, v0, tN, N)
% initialisation block:
…
% integration block:
…
end
```

• Note:

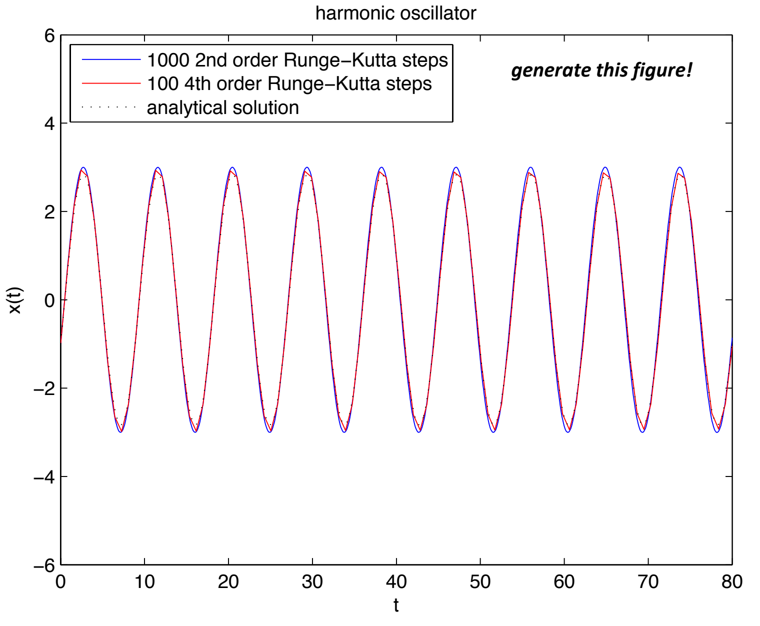
• your integration function now returns the vectors `x`, `v`, and `t`

• the solution becomes numerically instable when choosing too few integration steps as can be seen for the Euler method

Systems of Differential Equations *harmonic oscillator*

➢ **exercise:**

- consider a spring with the following specifications: k=1kg/s^2 and m=2kg
- use the initial conditions t_0=0s, x_0=-1m, v_0=2m/s
- solve the differential equation for the harmonic oscillator using the 4th order Runge-Kutta and the 2nd order Runge-Kutta scheme
- plot the numerical solutions alongside the analytical solution up to the end point t_N=80s:



▪ Note:

- this is a **highly advanced** exercise as it is **not simple** to write the formulae for the predictor steps!
- if you plan to do this exercise remember the way the predictor steps have been done in the case of the modified Euler or the 2nd order Runge-Kuta scheme method: one first makes a predictor step in both variables before doing the second predictor step and so on…

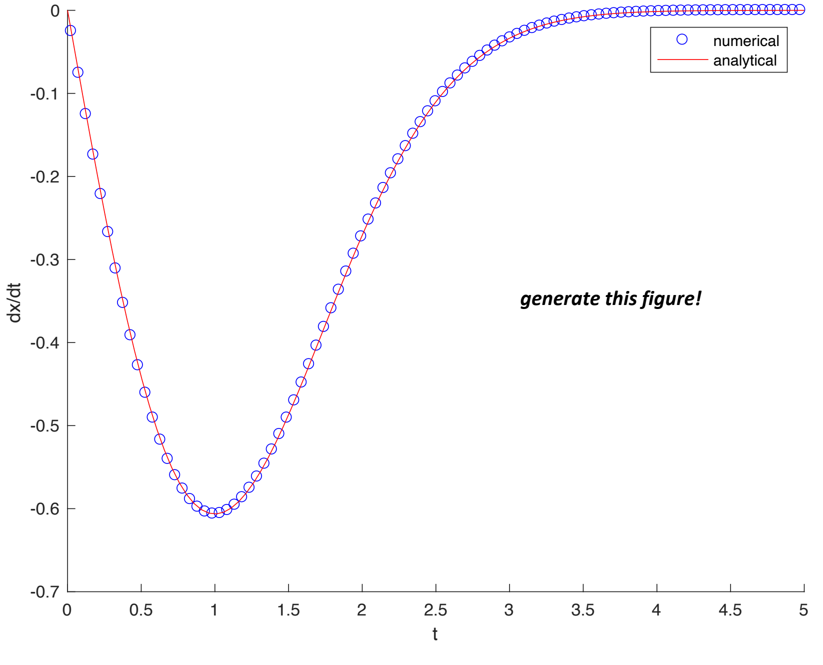
Applications *decay equation*

- consider the following ordinary differential equation

$$\frac{dx}{dt} = -xt$$

➢ **exercise:**

- solve the equation on the interval t = [0,5] using t_0=0, $x(t_0)$=1
- numerically differentiate the solution $x(t)$ with respect to t.
