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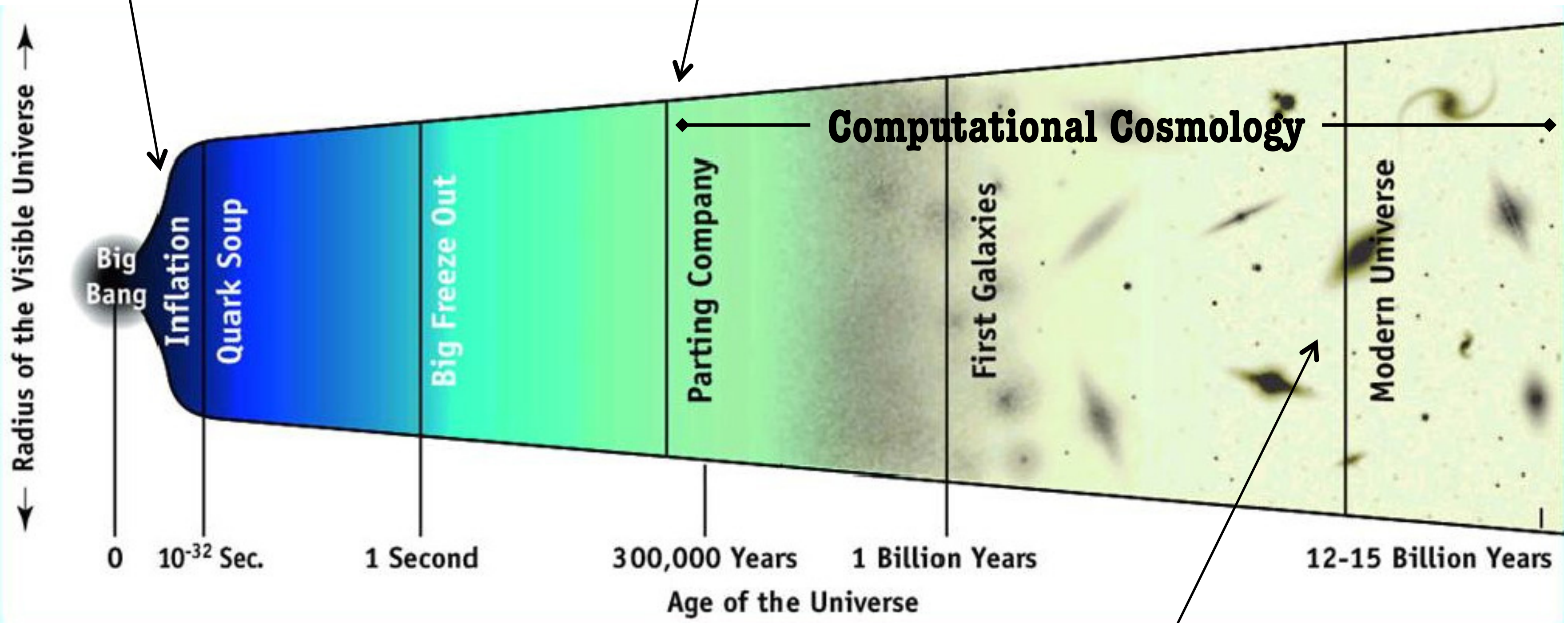
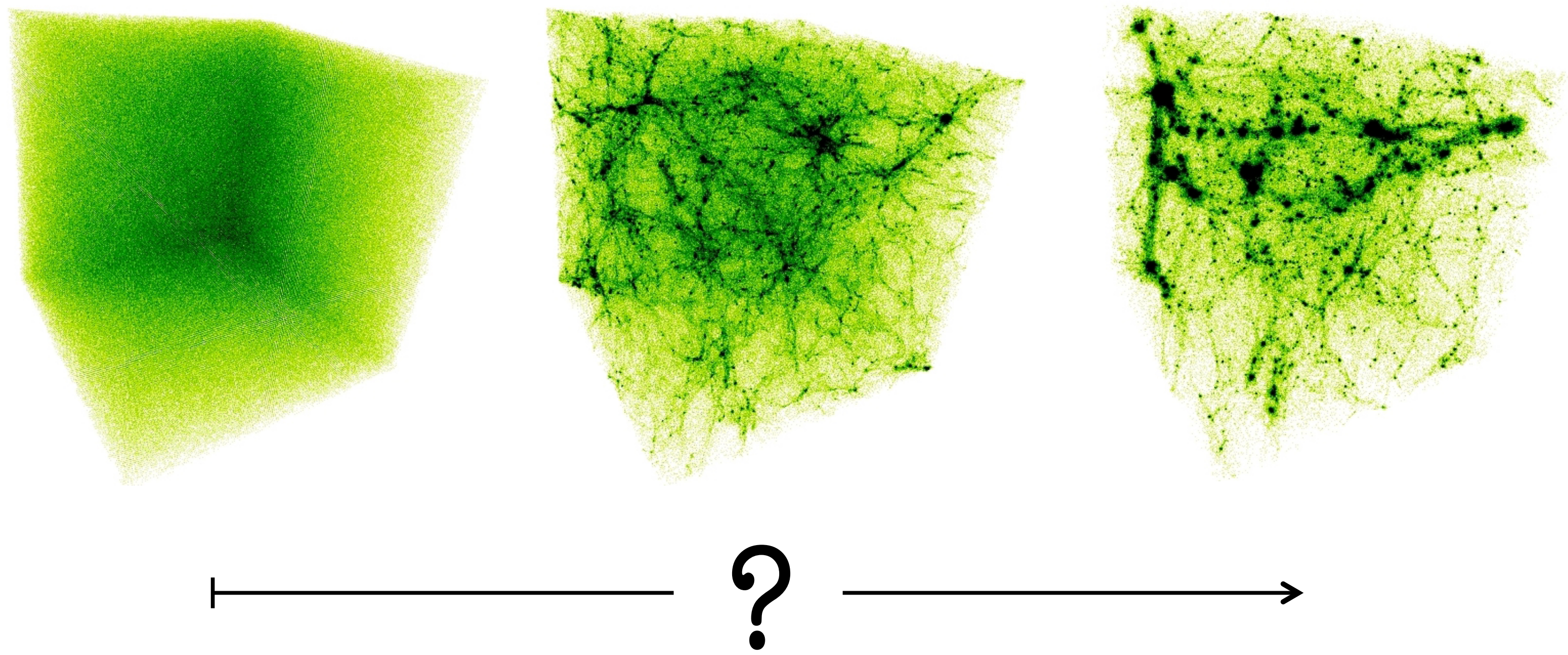
Cosmic Microwave Background Radiation

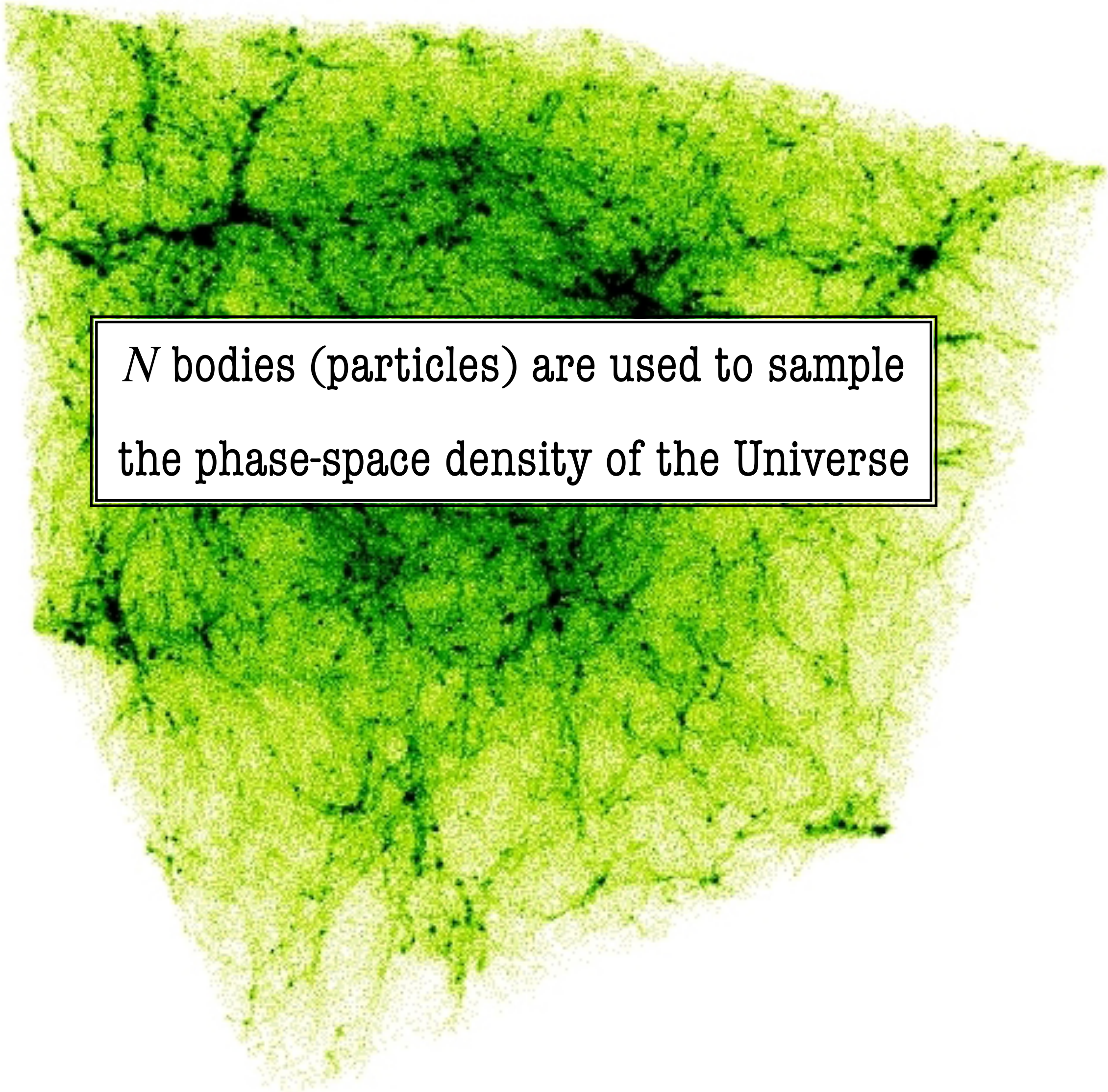
origin of seed inhomogeneities



(Cosmological) Structures: Galaxies and Galaxy Clusters

The Time Integration





N bodies (particles) are used to sample the phase-space density of the Universe

Following the trajectories of N particles
under their mutual gravity
(in expanding Universe)

