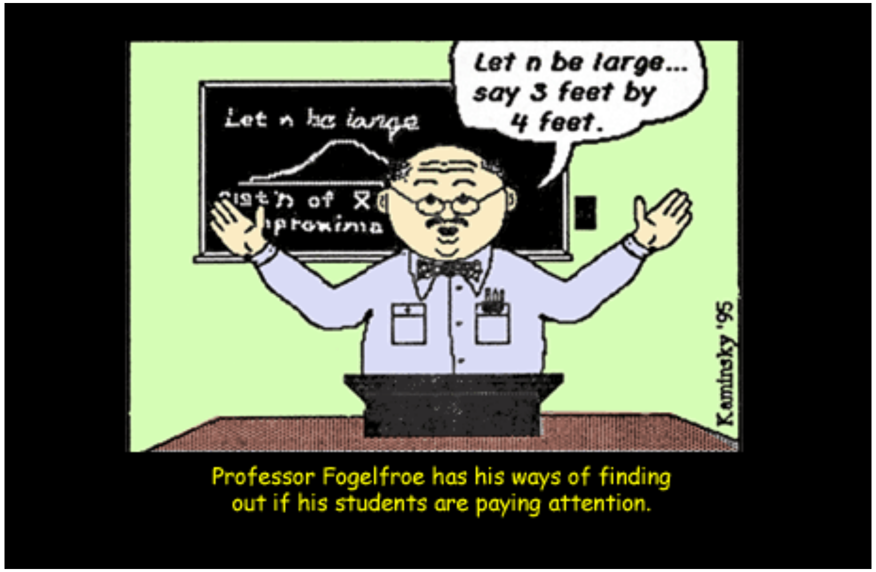
Alexander Knebe, *Universidad Autonoma de Madrid*



- forward
- N-bodies
- Boltzmann equation
- collisions
- magnetohydrodynamics
- summary

- **forward**
- N-bodies
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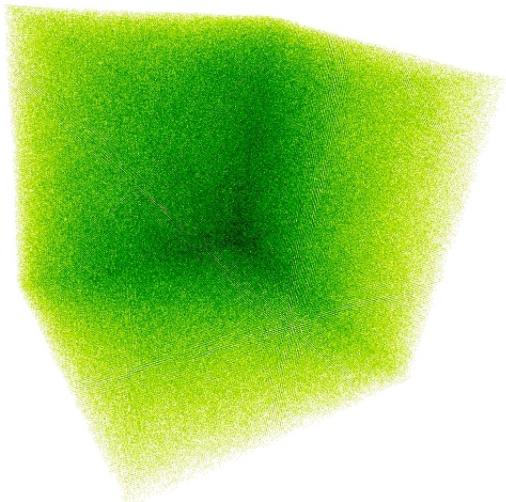
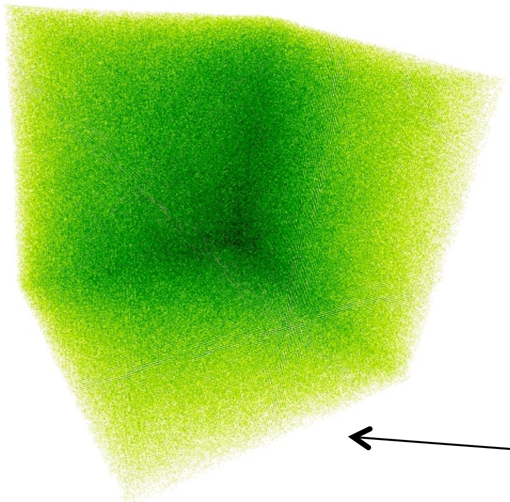
- simulation of cosmic structure formation

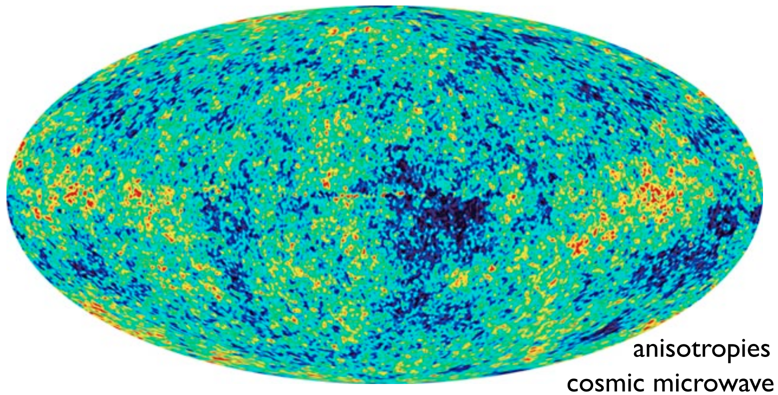
- simulation of cosmic structure formation – *initial conditions*



1. primordial matter density field

- simulation of cosmic structure formation – *initial conditions*



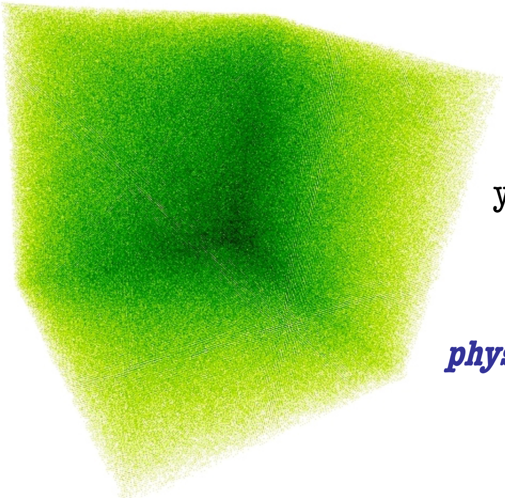


anisotropies in the cosmic microwave background

physics of "the early Universe"