- compare the analytical function for dx/dt against the numerically derived one.



▪ the number of decay events of unstable radioactive nuclei is proportional to the number of nuclei present in a given sample

$$\frac{dN}{dt} = -\lambda N$$

where N is the number of nuclei, and the decay constant λ depends on the type of (radioactive) nuclei.

▪ example:

for ^{232}U the half-life time is 68.9 days giving a decay constant of λ=ln(2)/68.9days=0.01006/day

➢ **exercise:**

• numerically solve the decay equation for ^{232}U using the 2nd order Runge-Kutta scheme.

• determine the minimum number of integration steps needed to obtain a numerical solution close to the correct analytical one for the following setup:

t_0 = 0 years, $N(t_0)$ = 5 million nuclei,

t_{end} = 4 years, $N^{analytical}(t_{end})$ = 2.09 nulcei

• use a while-loop to determine the required number of integration points

hints:

• when determining the number of integration steps to match the correct analytical solution, it is sufficient to achieve an absolute accuracy of 0.1 nuclei:

• when using a while-loop, you gradually need to increase the number of integration step inside the loop.

• do not mix days and years but use a unique time unit

more →

➢ **exercise:**

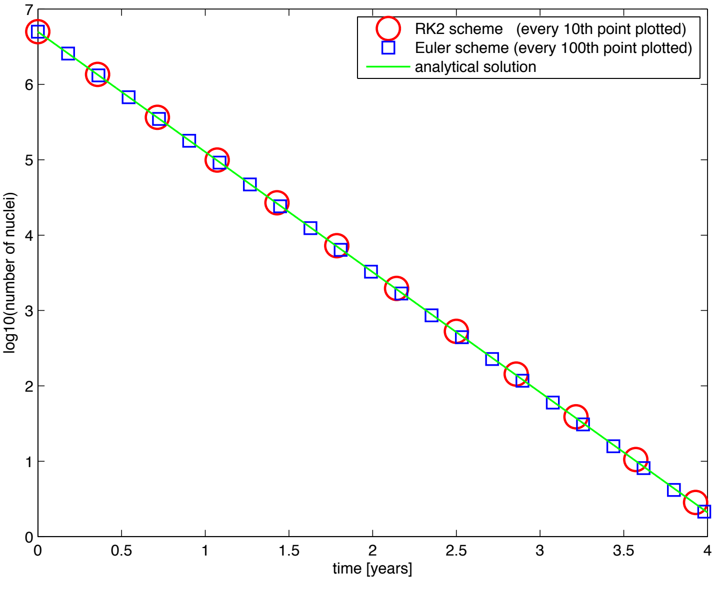
• numerically solve the decay equation using the Euler scheme.

• how many integration steps do you require to obtain a numerical solution close to the analytical solution when using the Euler scheme? An absolute accuracy of 0.1 nuclei is sufficient again.

➢ **exercise:**

• plot both numerical solutions into the same figure using a logarithmic y-axis.

• plot only every 10^{th} point for the RK2 solution and every 100^{th} for the Euler solution, i.e. generate a figure similar to the following one:



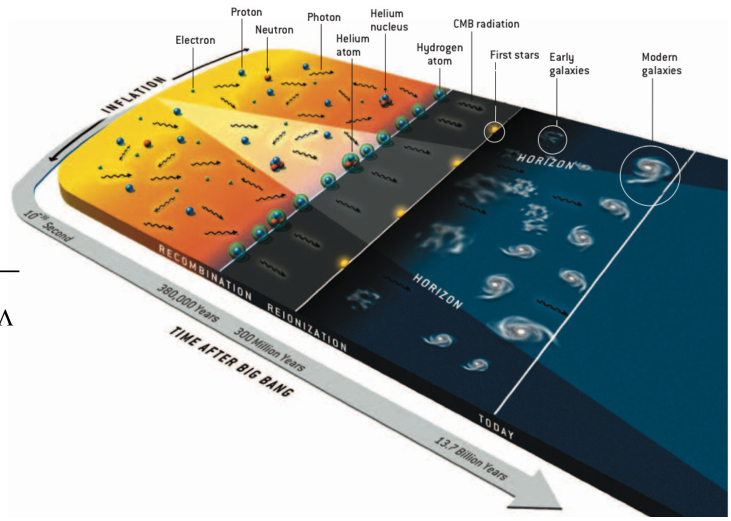
Notes:

• the reduction of the number of points for the numerical RK2 and Euler solution depends on the absolute accuracy chosen to match the analytical. The suggested values of 10^{th} and 100^{th} are best suited for an absolute accuracy of 0.1!