Following the trajectories of N particles under their mutual gravity
(in expanding Universe)

- collisionless system of N-bodies
  - equations-of-motion

$$\frac{d\vec{r}}{dt} = \vec{v}$$
$$\frac{d\vec{v}}{dt} = -\nabla\Phi = \vec{F}(\vec{r},t)$$

  - the potential

$$\Delta\Phi = 4\pi G\rho$$

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$$\frac{d\vec{r}}{dt} = \vec{v}$$

$$\frac{d\vec{v}}{dt} = -\nabla\Phi = \vec{F}(\vec{r},t)$$

1. obtain force at each particle position

  - the potential

$$\Delta\Phi = 4\pi G\rho$$

Following the trajectories of N particles under their mutual gravity
(in expanding Universe)

- collisionless system of N-bodies
  - equations-of-motion

$$\frac{d\vec{r}}{dt} = \vec{v}$$
$$\frac{d\vec{v}}{dt} = -\nabla\Phi = \vec{F}(\vec{r},t)$$

  - the potential

$$\Delta\Phi = 4\pi G\rho$$

1. obtain force at each particle position
2. integrate equations of motion

Following the trajectories of N particles
under their mutual gravity
(in expanding Universe)

- collisionless system of N-bodies

  - equations-of-motion

$$\frac{d\vec{r}}{dt} = \vec{v}$$

$$\frac{d\vec{v}}{dt} = -\nabla\Phi = \vec{F}(\vec{r},t)$$

1. obtain force at each particle position
2. integrate equations of motion

in expanding Universe

  - the potential

$$\Delta\Phi = 4\pi G\rho$$

- equations of motion in expanding Universe

  - physical coordinates r and u

$$\vec{r},\ \vec{v} \xrightarrow{\text{equations of motion}} \dot{\vec{r}} = \vec{v}, \quad \dot{\vec{v}} = -\vec{\nabla}_r \varphi_r \qquad \Delta_r \varphi_r = 4\pi G(\rho_r - \rho_\Lambda)$$

  - comoving coordinates x and p

$$\vec{x} \quad \left(= \frac{\vec{r}}{a}\right), \quad \vec{p} \quad \left(= a(\vec{v} - H\vec{r})\right) \xrightarrow{\text{equations of motion}} \dot{\vec{x}} = ?, \quad \dot{\vec{p}} = ? \qquad \Delta_x \Phi_x = ?$$

$$\dot{\vec{x}} = ?$$
$$\dot{\vec{p}} = ?$$
$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe
  - avoiding Poisson's equation
  - using Poisson's equation
    - the easy way: Hamilton formalism
    - the hard way: re-writing the Newtonian equations of motion

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe

    - avoiding Poisson's equation...

...let's just do the transformation to comoving coordinates

$\dot{\vec{x}} = ?$

$\dot{\vec{u}} = ?$

$\vec{f} = ?$

- equations of motion in expanding Universe

  - physical coordinates – no expansion

$$\vec{r}(t), \quad \vec{v}(t)$$

$$\dot{\vec{r}} = \vec{v}$$

$$\dot{\vec{v}} = \vec{F}$$

with: $$\vec{F}_i = -G \sum_{i \neq j}^{N} \frac{m_j}{\left|\vec{r}_i - \vec{r}_j\right|^3} (\vec{r}_i - \vec{r}_j)$$

(F_i = specific force)

$\dot{\vec{x}} = ?$

$\dot{\vec{u}} = ?$

$\vec{f} = ?$

- equations of motion in expanding Universe

  - physical coordinates – with expansion

time-dependent transformation of coordinates!

$$\vec{r} = a\vec{x}$$

$$\dot{\vec{r}} = \vec{v} = a\dot{\vec{x}} + \dot{a}\vec{x} = \vec{u} + \dot{a}\vec{x}$$

$$\dot{\vec{v}} = \dot{\vec{u}} + \dot{a}\dot{\vec{x}} + \ddot{a}\vec{x} = \dot{\vec{u}} + H\vec{u} + \ddot{a}\vec{x} = \vec{F}$$

with: $$\vec{F}_i = -G\frac{1}{a^2}\sum_{i\neq j}^{N}\frac{m_j}{\left|\vec{x}_i - \vec{x}_j\right|^3}(\vec{x}_i - \vec{x}_j)$$

$\dot{\vec{x}} = ?$

$\dot{\vec{u}} = ?$

$\vec{f} = ?$

- equations of motion in expanding Universe

  - physical coordinates – with expansion

$$\vec{r} = a\vec{x}, \quad \vec{v} = \vec{u} + H\vec{r}$$

$$\dot{\vec{x}} = \vec{u}/a$$

$$\dot{\vec{u}} = \vec{f} - H\vec{u}$$

with: $$\vec{f}_i = -G\frac{1}{a^2}\sum_{i\neq j}^{N}\frac{m_j}{\left|\vec{x}_i - \vec{x}_j\right|^3}(\vec{x}_i - \vec{x}_j) - \ddot{a}\vec{x}_i$$

$\dot{\vec{x}} = \checkmark$

$\dot{\vec{u}} = \checkmark$

$\vec{f} = \checkmark$

- equations of motion in expanding Universe

  - physical coordinates – with expansion

equations of motion in comoving coordinates:

$$\dot{\vec{x}} = \vec{u}/a$$

$$\dot{\vec{u}} = \vec{f} - H\vec{u}$$

$$\vec{f} = -G\frac{1}{a^2}\sum_{x \neq x_j}^{N}\frac{m_j}{\left|\vec{x}-\vec{x}_j\right|^3}\left(\vec{x}-\vec{x}_j\right) - \ddot{a}\vec{x}$$

transformation back to physical coordinates:

$$\vec{r} = a\vec{x}, \quad \vec{v} = \vec{u} + H\vec{r}$$

$\dot{\vec{x}} = \checkmark$

$\dot{\vec{u}} = \checkmark$

$\vec{f} = \checkmark$

- equations of motion in expanding Universe

  - physical coordinates – with expansion

equations of motion in comoving coordinates:

$$\dot{\vec{x}} = \vec{u}/a$$

$$\dot{\vec{u}} =$$

careful:

periodic boundaries not considered yet...

(we will revisit this situation later when dealing with tree codes)

$$\vec{f} = -G\frac{1}{a^2}\sum_{x \neq x_j}^{N} \frac{m_j}{\left|\vec{x} - \vec{x}_j\right|^3}\left(\vec{x} - \vec{x}_j\right) - \ddot{a}\vec{x}$$

transformation back to physical coordinates:

$$\vec{r} = a\vec{x}, \quad \vec{v} = \vec{u} + H\vec{r}$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe
    - using Poisson's equation...

...and now for a more general treatment

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe
  - comoving coordinates – Hamilton formalism

$$\mathcal{L} = \frac{1}{2} m v^2 - m\varphi(\vec{r})$$

$$\Delta\varphi(\vec{r}) = 4\pi G(\rho - \rho_\Lambda)$$

→ proper potential in physical coordinates

$$\dot{\vec{x}} = ?$$
$$\dot{\vec{p}} = ?$$
$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe

  - comoving coordinates – Hamilton formalism

comoving coordinates

$$\vec{r} = a\vec{x}$$
$$\vec{v} = a\dot{\vec{x}} + \dot{a}\vec{x}$$

$$\mathcal{L} = \frac{1}{2} m v^2 - m\varphi(\vec{r})$$

$$\mathcal{L} = \frac{1}{2} m(a\dot{\vec{x}} + \dot{a}\vec{x})^2 - m\varphi(\vec{x})$$

$$\Delta\varphi(\vec{r}) = 4\pi G(\rho - \rho_\Lambda)$$

proper potential in physical coordinates

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe

  - comoving coordinates – Hamilton formalism

$$\mathcal{L} = \frac{1}{2} m v^2 - m\varphi(\vec{r})$$

comoving coordinates

$\vec{r} = a\vec{x}$

$\vec{v} = a\dot{\vec{x}} + \dot{a}\vec{x}$

$$\mathcal{L} = \frac{1}{2} m(a\dot{\vec{x}} + \dot{a}\vec{x})^2 - m\varphi(\vec{x})$$

$$\Delta\varphi(\vec{r}) = 4\pi G(\rho - \rho_\Lambda)$$

proper potential
in physical coordinates

how to actually derive Poisson's equation?

$$\vec{F}_i = -G \sum_{i \neq j}^{N} \frac{m_j}{\left|\vec{r}_i - \vec{r}_j\right|^3} (\vec{r}_i - \vec{r}_j) \xrightarrow{?} \Delta\varphi(\vec{r}) = 4\pi G\rho$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe
  - comoving coordinates – Hamilton formalism
    - » Poisson's equation

$$\vec{F}(\vec{r}) = -\sum_{i=1}^{N} \frac{G m_i m}{\left|\vec{r}_i - \vec{r}\right|^3} \left(\vec{r}_i - \vec{r}\right)$$

continuum

$$m_i \rightarrow \Delta m = \rho \Delta V$$

$$\downarrow$$

$$\vec{F}(\vec{r}) = -Gm \int_V \rho(\vec{r}') \frac{\left(\vec{r}' - \vec{r}\right)}{\left|\vec{r}' - \vec{r}\right|^3} d^3 r'$$

$$\dot{\vec{x}} = ?$$
$$\dot{\vec{p}} = ?$$
$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe
  - comoving coordinates – Hamilton formalism
    - » Poisson's equation

$$\nabla_r \cdot \vec{F}(\vec{r}) = -Gm \int_V \rho(\vec{r}') \nabla_r \cdot \frac{(\vec{r}' - \vec{r})}{|\vec{r}' - \vec{r}|^3} d^3 r'$$
$$= -Gm \int_V \rho(\vec{r}') 4\pi \delta_{\text{Dirac}}(\vec{r}' - \vec{r}) d^3 r'$$
$$= -4\pi G m \rho(\vec{r})$$

$$\downarrow \quad \vec{F} = -m \nabla \varphi$$

$$\Delta \varphi = 4\pi G \rho(\vec{r})$$

$$\dot{\vec{x}} = ?$$

$$\dot{\vec{p}} = ?$$

$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe

  - comoving coordinates – Hamilton formalism

$$\mathcal{L} = \frac{1}{2} m(a\dot{\vec{x}} + \dot{a}\vec{x})^2 - m\varphi(\vec{x}) \qquad \Delta\varphi(\vec{r}) = 4\pi G(\rho - \rho_\Lambda)$$

perform canonical transformation:

$$\mathcal{L} = \mathcal{L} - \frac{dF}{dt} \quad \text{with} \quad F = \frac{1}{2} ma\dot{a}x^2$$

one can always add a total time derivative to the Lagrangian as it will only add a constant to the action; and the equations of motions are derived by requiring that the time variation of the action vanishes...