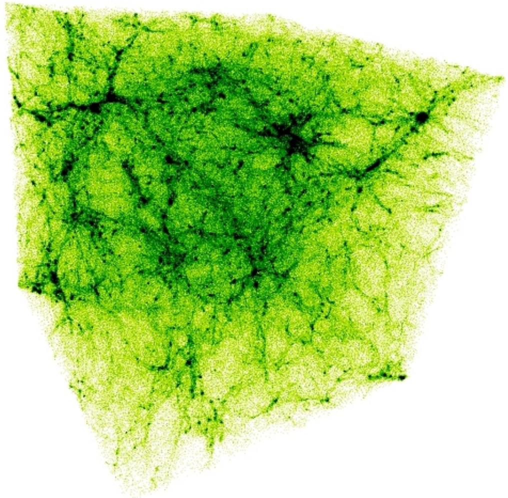
1. primordial matter density field

- simulation of cosmic structure formation – *temporal evolution*



your favourite simulation code →

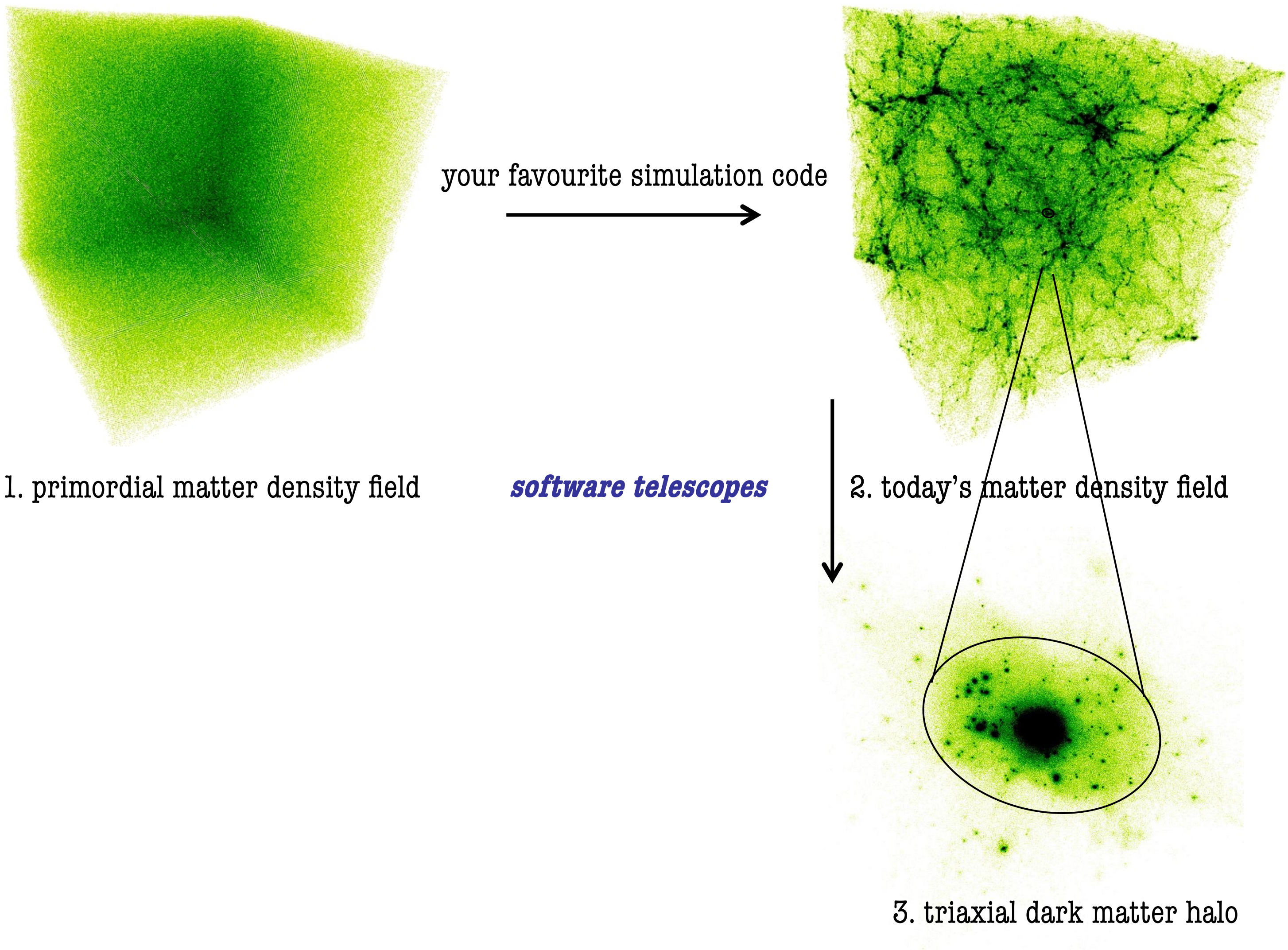
physics of "galaxy formation"