• the analytical solution is $N(t) = N_0 \, e^{-\lambda t}$

Applications *cosmic expansion*

$$\frac{da}{dt} = H_0 a\sqrt{\Omega_0 a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda}$$



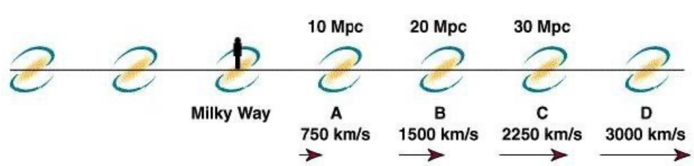
- **the expansion of the Universe is described by the *Friedmann equation* given above**

- **the constants in this equation have the following meaning…**

$$\begin{aligned} H_0 &= \text{expansion rate at present time} \\ \Omega_0 &= \text{matter content in the Universe} \\ \Omega_\Lambda &= \text{dark energy content in the Universe} \\ \Omega_k &= 1-(\Omega_0+\Omega_\Lambda) \;\; \text{curvature of space-time} \end{aligned}$$

- **…and the solution $a(t)$ describes how the Universe grows with time since the Big Bang**

- **H_0 can be obtained by measuring the recessional speed of distant galaxies…**



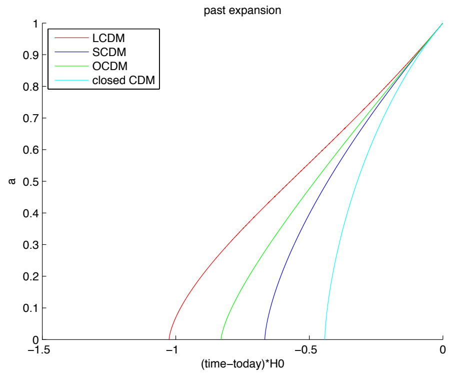
…and is found to be H_0=72km/sec/Mpc.

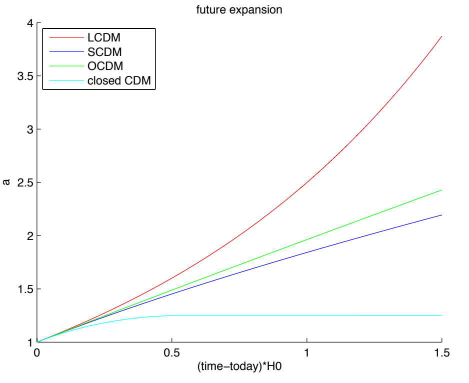
- **the *Friedmann equation* can be re-written as follows (by transforming t to $\tau = H_0 t$):**

$$\frac{da}{d\tau} = a\sqrt{\Omega_0 a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda}$$

Applications

cosmic expansion





$$\frac{da}{d\tau} = a\sqrt{\Omega_0 a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda}$$

➢ **exercise:**

- consider the following parameter sets…
  - LCDM model: $\Omega_0 = 0.28$, $\Omega_\Lambda = 0.72$ (the currently favoured model)
  - SCDM model: $\Omega_0 = 1.00$, $\Omega_\Lambda = 0.00$ (the old standard model)
  - OCDM model: $\Omega_0 = 0.24$, $\Omega_\Lambda = 0.00$ (an "open" model)
  - closed model: $\Omega_0 = 5.00$, $\Omega_\Lambda = 0.00$ (a "Big Crunch" model)
- using as initial values $t_0=0$, $a(t_0)=1$ …
- …study the past expansion by solving the *Friedmann equation* backwards in time
- …study the future expansion by solving the *Friedmann equation* forward in time

➢ **exercise:**

- why is the closed model not giving the expected recollapse in the future?
- why can't you start the integration at the Big Bang?

Applications *cannonball – numerically revisited*

We are reconsidering the cannonball exercise from Unit 1 again. So far, we have treated the solution in the dimensions x and y independently, but now the plan is to solve the coupled system of 2D 2nd order differential equations numerically, also adding a frictional force term:

$$\frac{d^2\vec{r}}{dt^2} = bv^2\frac{\vec{v}}{|v|} + \vec{g} \quad \Leftrightarrow \quad \begin{aligned} \frac{d\vec{r}}{dt} &= \vec{v} &&= g(\vec{r},\vec{v},t) \\ \frac{d\vec{v}}{dt} &= bv^2\frac{\vec{v}}{|v|} + \vec{g}_0 &&= h(\vec{r},\vec{v},t) \end{aligned}$$

where b is the frictional constant and $\vec{g}_0 = (0, -g_0), |\vec{g}_0| = 9.81\mathrm{m/s^2}$.