$$\dot{\vec{x}} = ?$$
$$\dot{\vec{p}} = ?$$
$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe

  - comoving coordinates – Hamilton formalism

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2} m(a\dot{\vec{x}} + \dot{a}\vec{x})^2 - m\varphi(\vec{x}) - \frac{dF}{dt} \\
&= \frac{1}{2} m(a^2\dot{\vec{x}}^2 + 2a\dot{a}\dot{\vec{x}}\cdot\vec{x} + \dot{a}^2\vec{x}^2) - m\varphi(\vec{x}) - \frac{dF}{dt} \\
&= \frac{1}{2} m(a^2\dot{\vec{x}}^2 + 2a\dot{a}\dot{\vec{x}}\cdot\vec{x} + \dot{a}^2\vec{x}^2) - m\varphi(\vec{x}) \\
&\quad - \frac{1}{2} m(a^2\vec{x}^2 + a\ddot{a}\vec{x}^2 + 2a\dot{a}\dot{\vec{x}}\cdot\vec{x}) \\
&= \frac{1}{2} ma^2\dot{\vec{x}}^2 - m\underbrace{(\varphi + \frac{1}{2} a\ddot{a}\vec{x}^2)}_{\psi = \varphi + \frac{1}{2} a\ddot{a}x^2}
\end{aligned}$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe

  - comoving coordinates – Hamilton formalism

$$\mathcal{L} = \frac{1}{2} m a^2 \dot{x}^2 - m\psi(\vec{x}) \qquad \psi = \varphi + \frac{1}{2} a\ddot{a}x^2$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe
  - comoving coordinates – Hamilton formalism

$$\mathcal{L} = \frac{1}{2} m a^2 \dot{x}^2 - m\psi(\vec{x})$$

$$\Delta_x \psi = ?$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe

  - comoving coordinates – Hamilton formalism

$$\mathcal{L} = \frac{1}{2} m a^2 \dot{x}^2 - m\psi(\vec{x})$$

$\Delta_x \psi = ?$

$$\Delta_x \psi(\vec{x}) = \Delta_x(\varphi + \frac{1}{2} a\ddot{a}x^2)$$

$\nabla_x = \frac{dr}{dx}\nabla = a\nabla$

$\Delta_x = \ldots = a^2\Delta$

$$= \Delta_x \varphi + \frac{1}{2} a\ddot{a}\Delta_x x^2$$

$$= a^2 \Delta\varphi + 3a\ddot{a}$$

$$= a^2\left[4\pi G(\rho - \rho_\Lambda)\right] + 3a\ddot{a}$$

$$= a^2\left[4\pi G(\rho - \rho_\Lambda)\right] - 4\pi G a^2(\overline{\rho} - \rho_\Lambda)$$

$$= 4\pi G a^2(\rho - \overline{\rho})$$

$$= \frac{4\pi G}{a}(\rho_x - \overline{\rho}_x)$$

<u>2nd Friedmann equation:</u>

$\ddot{a} = -\frac{4\pi G}{3} a(\overline{\rho} - \rho_\Lambda)$

$$\dot{\vec{x}} = ?$$

$$\dot{\vec{p}} = ?$$

$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe

  - comoving coordinates – Hamilton formalism

$$\mathcal{L} = \frac{1}{2} m a^2 \dot{x}^2 - m\psi(\vec{x}) \qquad \Delta_x \psi = \frac{4\pi G}{a}(\rho_x - \overline{\rho}_x)$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe

  - comoving coordinates – Hamilton formalism

$$\mathcal{L} = \frac{1}{2} m a^2 \dot{x}^2 - m\psi(\vec{x})$$

$$\Delta_x \psi = \frac{4\pi G}{a}(\rho_x - \overline{\rho}_x)$$

obtain canonical momentum

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} = m a^2 \dot{\vec{x}}$$

$\mathcal{H} = \dot{\vec{x}}\,\vec{p} - \mathcal{L}$

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m\psi(\vec{x})$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe
  - comoving coordinates – Hamilton formalism

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m\psi(\vec{x}) \qquad \Delta_x \psi = \frac{4\pi G}{a}(\rho_x - \overline{\rho}_x)$$

equations of motion:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} \qquad \dot{\vec{x}} = \frac{\vec{p}}{ma^2}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \qquad \dot{\vec{p}} = -m\nabla_x \psi$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe

  - comoving coordinates – Hamilton formalism

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m\psi(\vec{x}) \qquad \Delta_x \psi = \frac{4\pi G}{a}(\rho_x - \bar{\rho}_x)$$

introduce $\Phi = a\psi$

equations of motion:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}} \qquad \dot{\vec{x}} = \frac{\vec{p}}{ma^2}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \qquad \dot{\vec{p}} = -m\nabla_x \psi$$

$\dot{\vec{x}} = \checkmark$

$\dot{\vec{p}} = \checkmark$

$\Delta_x \Phi_x = \checkmark$

- equations of motion in expanding Universe

  - comoving coordinates – Hamilton formalism

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m \frac{\Phi(\vec{x})}{a}$$

equations of motion:

$$\dot{\vec{x}} = \frac{\partial \mathcal{H}}{\partial \vec{p}}$$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}}$$

$$\dot{\vec{x}} = \frac{\vec{p}}{ma^2}$$

$$\dot{\vec{p}} = -\frac{m}{a} \nabla_x \Phi$$

$$\Delta_x \Phi = 4\pi G(\rho_x - \overline{\rho})$$

$$\dot{\vec{x}} = ?$$
$$\dot{\vec{p}} = ?$$
$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

just for completeness' sake...

(setting m=1 for convenience)

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe
  - comoving coordinates – re-writing the Newtonian equations of motion
    - Newtonian physics in r and u

$$\vec{r}, \quad \vec{v} \quad \xrightarrow{\text{equations of motion}} \quad \dot{\vec{r}} = \vec{v}, \quad \dot{\vec{v}} = -\vec{\nabla}_r \varphi_r \qquad \Delta_r \varphi_r = 4\pi G(\rho_r - \rho_\Lambda)$$

    - comoving coordinates x and p

$$\vec{x} \quad \left(= \frac{\vec{r}}{a}\right), \quad \vec{p} \quad \left(= a(\vec{v} - H\vec{r})\right) \quad \xrightarrow{\text{equations of motion}} \quad \dot{\vec{x}} = ?, \quad \dot{\vec{p}} = ? \qquad \Delta_x \Phi_x = ?$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

    – Newtonian physics in r and u

$$\vec{r}, \quad \vec{v} \quad \xrightarrow{\text{equations of motion}} \quad \dot{\vec{r}} = \vec{v}, \quad \dot{\vec{v}} = -\vec{\nabla}_r \varphi_r \qquad \Delta_r \varphi_r = 4\pi G(\rho_r - \rho_\Lambda)$$

"brute force"

    – comoving coordinates x and p

$$\vec{x} \quad \left(= \frac{\vec{r}}{a}\right), \quad \vec{p} \quad \left(= a(\vec{v} - H\vec{r})\right) \quad \xrightarrow{\text{equations of motion}} \quad \dot{\vec{x}} = ?, \quad \dot{\vec{p}} = ? \qquad \Delta_x \Phi_x = ?$$

$$\dot{\vec{x}} = ?$$

$$\dot{\vec{p}} = ?$$

$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

    – comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a} \qquad \qquad \dot{\vec{x}} = ?$$

$$\vec{p} = a(\vec{v} - H\vec{r}) \xrightarrow{\text{equations of motion}} \dot{\vec{p}} = ?$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

    – comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a}$$

$$\vec{p} = a(\vec{v} - H\vec{r})$$

$$\dot{\vec{x}} = \frac{\dot{\vec{r}}}{a} - \frac{\vec{r}}{a^2}\dot{a} = \frac{\dot{\vec{r}}}{a} - H\frac{\vec{r}}{a}$$

$$a^2\dot{\vec{x}} = a(\dot{\vec{r}} - H\vec{r}) = \vec{p}$$

$$\Rightarrow \quad \dot{\vec{x}} = \frac{\vec{p}}{a^2}$$

✓ agrees with Hamiltonian momentum

$$\dot{\vec{x}} = ?$$
$$\dot{\vec{p}} = ?$$
$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

    – comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a}$$
$$\vec{p} = a(\vec{v} - H\vec{r})$$

$$\xrightarrow{\text{equations of motion}}$$

$$\dot{\vec{x}} = \frac{\vec{p}}{a^2}$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe
  - comoving coordinates – re-writing the Newtonian equations of motion
    - comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a} \qquad \xrightarrow{\text{equations of motion}} \qquad \dot{\vec{x}} = \frac{\vec{p}}{a^2}$$

$$\vec{p} = a(\vec{v} - H\vec{r})$$

$$\begin{aligned}\dot{\vec{p}} &= \dot{a}(\vec{v} - H\vec{r}) + a(\dot{\vec{v}} - \dot{H}\vec{r} - H\dot{\vec{r}}) \\ &= \dot{a}\vec{v} - \dot{a}H\vec{r} + a\dot{\vec{v}} - a\dot{H}\vec{r} - \dot{a}\dot{\vec{r}} \\ &= a\dot{\vec{v}} - \dot{a}H\vec{r} - a\dot{H}\vec{r} = a\dot{\vec{v}} - \dot{a}^2\vec{x} - a\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\vec{r} \\ &= a\dot{\vec{v}} - \dot{a}\vec{x} - a\ddot{a}\vec{x} + \dot{a}^2\vec{x} = a\dot{\vec{v}} - a\ddot{a}\vec{x} \\ &= -a(\nabla_r \varphi + a\ddot{a}\vec{x})\end{aligned}$$

r and x still mixed...