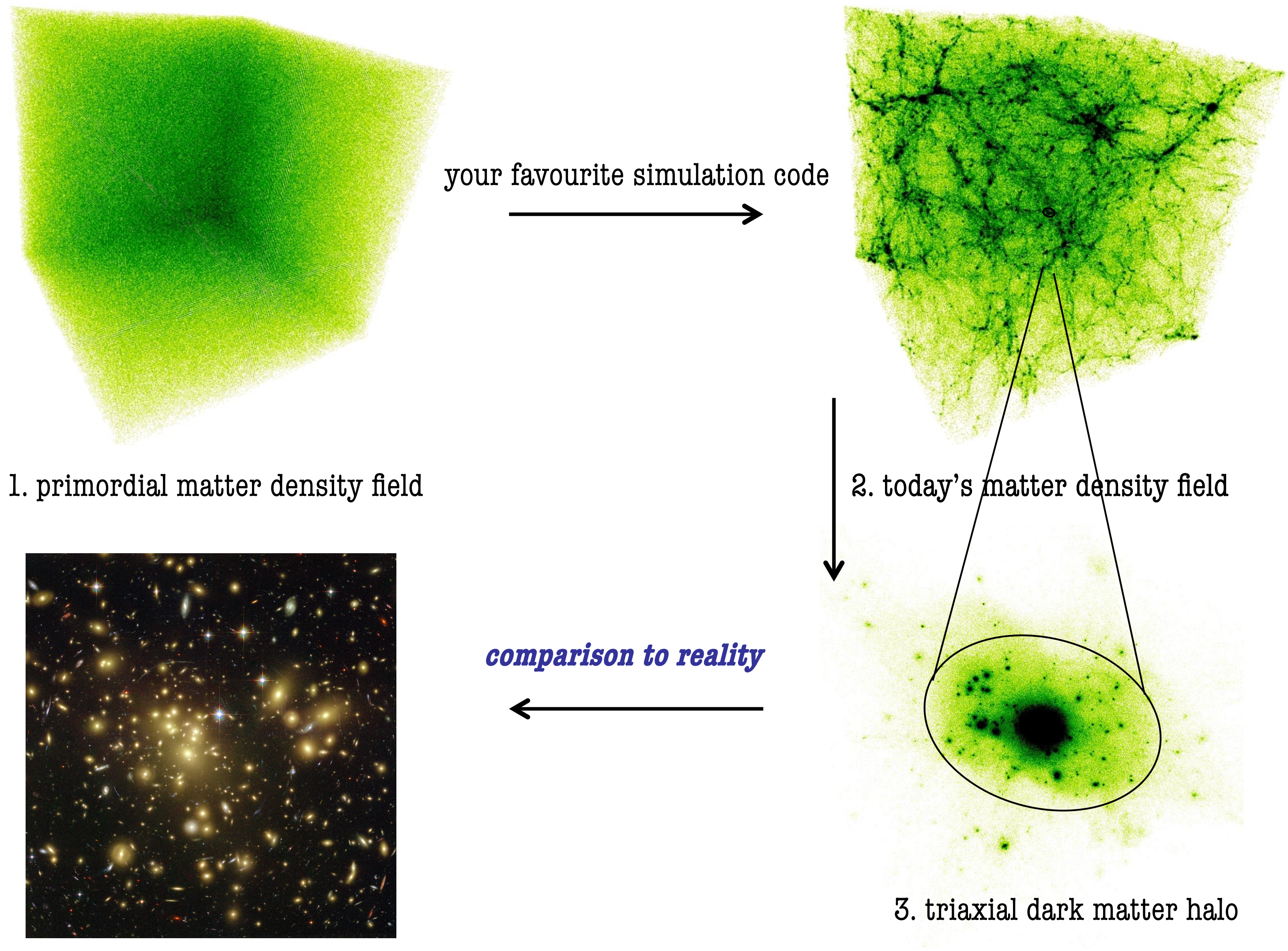
1. primordial matter density field

2. today's matter density field

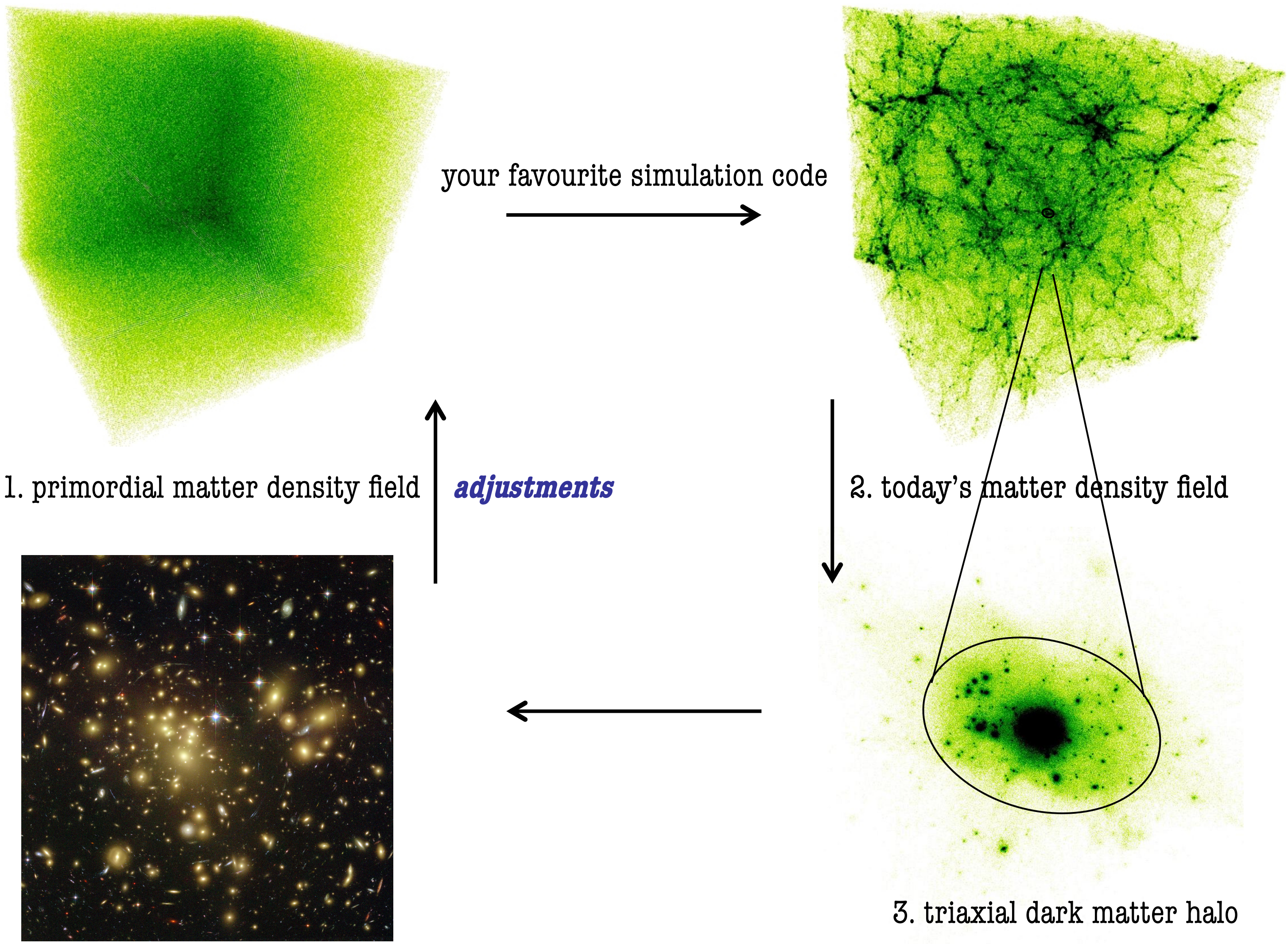
- simulation of cosmic structure formation – *analysis of outputs*



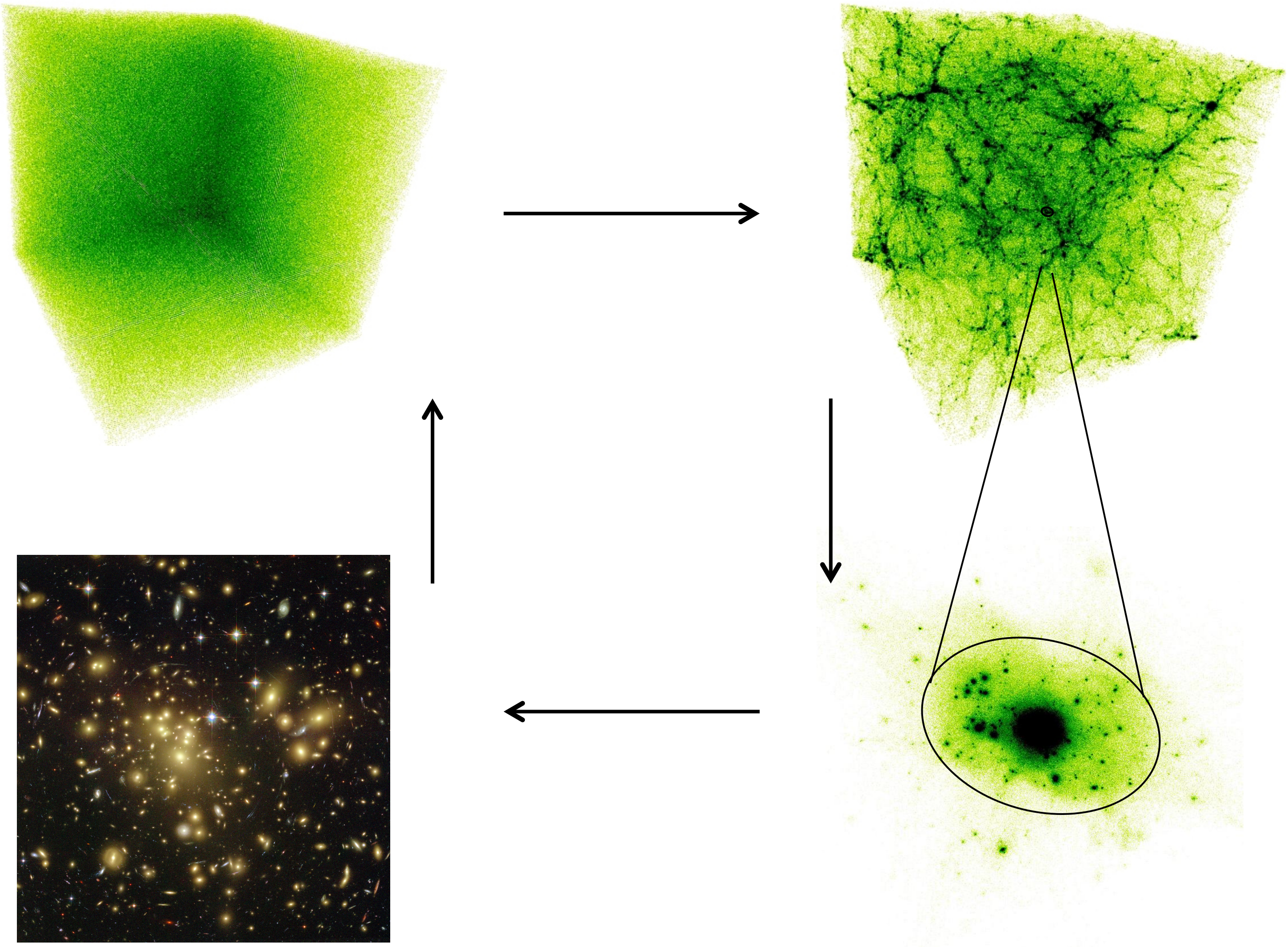
- simulation of cosmic structure formation – *analysis of outputs*