➢ **exercise #1:**

• Write a function "rk22_2D.m" that numerically integrates a 2nd order 2D ordinary differential equation using a 2nd order Runge-Kutta scheme. The function should work like this:

```
function [r,v,t] = rk22_2D(g,h,r0,v0,t0,tN,N)
  % input:
  % g: anonymous function for g(r,v,t)
  % h: anonymous function for h(r,v,t)
  % r0: 2D column vector with initial position
  % v0: 2D column vector with initial velocity
  % t0: integration start time
  % tN: integration end time
  % N: number of integration steps
  %
  % output:
  % r: 2xN matrix with positions
  % v: 2xN matrix with velocities
  % t: vector of length N with time
```

more ->

Applications *cannonball – numerically revisited*

We are reconsidering the cannonball exercise from Unit 1 again. So far, we have treated the solution in the dimensions x and y independently, but now the plan is to solve the coupled system of 2D 2nd order differential equations numerically, also adding a frictional force term:

$$\frac{d^2\vec{r}}{dt^2} = bv^2\frac{\vec{v}}{|v|} + \vec{g} \quad \Leftrightarrow \quad \begin{aligned} \frac{d\vec{r}}{dt} &= \vec{v} &&= g(\vec{r},\vec{v},t) \\ \frac{d\vec{v}}{dt} &= bv^2\frac{\vec{v}}{|v|} + \vec{g}_0 &&= h(\vec{r},\vec{v},t) \end{aligned}$$

where b is the frictional constant and $\vec{g}_0 = (0, -g_0), |\vec{g}_0| = 9.81\mathrm{m/s^2}$.

➢ **exercise #2:**

- Use rk22_2D() to solve the equations of motion for the following conditions:

  x_0 = 3m, y_0 = 2m, v_0 = 10 m/s, α = 23° , g=9.81 kg m/s^2, β=-0.05

  Show the analytical solution (without friction!) in the same plot.

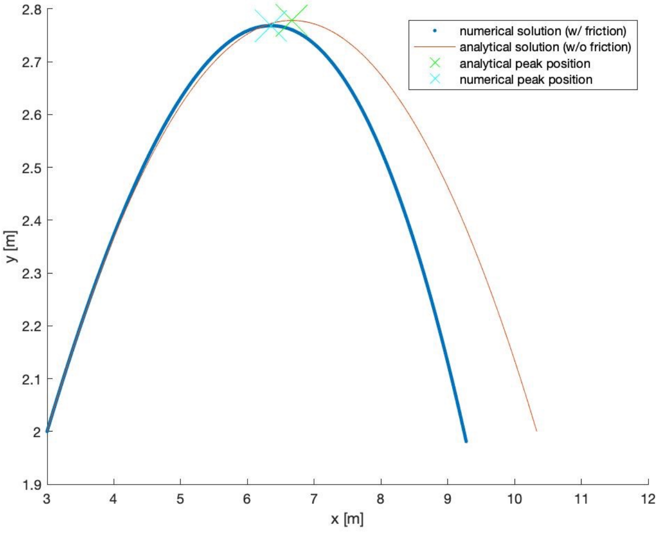
➢ **exercise #3:**

- using the formula for the analytical solution $\vec{v}(t) = \vec{v}_0 + \vec{g}t$ calculate the peak height y_{max}=y(t_{max}) of the analytical solution via the condition $v_y(t_{max})$=0 using a bi-section method and mark it in the same plot using a large X.

➢ **exercise #4:**

- find the peak position of the numerically obtained solution and also mark it with a X.

the final plot should look similar to this →



$$\frac{df}{dx} = g(f,x)$$

$$f_{i+1} - f_i = \int_{x_i}^{x_{i+1}} g(f,x)dx$$

Simpson's rule for numerical integration

$$= \frac{\Delta x}{6}\left[g(f_i,x_i) + 4g(f_{i+1/2},x_{i+1/2}) + g(f_{i+1},x_{i+1})\right]$$

$$= \frac{\Delta x}{6}\left[g(f_i,x_i) + 2g(f_{i+1/2}^{(1)},x_{i+1/2}) + 2g(f_{i+1/2}^{(2)},x_{i+1/2}) + g(f_{i+1},x_{i+1})\right]$$

? ? ?

$$f_{i+1/2}^{(1)} = f_i + \frac{\Delta x}{2}k_1 \qquad , k_1 = g(f_i,x_i)$$

$$f_{i+1/2}^{(2)} = f_i + \frac{\Delta x}{2}k_2 \qquad , k_2 = g(f_{i+1/2}^{(1)},x_{i+1/2})$$

$$\tilde{f}_{i+1} = f_i + \Delta x\, k_3 \qquad , k_3 = g(f_{i+1/2}^{(2)},x_{i+1/2})$$

$$k_4 = g(\tilde{f}_{i+1},x_{i+1})$$

$$= \frac{\Delta x}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$f_{i+1} = f_i + \frac{\Delta x}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$