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

    – comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a}$$

$$\vec{p} = a(\vec{v} - H\vec{r})$$

equations of motion →

$$\dot{\vec{x}} = \frac{\vec{p}}{a^2}$$

$$\begin{aligned}\dot{\vec{p}} &= -a(\nabla_r \varphi + a\ddot{a}\vec{x}) \\ &= -\nabla_x \varphi + a\ddot{a}\vec{x} \\ &= -\nabla_x (\varphi + \frac{1}{2} a\ddot{a}x^2) \\ &= -\nabla_x \psi\end{aligned}$$

$$\nabla_x = \frac{dr}{dx}\nabla = a\nabla$$

$$\Delta_x = \ldots = a^2\Delta$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

    – comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a} \qquad \dot{\vec{x}} = \frac{\vec{p}}{a^2}$$

$$\vec{p} = a(\vec{v} - H\vec{r}) \xrightarrow{\text{equations of motion}} \dot{\vec{p}} = -\nabla_x \psi$$

$$\dot{\vec{x}} = ?$$
$$\dot{\vec{p}} = ?$$
$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

    – comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a} \qquad \xrightarrow{\text{equations of motion}} \qquad \dot{\vec{x}} = \frac{\vec{p}}{a^2}$$

$$\vec{p} = a(\vec{v} - H\vec{r}) \qquad\qquad \dot{\vec{p}} = -\nabla_x \psi \qquad \Delta_x \psi = ?$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe
  - comoving coordinates – re-writing the Newtonian equations of motion
    - comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a} \qquad \vec{p} = a(\vec{v} - H\vec{r}) \quad \xrightarrow{\text{equations of motion}} \quad \dot{\vec{x}} = \frac{\vec{p}}{a^2} \qquad \dot{\vec{p}} = -\nabla_x \psi \qquad \Delta_x \psi = ?$$

$$\psi = \varphi + \frac{1}{2} a \ddot{a} x^2$$

$$\dot{\vec{x}} = ?$$
$$\dot{\vec{p}} = ?$$
$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

    – comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a} \qquad \vec{p} = a(\vec{v} - H\vec{r}) \quad \xrightarrow{\text{equations of motion}} \quad \dot{\vec{x}} = \frac{\vec{p}}{a^2} \qquad \dot{\vec{p}} = -\nabla_x \psi \qquad \Delta_x \psi = ?$$

$$\psi = \varphi + \frac{1}{2} a\ddot{a}x^2 \quad \xrightarrow[\Delta_x = a^2 \Delta_r]{\rho_x = a^3 \rho_r} \quad \Delta_r \varphi = \frac{1}{a^2}\Delta_x(\psi - \frac{1}{2}a\ddot{a}x^2) = \frac{1}{a^2}\Delta_x \psi - \frac{1}{2}\frac{\ddot{a}}{a}\Delta_x x^2$$

$$= \frac{1}{a^2}\Delta_x \psi - 3\frac{\ddot{a}}{a}$$

$$= 4\pi G(\rho_r - \rho_\Lambda) = 4\pi G(\frac{1}{a^3}\rho_x - \rho_\Lambda)$$

$\dot{\vec{x}} = ?$

$\dot{\vec{p}} = ?$

$\Delta_x \Phi_x = ?$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

    – comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a} \qquad \vec{p} = a(\vec{v} - H\vec{r})$$

equations of motion →

$$\dot{\vec{x}} = \frac{\vec{p}}{a^2} \qquad \dot{\vec{p}} = -\nabla_x \psi \qquad \Delta_x \psi = ?$$

$$\Delta_x \psi = 4\pi G\left(\frac{1}{a}\rho_x - a^2 \rho_\Lambda\right) + 3\ddot{a}a$$

2nd Friedmann equation:

$$\ddot{a} = -\frac{4\pi G}{3} a(\overline{\rho}_r - \rho_\Lambda) = -\frac{4\pi G}{3} a\left(\frac{1}{a^3}\overline{\rho}_x - \rho_\Lambda\right)$$

$$\dot{\vec{x}} = ?$$
$$\dot{\vec{p}} = ?$$
$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

    – comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a} \qquad \vec{p} = a(\vec{v} - H\vec{r}) \quad \xrightarrow{\text{equations of motion}} \quad \dot{\vec{x}} = \frac{\vec{p}}{a^2} \qquad \dot{\vec{p}} = -\nabla_x \psi \qquad \Delta_x \psi = ?$$

$$\begin{aligned}
\Delta_x \psi &= 4\pi G(\frac{1}{a}\rho_x - a^2 \rho_\Lambda) + 3\ddot{a}a \\
&= 4\pi G(\frac{1}{a}\rho_x - a^2 \rho_\Lambda) - 4\pi G a^2 (\frac{1}{a^3}\overline{\rho}_x - \rho_\Lambda) \\
&= \frac{4\pi G}{a}(\rho_x - \overline{\rho}_x)
\end{aligned}$$

$$\dot{\vec{x}} = ?$$
$$\dot{\vec{p}} = ?$$
$$\Delta_x \Phi_x = ?$$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

    – comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a} \qquad \dot{\vec{x}} = \frac{\vec{p}}{a^2}$$

equations of motion →

$$\vec{p} = a(\vec{v} - H\vec{r}) \qquad \dot{\vec{p}} = -\nabla_x \psi$$

$$\Delta_x \psi = \frac{4\pi G}{a}(\rho_x - \bar{\rho}_x)$$

– final re-definition:

$$\Phi = a\psi$$

$\dot{\vec{x}} = \checkmark$

$\dot{\vec{p}} = \checkmark$

$\Delta_x \Phi_x = \checkmark$

- equations of motion in expanding Universe

  - comoving coordinates – re-writing the Newtonian equations of motion

    – comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a}$$

$$\vec{p} = a(\vec{v} - H\vec{r})$$

equations of motion →

$$\dot{\vec{x}} = \frac{\vec{p}}{a^2}$$

$$\dot{\vec{p}} = -\frac{1}{a}\nabla_x \Phi$$

$$\Delta_x \Phi = 4\pi G(\rho_x - \overline{\rho}_x)$$

identical with previous formulae…

(except that m is missing due to convention m=1)

- **equations of motion in expanding Universe:**

  - comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a}$$

$$\vec{p} = ma(\vec{v} - H\vec{r})$$

  - equations of motions in comoving coordinates x and p

$$\dot{\vec{x}} = \frac{\vec{p}}{ma^2}$$

$$\dot{\vec{p}} = -\frac{m}{a}\nabla_x\Phi$$

$$\Delta_x\Phi = 4\pi G(\rho_x - \overline{\rho}_x)$$

- **equations of motion in expanding Universe:**

  - comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a}$$

$$\vec{p} = ma(\vec{v} - H\vec{r})$$

r = physical coordinate
v = physical velocity

  - equations of motions in comoving coordinates x and p

$$\dot{\vec{x}} = \frac{\vec{p}}{ma^2}$$

$$\dot{\vec{p}} = -\frac{m}{a}\nabla_x\Phi$$

$$\Delta_x\Phi = 4\pi G(\rho_x - \overline{\rho}_x)$$

▪ equations of motion in expanding Universe:

- comoving coordinates x and p

$$\vec{x} = \frac{\vec{r}}{a}$$

$$\vec{p} = ma(\vec{v} - H\vec{r})$$

x = comoving coordinate
p = comoving (canonical) momentum

$u = p/ma = a\dot{x}$ peculiar velocity

- equations of motions in comoving coordinates x and p

$$\dot{\vec{x}} = \frac{\vec{p}}{ma^2}$$

$$\dot{\vec{p}} = -\frac{m}{a}\nabla_x\Phi$$

$$\Delta_x\Phi = 4\pi G(\rho_x - \overline{\rho}_x)$$

Φ = peculiar potential
ρ_x = comoving matter density

symplectic integrator for the equations of motion

- symplectic integrators

the (numerical) evolution of the system (i.e. x and p) from t_0 to t_n

$$\begin{matrix} \vec{x}_0 \\ \vec{p}_0 \end{matrix} \longrightarrow \begin{matrix} \vec{x}_n \\ \vec{p}_n \end{matrix}$$

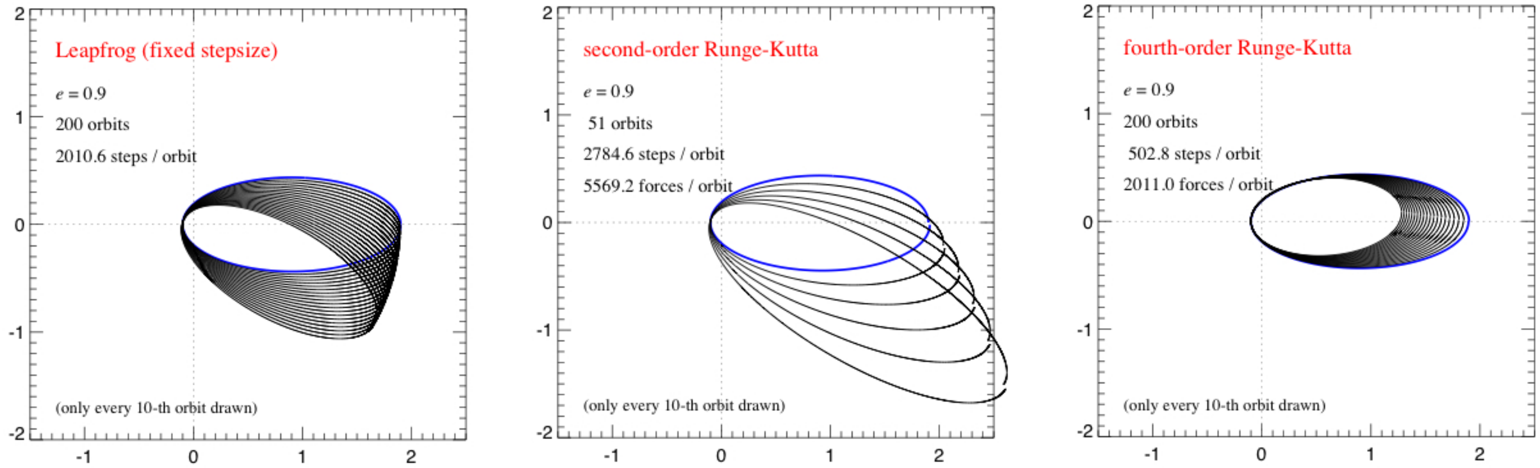
is a canonical coordinate transformation

"symplectic" says nothing about the accuracy of the integrator,

but rather preserves the geometric structure of the original Hamiltonian flow!