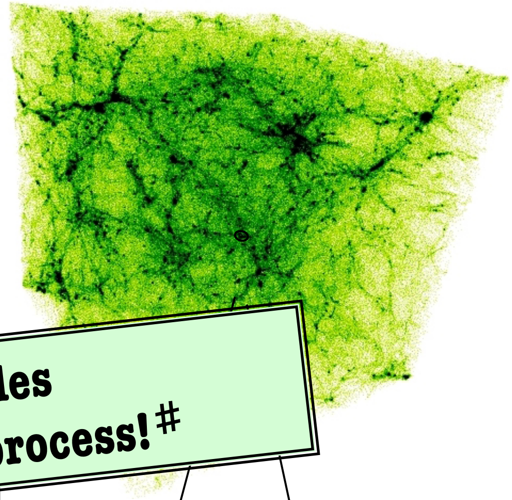
- simulation of cosmic structure formation – *feedback!?*



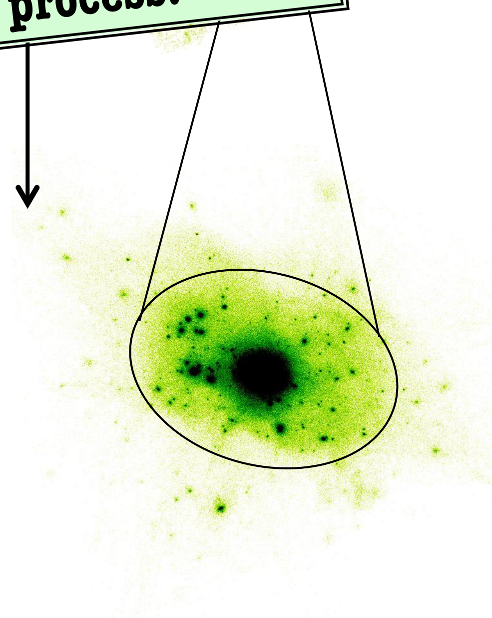
- simulation of cosmic structure formation

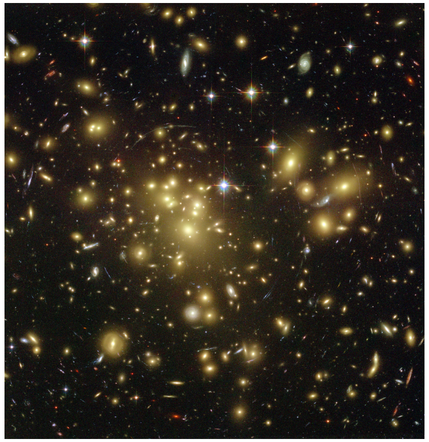


- simulation of cosmic structure formation



on most astrophysical scales
***gravity* is a dominant physical process!**#





#not a single stellar astrophysicist would agree on this...

- forward
- **N-bodies**
- Boltzmann equation
- collisions
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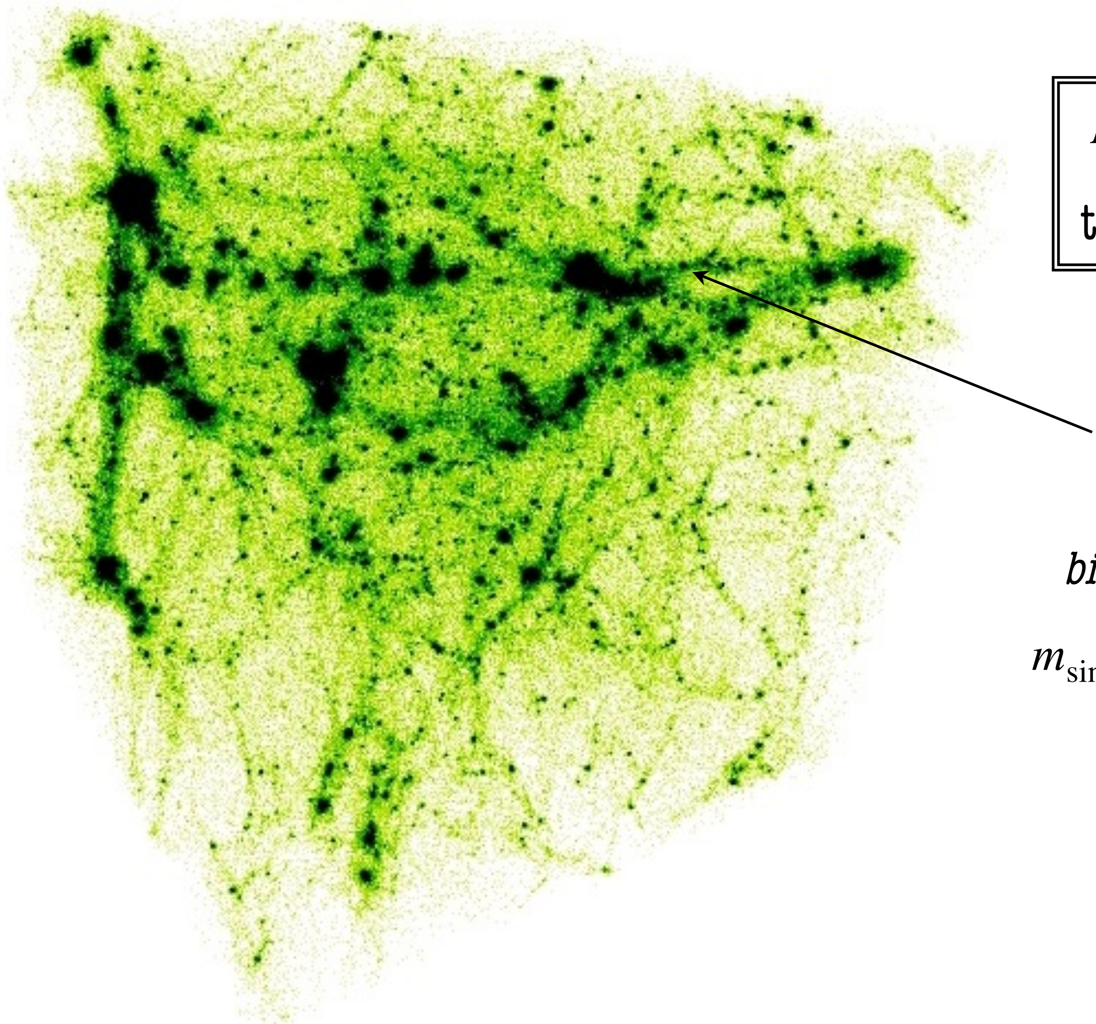
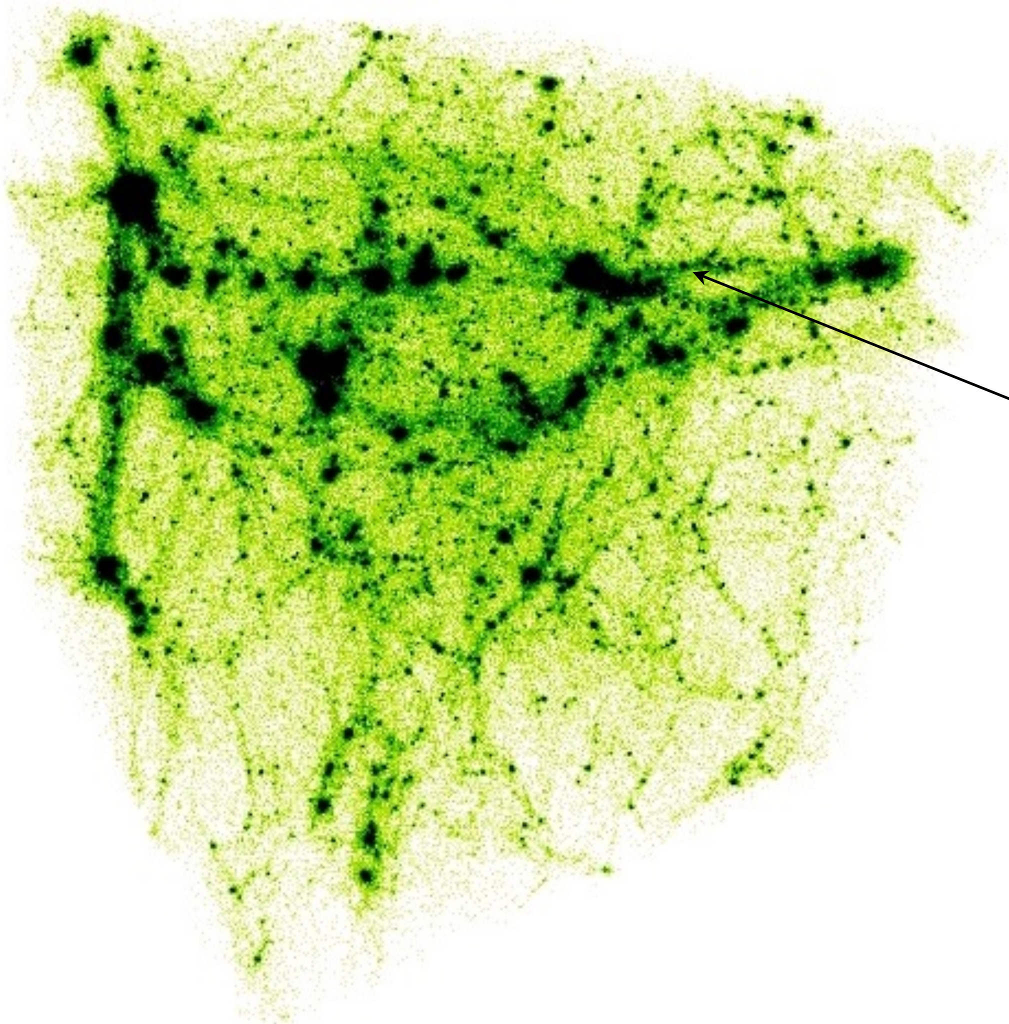


N bodies are used to sample the evolution of the Universe

one "simulation particle" represents *billions* of dark matter particles:

$m_{\text{simu}} \sim 10^7 M_\odot$ vs $m_{\text{DM}} << 10^{-27} M_\odot$



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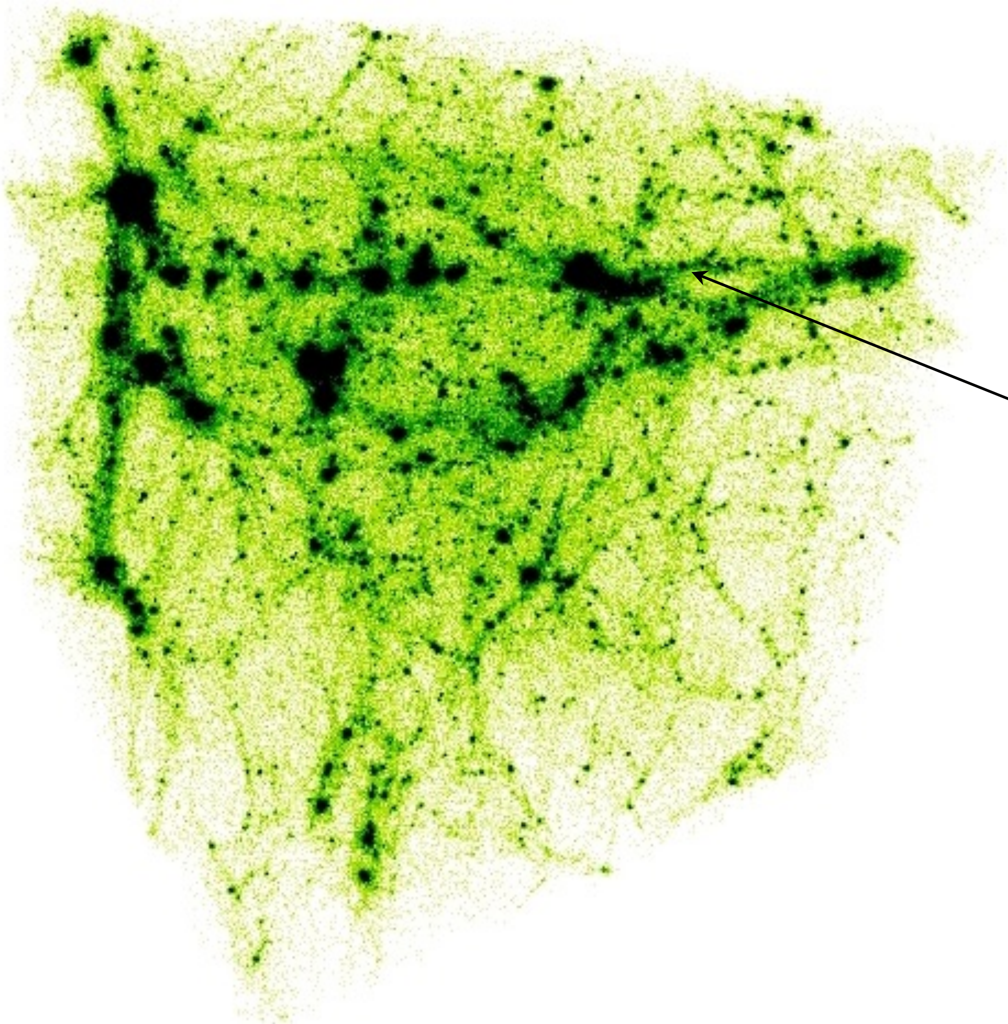
$$m_{\text{simu}} \sim 10^7 M_\odot \quad \text{vs} \quad m_{\text{DM}} << \underbrace{10^{-27}\, M_\odot}_{\approx 1\text{g}}$$

(non-baryonic) dark matter candidates

axion:	10^{-5} eV	
neutrino:	10eV	
WIMP:	$1\text{-}10^{3}$ GeV	
monopoles:	10^{16} GeV	
Planck relics:	10^{19} GeV	<< 1g
???		