- symplectic integrators

  - motivation:



symplectic integrator, 2n order accurate

non-symplectic integrators, 2nd order accurate — 4th order accurate

taken from Volker Springel's GADGET-2 paper (astro-ph/0505010)

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m \frac{\Phi(\vec{x})}{a}$$

  - equations of motion:

$$\dot{\vec{x}} = \{\vec{x}, \mathcal{H}\}$$
$$\dot{\vec{p}} = \{\vec{p}, \mathcal{H}\}$$

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2}\,p^2 + m\frac{\Phi(\vec{x})}{a}$$

- equations of motion:

$$\dot{\vec{x}} = \{\vec{x},\mathcal{H}\} = H\ \vec{x}$$
$$\dot{\vec{p}} = \{\vec{p},\mathcal{H}\} = H\ \vec{p}$$

Poisson bracket = linear operator

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m \frac{\Phi(\vec{x})}{a}$$

- equations of motion:

$$\dot{\vec{x}} = \{\vec{x}, \mathcal{H}\} = H\,\vec{x}$$
$$\dot{\vec{p}} = \{\vec{p}, \mathcal{H}\} = H\,\vec{p}$$

evolution of system →

$$\vec{x}(t) = e^{tH}\vec{x}_0$$
$$\vec{p}(t) = e^{tH}\vec{p}_0$$

Poisson bracket = linear operator

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m \frac{\Phi(\vec{x})}{a}$$

- equations of motion:

$$\dot{\vec{x}} = \{\vec{x}, \mathcal{H}\} = H\ \vec{x} \qquad \xrightarrow{\text{evolution of system}} \qquad \vec{x}(t) = e^{tH}\vec{x}_0$$

$$\dot{\vec{p}} = \{\vec{p}, \mathcal{H}\} = H\ \vec{p} \qquad\qquad \vec{p}(t) = e^{tH}\vec{p}_0$$

$$\mathcal{H} = \mathcal{H}_p + \mathcal{H}_x = T + V$$

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m \frac{\Phi(\vec{x})}{a}$$

- equations of motion:

$$\dot{\vec{x}} = \{\vec{x}, \mathcal{H}\} = H\,\vec{x} \qquad \xrightarrow{\text{evolution of system}} \qquad \vec{x}(t) = e^{t(T+V)}\vec{x}_0$$

$$\dot{\vec{p}} = \{\vec{p}, \mathcal{H}\} = H\,\vec{p} \qquad\qquad \vec{p}(t) = e^{t(T+V)}\vec{p}_0$$

$$\mathcal{H} = \mathcal{H}_p + \mathcal{H}_x = T + V$$

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m\frac{\Phi(\vec{x})}{a}$$

- equations of motion:

$$\dot{\vec{x}} = \{\vec{x},\mathcal{H}\} = H\ \vec{x} \qquad \xrightarrow{\text{evolution of system}} \qquad \vec{x}(t) = e^{t(T+V)}\vec{x}_0$$

$$\dot{\vec{p}} = \{\vec{p},\mathcal{H}\} = H\ \vec{p} \qquad \qquad \vec{p}(t) = e^{t(T+V)}\vec{p}_0$$

$$\mathcal{H} = \mathcal{H}_p + \mathcal{H}_x = T + V$$

try to split $e^{t(T+V)}$ into something like $e^{tT}e^{tV}$?!

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m\frac{\Phi(\vec{x})}{a}$$

- equations of motion:

$$\dot{\vec{x}} = \{\vec{x},\mathcal{H}\} = H\,\vec{x} \qquad \xrightarrow{\text{evolution of system}} \qquad \vec{x}(t) = e^{t(T+V)}\vec{x}_0$$

$$\dot{\vec{p}} = \{\vec{p},\mathcal{H}\} = H\,\vec{p} \qquad\qquad \vec{p}(t) = e^{t(T+V)}\vec{p}_0$$

$$\mathcal{H} = \mathcal{H}_p + \mathcal{H}_x = T + V$$

- Baker-Campbell-Hausdorff identity:

$$e^A e^B \neq e^{A+B}$$

$$e^A e^B = e^C \ \text{ with } \ C = A + B + \frac{1}{2}\{A,B\} + \ldots$$

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m\frac{\Phi(\vec{x})}{a}$$

- equations of motion:

$$\dot{\vec{x}} = \{\vec{x},\mathcal{H}\} = H\ \vec{x}$$
$$\dot{\vec{p}} = \{\vec{p},\mathcal{H}\} = H\ \vec{p}$$

evolution of system $\longrightarrow$

$$\vec{x}(t) = e^{t(T+V)}\vec{x}_0$$
$$\vec{p}(t) = e^{t(T+V)}\vec{p}_0$$

$$\mathcal{H} = \mathcal{H}_p + \mathcal{H}_x = T + V$$

- Baker-Campbell-Hausdorff identity:

$$e^{t(T+V)} = e^{tT/2}e^{tV}e^{tT/2} + O(t^3)$$

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m\frac{\Phi(\vec{x})}{a}$$

- equations of motion:

$$\dot{\vec{x}} = \{\vec{x},\mathcal{H}\} = H\,\vec{x} \qquad \xrightarrow{\text{evolution of system}} \qquad \vec{x}(t) = e^{tT/2}e^{tV}e^{tT/2}\vec{x}_0 + O(t^3)$$

$$\dot{\vec{p}} = \{\vec{p},\mathcal{H}\} = H\,\vec{p} \qquad\qquad \vec{p}(t) = e^{tT/2}e^{tV}e^{tT/2}\vec{p}_0 + O(t^3)$$

$$\mathcal{H} = \mathcal{H}_p + \mathcal{H}_x = T + V$$

because the Hamiltonian can be split into two independent parts (i.e. kinetic energy T and potential energy V) ,

we are able to approximate the evolution of the system by this special choice of operators...

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m \frac{\Phi(\vec{x})}{a}$$

- equations of motion:

$$\dot{\vec{x}} = \{\vec{x}, \mathcal{H}\} = H\,\vec{x}$$
$$\dot{\vec{p}} = \{\vec{p}, \mathcal{H}\} = H\,\vec{p}$$

evolution of system →

$$\vec{x}(t) = e^{tT/2} e^{tV} e^{tT/2} \vec{x}_0 + O(t^3)$$
$$\vec{p}(t) = e^{tT/2} e^{tV} e^{tT/2} \vec{p}_0 + O(t^3)$$

$$\mathcal{H} = \mathcal{H}_p + \mathcal{H}_x = T + V$$

we rather approximate the true Hamiltonian
than discretizing the equations of motions

=> **symplectic integration scheme**!

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m \frac{\Phi(\vec{x})}{a}$$

- second-order accurate scheme:

$$\vec{x}(t) = e^{tT/2} e^{tV} \left(e^{tT/2} \vec{x}_0 \right)$$
$$\vec{p}(t) = e^{tT/2} e^{tV} \left(e^{tT/2} \vec{p}_0 \right)$$

↘

1. evolve the system for $\Delta t/2$ under $\mathcal{H}_p$

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m\frac{\Phi(\vec{x})}{a}$$

- second-order accurate scheme:

$$\vec{x}(t) = e^{tT/2}\left(e^{tV}\left(e^{tT/2}\vec{x}_0\right)\right)$$

$$\vec{p}(t) = e^{tT/2}\left(e^{tV}\left(e^{tT/2}\vec{p}_0\right)\right)$$

1. evolve the system for $\Delta t/2$ under $\mathcal{H}_p$
2. evolve the system for Δt under $\mathcal{H}_x$

- symplectic integrators

$$\mathcal{H} = \frac{1}{2ma^2} p^2 + m \frac{\Phi(\vec{x})}{a}$$

- second-order accurate scheme:

$$\vec{x}(t) = \left(e^{tT/2} \left(e^{tV} \left(e^{tT/2} \vec{x}_0 \right) \right) \right)$$

$$\vec{p}(t) = \left(e^{tT/2} \left(e^{tV} \left(e^{tT/2} \vec{p}_0 \right) \right) \right)$$

1. evolve the system for $\Delta t/2$ under $\mathcal{H}_p$
2. evolve the system for Δt under $\mathcal{H}_x$
3. evolve the system for $\Delta t/2$ under $\mathcal{H}_p$

- Drift-Kick-Drift time integration

$$
\begin{array}{ccc}
\vec{x}^{\,n} & & \vec{x}^{\,n+1} \\
 & \curvearrowright & \\
\vec{p}^{\,n} & & \vec{p}^{\,n+1}
\end{array}
$$

superscript indicates t, i.e. $x^n = x(t_n)$

- Drift-Kick-Drift time integration

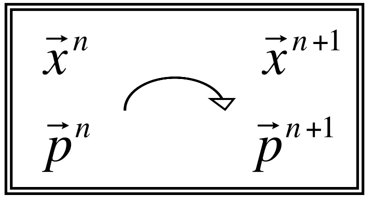
$$\vec{x}^{\,n} \quad \curvearrowright \quad \vec{x}^{\,n+1}$$
$$\vec{p}^{\,n} \qquad\qquad \vec{p}^{\,n+1}$$

$$\dot{\vec{x}} = \{\vec{x},\mathcal{H}\} = H\ \vec{x}$$

$$\dot{\vec{p}} = \{\vec{p},\mathcal{H}\} = H\ \vec{p}$$

superscript indicates t, i.e. $x^{n} = x(t_{n})$

- Drift-Kick-Drift time integration



$$\dot{\vec{x}} = \{\vec{x},\mathcal{H}\} = H\,\vec{x}$$

$$\dot{\vec{p}} = \{\vec{p},\mathcal{H}\} = H\,\vec{p}$$

$$\mathcal{H} = \frac{1}{2ma^2}p^2 + m\frac{\Phi(\vec{x})}{a}$$

superscript indicates t, i.e. $x^{n} = x(t_{n})$

- Drift-Kick-Drift time integration

$$\vec{x}^{\,n} \quad \longrightarrow \quad \vec{x}^{\,n+1}$$
$$\vec{p}^{\,n} \qquad\qquad \vec{p}^{\,n+1}$$

$$\dot{\vec{x}} = \frac{\vec{p}}{a^2}$$

$$\dot{\vec{p}} = -\frac{1}{a}\nabla\Phi$$

superscript indicates t, i.e. $x^n = x(t_n)$

- Drift-Kick-Drift time integration

$$\dot{\vec{x}} = \frac{\vec{p}}{a^2}$$

$$\dot{\vec{p}} = -\frac{1}{a}\nabla\Phi$$

symplectic integrators
=>

$$\vec{x}^{\,n},\ \vec{p}^{\,n} \curvearrowright \vec{x}^{\,n+1},\ \vec{p}^{\,n+1}$$

$$\vec{x}^{\,n+1/2} = \vec{x}^{\,n} + \vec{p}^{\,n} \int_{t_n}^{t_n+\Delta t/2} \frac{dt}{a^2}$$

$$\vec{p}^{\,n+1} = \vec{p}^{\,n} - \vec{\nabla}\Phi^{n+1/2} \int_{t_n}^{t_n+\Delta t} \frac{dt}{a}$$

$$\vec{x}^{\,n+1} = \vec{x}^{\,n+1/2} + \vec{p}^{\,n+1} \int_{t_n+\Delta t/2}^{t_n+\Delta t} \frac{dt}{a^2}$$

superscript indicates t, i.e. $x^n = x(t_n)$

- Drift-Kick-Drift time integration

  - Kick (K) and Drift (D) operators:

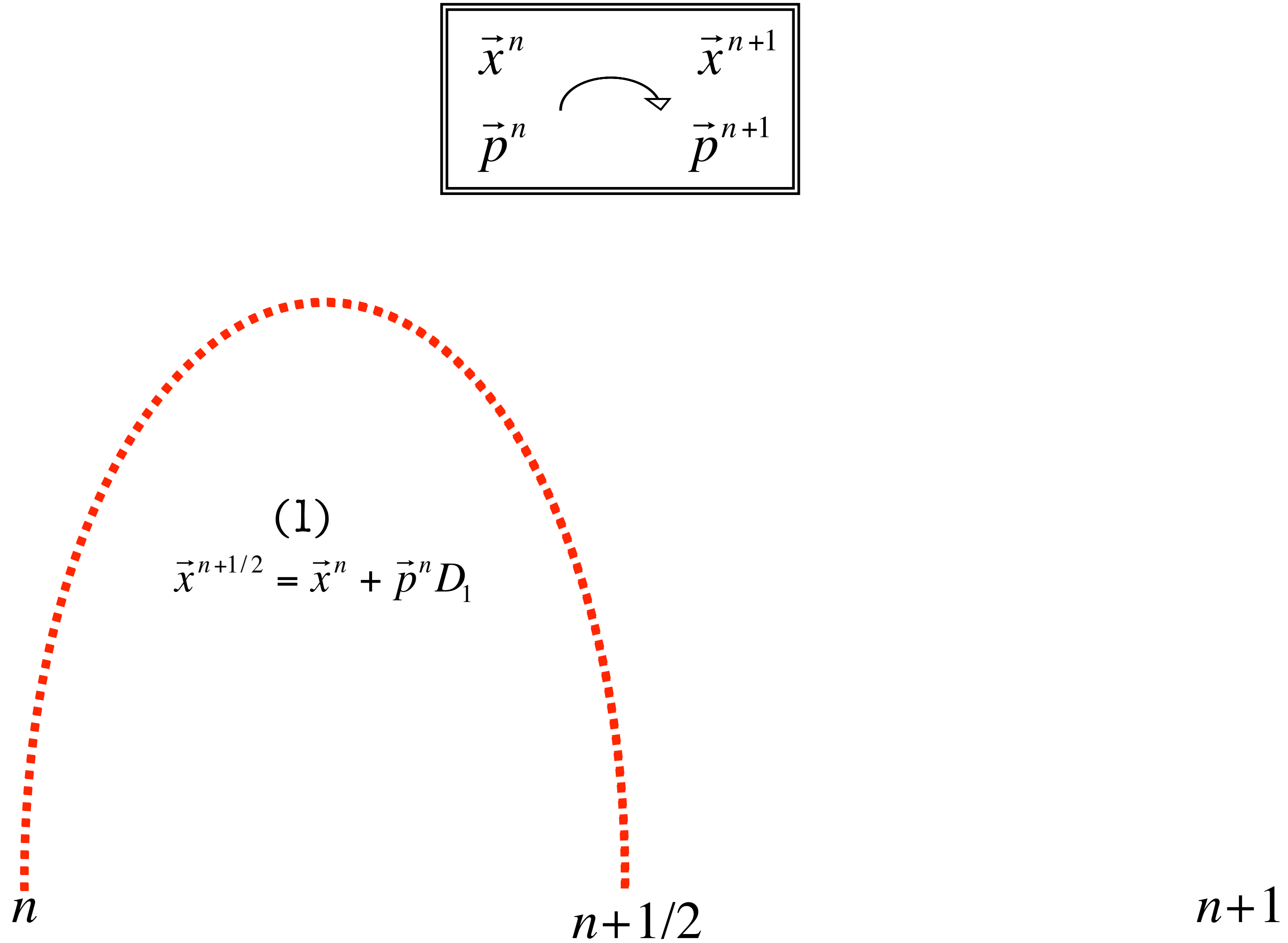
$$K^n = \int_{t_n}^{t_n+\Delta t} \frac{dt}{a} \qquad\qquad K^n = \int_{a_n}^{a_n+\Delta a} \frac{da}{a\dot{a}}$$

$$dt = \frac{1}{\dot{a}} da \quad \longleftrightarrow$$

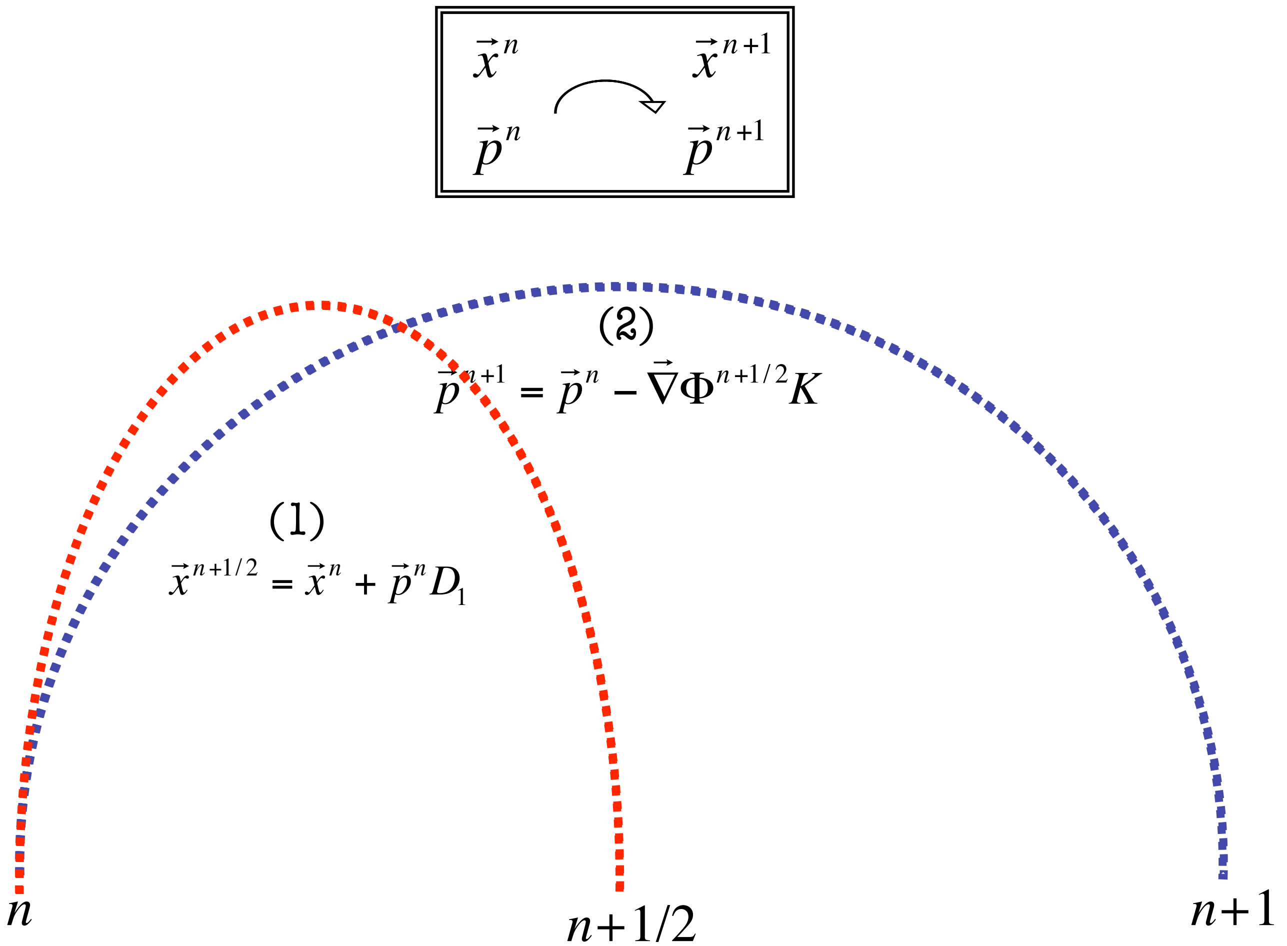
$$D^n = \int_{t_n}^{t_n+\Delta t} \frac{dt}{a^2} \qquad\qquad D^n = \int_{a_n}^{a_n+\Delta a} \frac{da}{a^2\dot{a}}$$

integration variable: time — expansion factor (not symplectic though!)

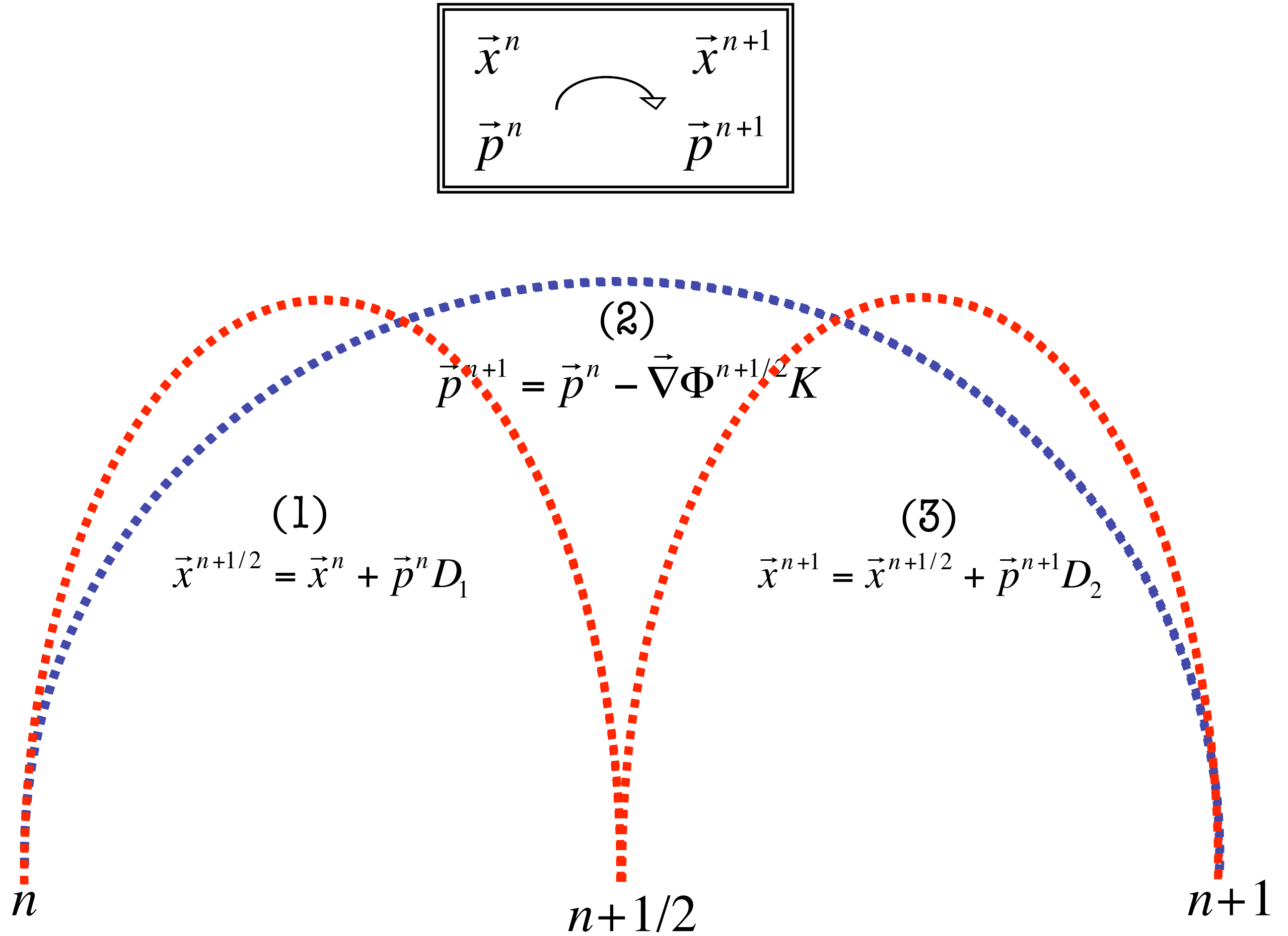
- Drift-Kick-Drift time integration



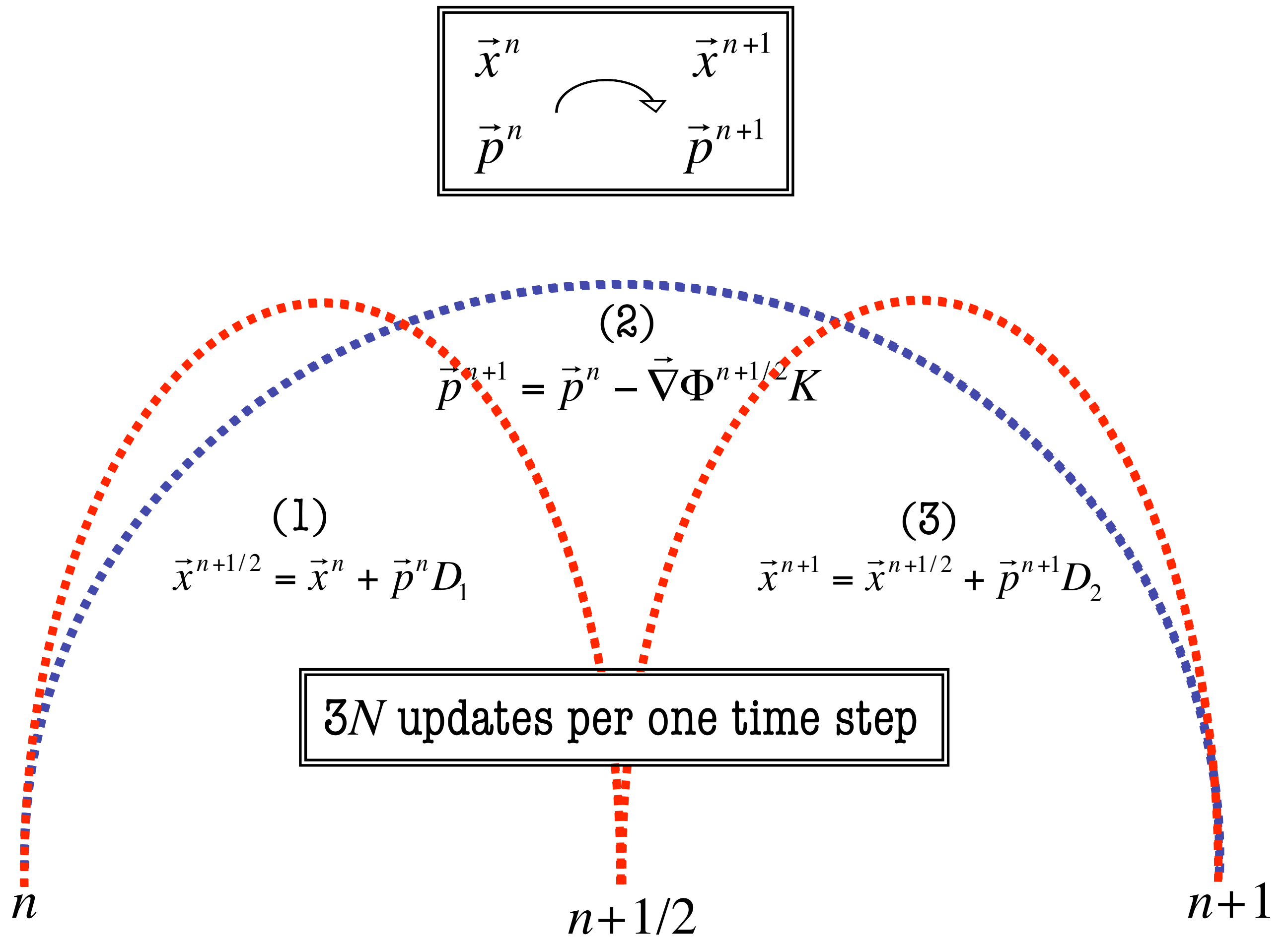
- Drift-Kick-Drift time integration



- Drift-Kick-Drift time integration



- Drift-Kick-Drift time integration



- recovering the leap-frog scheme

- recovering the leap-frog scheme

$$\ldots$$

n

$$\vec{x}^{n+1/2} = \vec{x}^n + \vec{p}^n \int_t^{t+\Delta t/2} \frac{dt}{a^2}$$

$$\vec{p}^{n+1} = \vec{p}^n - \vec{\nabla}\Phi^{n+1/2} \int_t^{t+\Delta t} \frac{dt}{a}$$

$$\vec{x}^{n+1} = \vec{x}^{n+1/2} + \vec{p}^{n+1} \int_{t+\Delta t/2}^{t+\Delta t} \frac{dt}{a^2}$$

$n+1$

$$\vec{x}^{n+3/2} = \vec{x}^{n+1/2} + \vec{p}^{n+1} \int_{t+\Delta t/2}^{t+3\Delta t/2} \frac{dt}{a^2}$$

$$\ldots$$

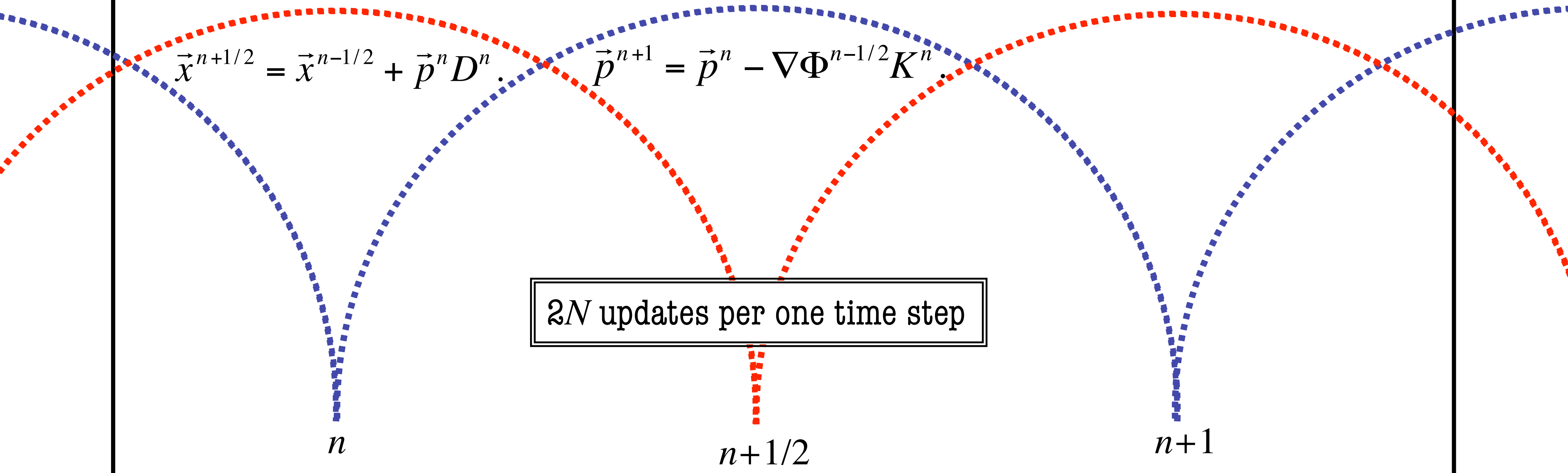
$$\vec{p}^{n+1} = \vec{p}^n - \vec{\nabla}\Phi^{n+1/2} \int_t^{t+\Delta t} \frac{dt}{a}$$

$$\vec{x}^{n+3/2} = \vec{x}^{n+1/2} + \vec{p}^{n+1} \int_{t+\Delta t/2}^{t+3\Delta t/2} \frac{dt}{a^2}$$