$0.5\,MeV \approx 9 \cdot 10^{-28}\, g$



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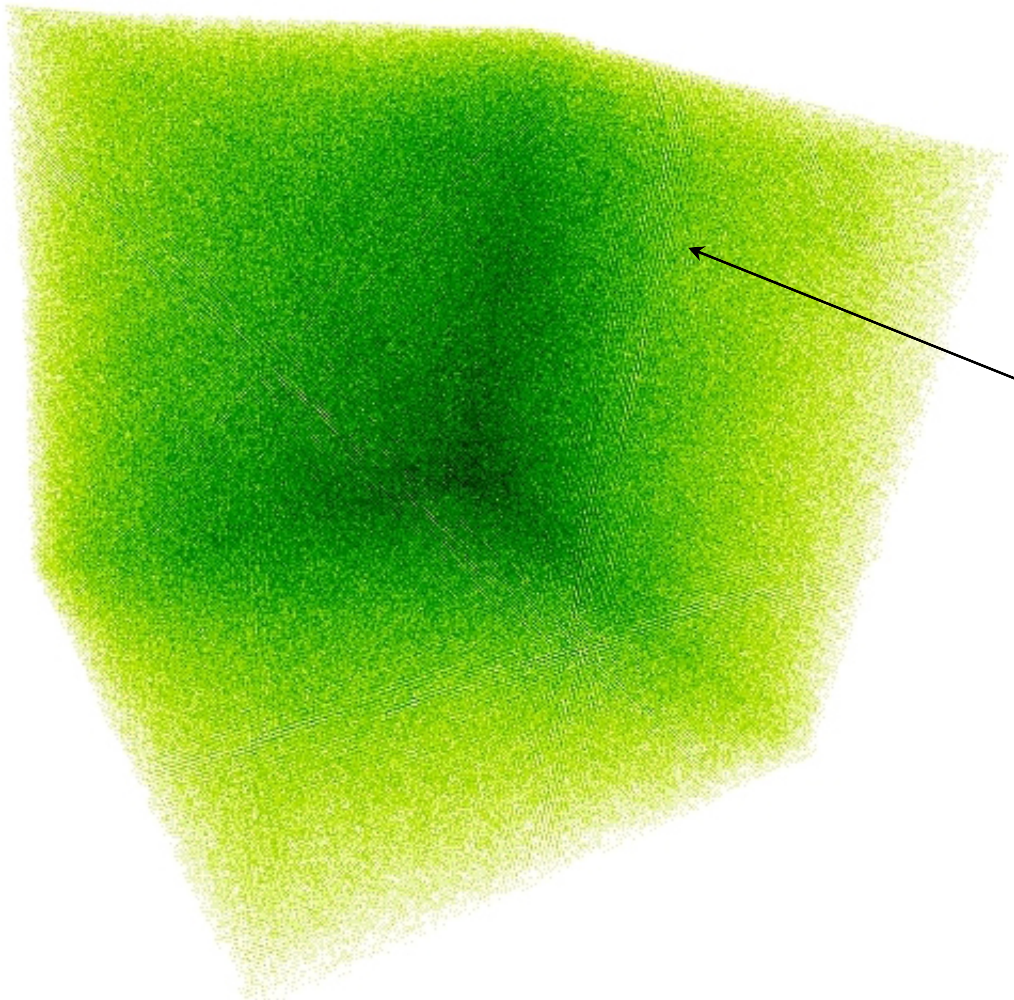
$$m_{\rm simu} \sim 10^7 M_\odot \quad \text{vs} \quad m_{\rm DM} << 10^{-27} M_\odot$$

$$\rho = \frac{N m_{simu}}{B^3} = \Omega_0 \rho_{crit,0} = \Omega_0 \frac{3H_0^2}{8\pi G}$$

$$\Rightarrow \quad m_{simu} = \Omega_0 \frac{3H_0^2}{8\pi G}\frac{B^3}{N}$$

$$B \approx 50\,Mpc\ , \quad N \approx 1024^3$$

$$\Rightarrow \quad m_{simu} \approx 10^7 M_\otimes$$



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one "simulation particle" represents *billions* of dark matter particles:

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how to obtain $\vec{r}(t_0)\,, \vec{v}(t_0)$?

("generating initial conditions" lecture...)

why N-bodies?

dark matter particles are collisionless!

dark matter particles are collisionless!

or in other words...

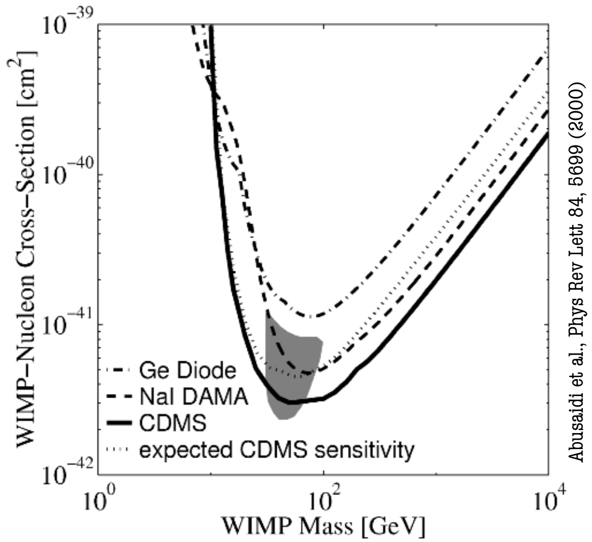
the evolution of the Universe is driven by the mean potential
rather than two-body interactions of dark matter particles

- mean free path of dark matter particles

$$\sigma \approx 10^{-42} cm^2$$

$$m_{DM} \approx 10^2 GeV \approx 10^{-22} g$$

$$\rho_{crit} \approx 10^{-30} \frac{g}{cm^3}$$



$$\rho_{crit} = \frac{N\, m_{DM}}{V} = n\; m_{DM}$$

$$n \approx 10^{-8} \frac{1}{cm^3}$$

$$\lambda = \frac{1}{n\,\sigma} \approx \frac{1}{10^{-8}\,10^{-42}} cm = 10^{50} cm \approx 10^{30} Mpc$$

how to describe a collisionless system?

- forward
- N-bodies
- **Boltzmann equation**
- collisions
- magnetohydrodynamics
- summary

- phase-space distribution function

$$f(\vec{r},\vec{v},t)\ d^3r\ d^3v$$

probability* of finding a dark matter particle in the interval:

$$[\vec{r}-\frac{d\vec{r}}{2},\ \vec{r}+\frac{d\vec{r}}{2}]$$
$$[\vec{v}-\frac{d\vec{v}}{2},\ \vec{v}+\frac{d\vec{v}}{2}]$$

* $\int f(\vec{r},\vec{v},t)\ d^3r\ d^3v = 1$

- phase-space distribution function

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example: particle with velocity v_1 and coordinate r_1: $f(\vec{r},\vec{v})=\delta(\vec{r}-\vec{r}_1)\delta(\vec{v}-\vec{v}_1)$

* $\int f(\vec{r},\vec{v},t)\ d^3r\ d^3v=1$

- phase-space distribution function

$$f(\vec{r},\vec{v},t)\ d^3r\ d^3v$$

probability* of finding a dark matter particle in the interval:

$$[\vec{r} - \frac{d\vec{r}}{\cdots}, \vec{\cdots}\ \frac{d\vec{r}}{\cdots}$$

we require a differential equation for *f* allowing us to compute its temporal evolution!

$$\frac{\cdots}{2}, v + \frac{\cdots}{2}]$$

example: particle with velocity v_1 and coordinate r_1: $f(\vec{r},\vec{v}) = \delta(\vec{r} - \vec{r}_1)\delta(\vec{v} - \vec{v}_1)$

$^* \int f(\vec{r},\vec{v},t)\ d^3r\ d^3v = 1$

- Boltzmann equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(v_i \frac{\partial f}{\partial r_i} - \frac{\partial \Phi}{\partial r_i}\frac{\partial f}{\partial v_i}\right) = \left(\frac{\delta f}{\delta t}\right)_c$$

- coupled with Poisson's equation

$$\Delta\Phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

- and the collisional integral

$$\left(\frac{\delta f}{\delta t}\right)_c = \int |\vec{v} - \vec{v}_2| \sigma(\Omega)\left[f(\vec{p}_2')f(\vec{p}') - f(\vec{p}_2)f(\vec{p})\right] d\Omega d^3 p_2$$

- Boltzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \left\{ f, H \right\}$$

$$\overbrace{\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(v_i \frac{\partial f}{\partial r_i} - \frac{\partial \Phi}{\partial r_i} \frac{\partial f}{\partial v_i} \right)} = \left(\frac{\delta f}{\delta t} \right)_c$$

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Monte-Carlo approach ➔ gravity equations

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moments ➔ MHD equations

- coupled with Poisson's equation

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use (collisionless) particles to sample *f(x,v,t)*

- coupled with Poisson's equation

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- and the collisional integral

$$\left(\frac{\delta f}{\delta t}\right)_c = \int |\vec{v} - \vec{v}_2| \sigma(\Omega)\left[f(\vec{p}_2')f(\vec{p}') - f(\vec{p}_2)f(\vec{p})\right] d\Omega d^3 p_2$$

- why particles?
  - astrophysics
    - solar system N ~ 10 planets
    - open clusters of stars N ~ 10-100 stars
    - globular clusters N ~ 10^6 stars
    - galaxies N ~ 10^{12} stars
    - clusters of galaxies N ~ 1000 galaxies
    - the universe N ~ 10^{34} dark matter particles
  - outside astrophysics
    - molecular dynamics
    - plasma physics
    - statistical mechanics
    - high-energy physics (accelerator simulation/beam physics)

- why particles?

  - astrophysics

    - solar system — N ~ 10 planets
    - open clusters of stars — N ~ 10-100 stars
    - globular clusters — N ~ 10^6 stars
    - galaxies — N ~ 10^{12} stars
    - clusters of galaxies — N ~ 1000 galaxies
    - the universe — N ~ 10^{34} dark matter particles

collisions are (supposed to be) un-important!

(the dynamics is dictated by the mean potential rather than close encounters...)

  - outside astrophysics

    - molecular dynamics
    - plasma physics
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    - high-energy physics (accelerator simulation/beam physics)

- why particles?
  - astrophysics]88
    - solar syster
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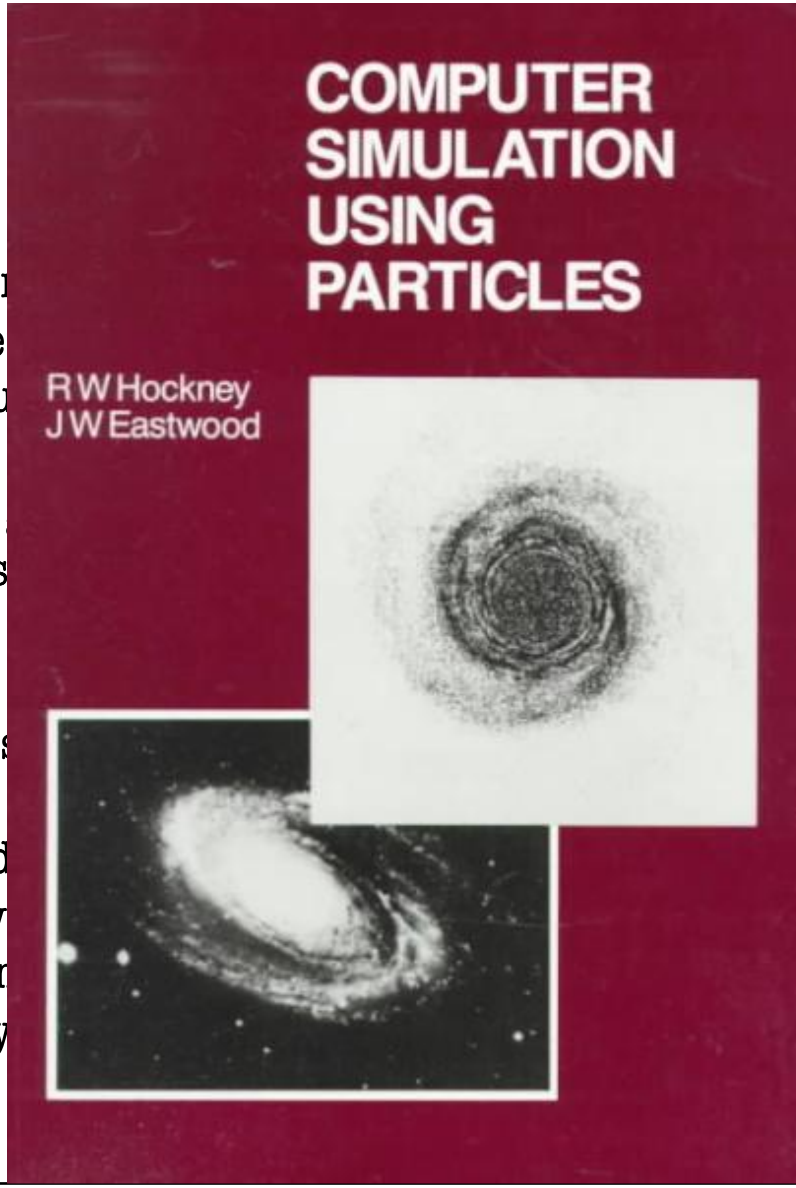
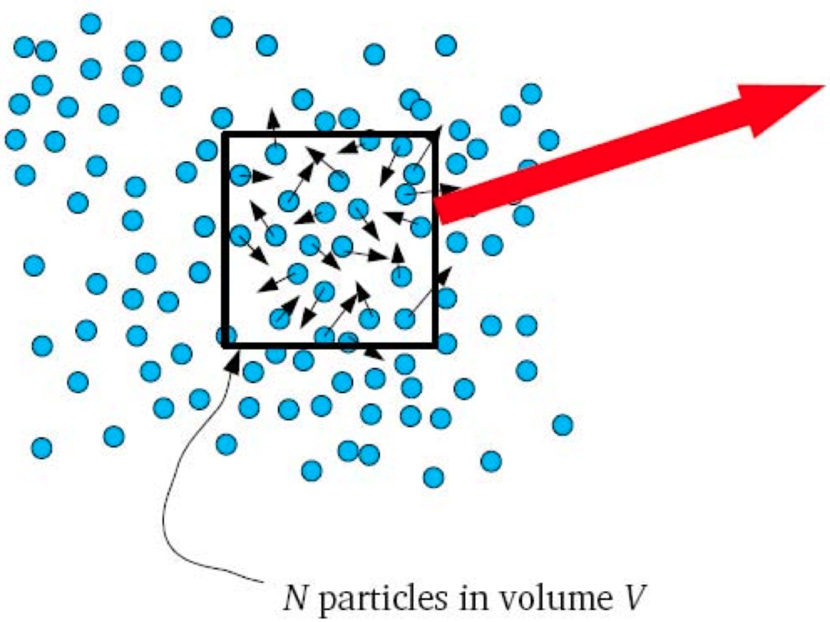