- recovering the leap-frog scheme

$$\vec{x}^{\,n},\ \vec{p}^{\,n} \curvearrowright \vec{x}^{\,n+1},\ \vec{p}^{\,n+1}$$

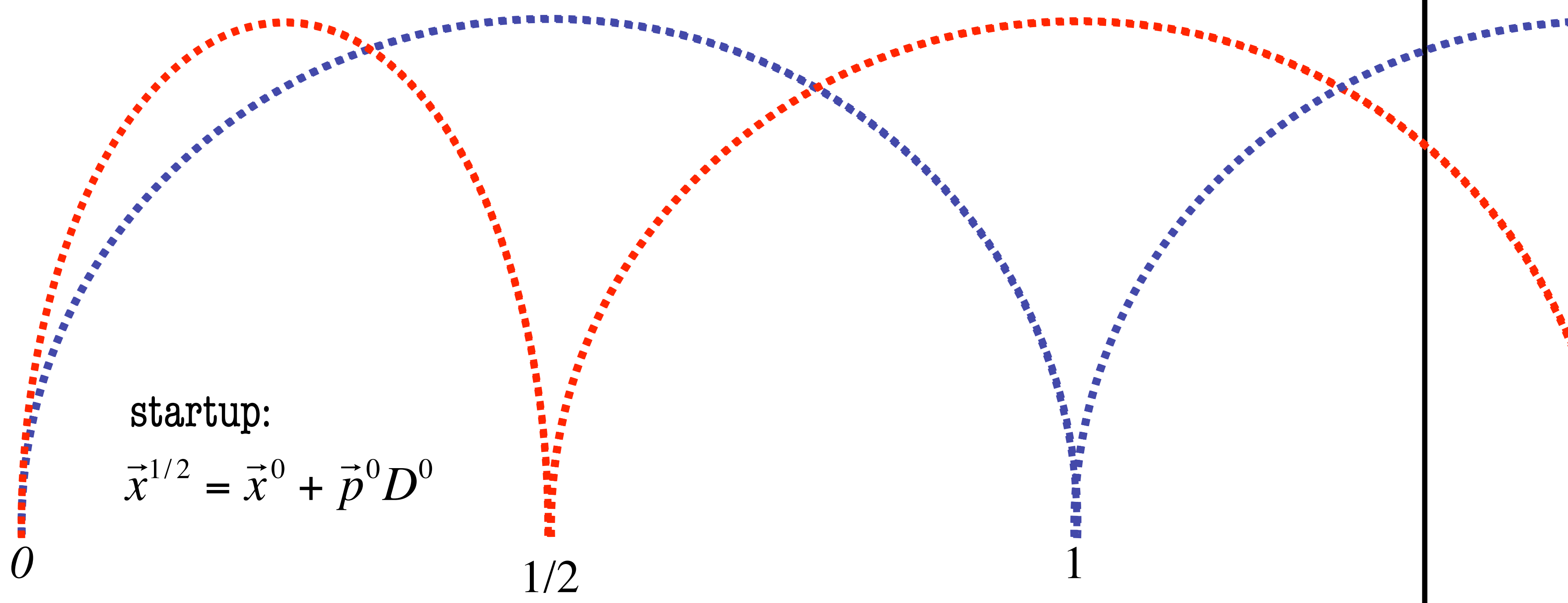
$$\vec{x}^{\,n+1/2} = \vec{x}^{\,n-1/2} + \vec{p}^{\,n} D^n . \qquad \vec{p}^{\,n+1} = \vec{p}^{\,n} - \nabla\Phi^{n-1/2} K^n .$$



$2N$ updates per one time step

- recovering the leap-frog scheme

$$\vec{x}^{n} \quad \vec{x}^{n+1}$$
$$\vec{p}^{n} \quad \vec{p}^{n+1}$$

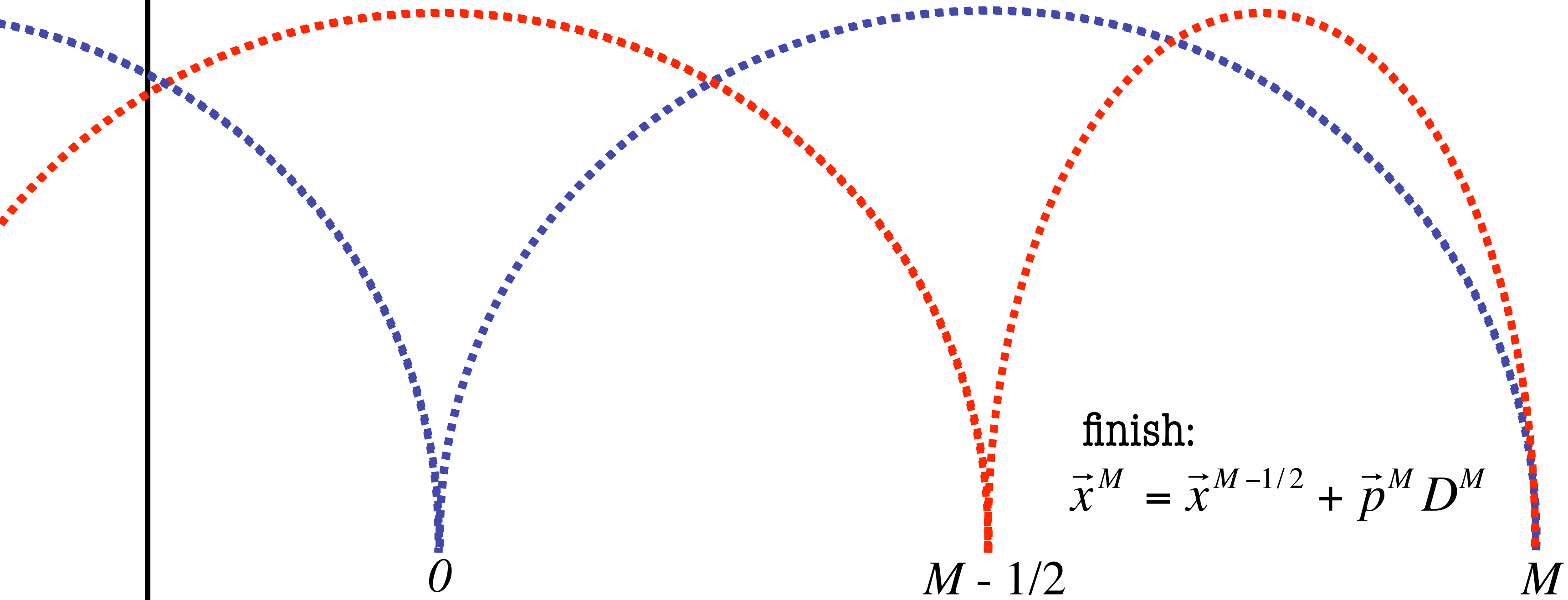


startup:

$$\vec{x}^{1/2} = \vec{x}^{0} + \vec{p}^{0} D^{0}$$

- recovering the leap-frog scheme

$$\begin{matrix} \vec{x}^{\,n} & & \vec{x}^{\,n+1} \\ & \curvearrowright & \\ \vec{p}^{\,n} & & \vec{p}^{\,n+1} \end{matrix}$$



finish:

$$\vec{x}^{M} = \vec{x}^{M-1/2} + \vec{p}^{M} D^{M}$$

- leap-frog scheme for minimal memory usage & minimal flops
  - requires only one force evaluation per time step
  - only one copy of variables stored

- Drift-Kick-Drift scheme for memory economy & synchronisation

=> • both schemes are 2nd order accurate in time

  - even though DKD scheme requires N more operations, it is favourable for adaptive mesh refinement codes...

- leap-frog scheme for minimal memory usage & minimal flops
    - requires only one force evaluation per time step
    - only one copy of variables stored
- Drift-Kick-Drift scheme for memory economy & synchronisation

=> • both schemes are 2nd order accurate in time

• even though DKD scheme requires N more operations, it is favourable for adaptive mesh refinement codes...

→ how to check your integrator?

→ how to choose the correct time step?

→ how to monitor the accuracy?

- time step criteria

  - cosmological criterion

$$\Delta t \leq \frac{1}{H}$$

≈ the time step should be smaller than the age of the Universe

  - acceleration/velocity criterion

$$\Delta t \leq \sqrt{\frac{\varepsilon}{a_{\max}}} \qquad \Delta t \leq \frac{\varepsilon}{v_{\max}}$$

≈ particles should not move farther than some preselected threshold ε →

ε of order the force resolution

- time step criteria

  - cosmological criterion

$$\Delta t \leq \frac{1}{H}$$

≈ the time step should be smaller than the age of the Universe

more details later when dealing with code testing

  - acceleration/velocity criterion

$$\Delta t \leq \sqrt{\frac{\varepsilon}{a_{\max}}} \qquad \Delta t \leq \frac{\varepsilon}{v_{\max}}$$

≈ particles should not move farther than some preselected threshold ε

→ ε of order the force resolution

- obtaining the forces
  - Poisson's equation in comoving coordinates

$$\Delta\Phi(\vec{x}) = 4\pi G\left(\rho(\vec{x}) - \overline{\rho}\right)$$

$$\vec{F}(\vec{x}) = -m\nabla\Phi(\vec{x})$$

heart and soul of every N-body code

- obtaining the forces
  - Poisson's equation

$$\Delta\Phi(\vec{x}) = 4\pi G\left(\rho(\vec{x}) - \bar{\rho}\right)$$

$$\vec{F}(\vec{x}) = -m\nabla\Phi(\vec{x})$$

particle approach ($\vec{x}_i$ =comoving position of ith particle)

$$\vec{F}(\vec{x}_i) = -\sum_{i\neq j}\frac{Gm_i m_j}{(x_i - x_j)^3}(\vec{x}_i - \vec{x}_j)$$

grid approach ($\vec{g}_{i,j,k}$ =position of centre of grid cell (i,j,k))

$$\Delta\Phi(\vec{g}_{i,j,k}) = 4\pi G\left(\rho(\vec{g}_{i,j,k}) - \bar{\rho}\right)$$

$$\vec{F}(\vec{g}_{i,j,k}) = -m\nabla\Phi(\vec{g}_{i,j,k})$$

heart and soul of every N-body code

- obtaining the forces
  - particle approach

    ⇒ tree codes

  - grid approach

    ⇒ AMR codes

  - hybrid approach

    ⇒ P^3M, tree-PM, ...