- particle representation
  - Monte Carlo sampling of phase-space distribution function



$$\frac{|f_N - f_{true}|}{|f_{true}|} \propto N^{-1/2}$$

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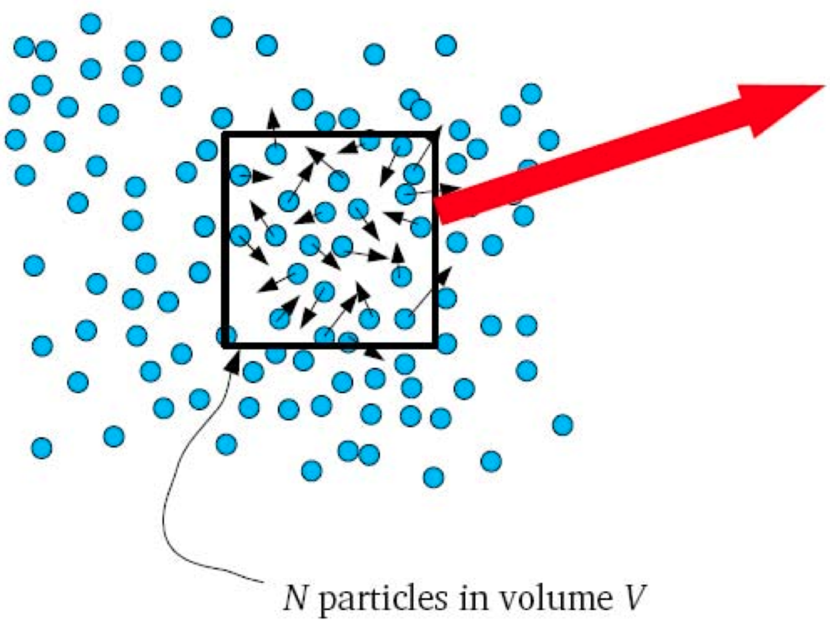
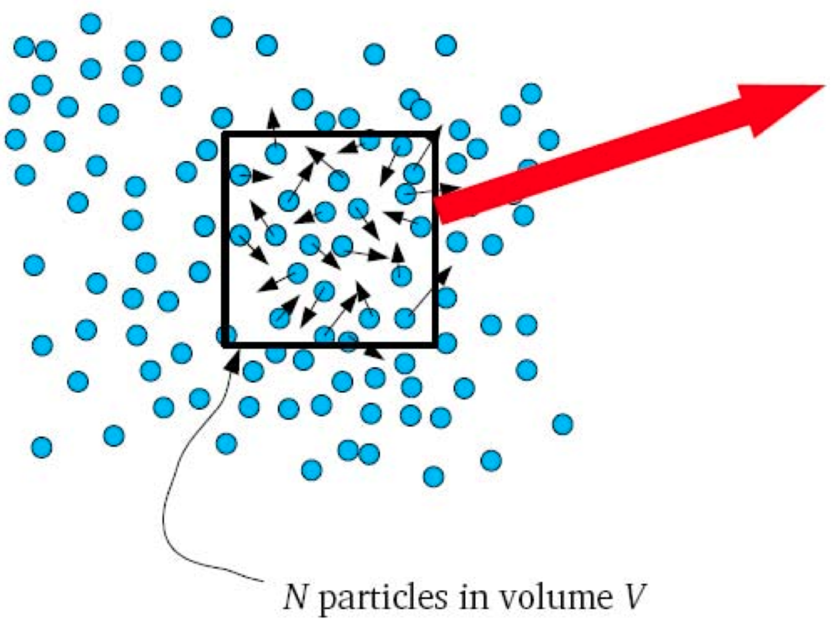


$$\frac{|f_N - f_{true}|}{|f_{true}|} \propto N^{-1/2}$$

"shot- noise": error goes to zero for large N

- particle representation
  - Monte Carlo sampling of phase-space distribution function

but how to solve for $f(r,v,t)$?



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- collisionless Boltzmann equation (CBE)

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(v_i \frac{\partial f}{\partial r_i} - \frac{\partial \Phi}{\partial r_i} \frac{\partial f}{\partial v_i} \right) = 0$$

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⇒ **impossible to solve numerically!**

- coupled with Poisson's equation

$$\Delta\Phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

⇒ **we'll deal with it later...**

- collisionless Boltzmann equation (CBE)

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(v_i \frac{\partial f}{\partial r_i} - \frac{\partial \Phi}{\partial r_i}\frac{\partial f}{\partial v_i}\right) = 0$$

$$\downarrow$$

- "method of characteristics":

$$\frac{\partial f}{\partial t} + \left\{ f, H \right\} = 0 \qquad H = \frac{1}{2}v^2 + \Phi(\vec{r})$$

$$\downarrow$$

$$\frac{df(\vec{r},\vec{v})}{dt} = 0$$

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f is constant along the possible trajectories $[\vec{r}(t),\vec{v}(t)]$

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↓

$$\frac{df(\vec{r},\vec{v})}{dt} = 0$$

↓

f is constant along the possible trajectories $[\vec{r}(t),\vec{v}(t)]$

$$f(\vec{r},\vec{v},t) = f(\vec{r}_0,\vec{v}_0,0) \quad \forall \vec{r},\vec{v} \text{ satisfying } \begin{array}{l} \{\vec{r},H\} = \dfrac{\partial H}{\partial \vec{v}} \\ \{\vec{v},H\} = -\dfrac{\partial H}{\partial \vec{r}} \end{array}$$

- "method of characteristics":

$$\frac{\partial f}{\partial t} + \left\{ f, H \right\} = 0 \qquad H = \frac{1}{2}v^2 + \Phi(\vec{r})$$

↓ solution to CBE

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↓

the problems "reduces" to finding $[r(t), v(t)]$ for a given initial value problem $f(r_0,v_0)$

- initial value problem

the initial values

$$f(\vec{r}(t_0),\vec{v}(t_0))$$

Hamiltonian of the system

$$H = \frac{1}{2}v^2 + \Phi(\vec{r})$$

the equations of motion

$$\{\vec{r},H\} = \frac{\partial H}{\partial \vec{v}}$$

$$\{\vec{v},H\} = -\frac{\partial H}{\partial \vec{r}}$$

- N-body approach

1. sample $f(r_i(t_0),v_i(t_0))$ with $i=1,\ldots,N$ points $[r_i(t_0),v_i(t_0)]$
2. those $[r_i(t),v_i(t)]$ obeying the equations-of-motion sample $f(r_i(t),v_i(t))$

- consistency check...

$$\{\vec{r},H\} = \frac{\partial H}{\partial \vec{v}} \qquad \{\vec{v},H\} = -\frac{\partial H}{\partial \vec{r}}$$

$$\xrightarrow{H=\frac{1}{2}v^2+\Phi(\vec{r})}$$

$$\frac{d\vec{r}}{dt} = \vec{v} \qquad \frac{d\vec{v}}{dt} = -\nabla\Phi$$

$$\xrightarrow{\vec{F}=-\nabla\Phi}$$

$$\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

- collisionless system of *N*-bodies
  - equations-of-motion

$$\frac{d\vec{r}}{dt} = \vec{v}$$
$$\frac{d\vec{v}}{dt} = -\nabla\Phi = \vec{F}(\vec{r},t)$$

  - the potential

$$\Delta\Phi = 4\pi G\rho$$

- collisionless system of *N*-bodies
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$$\left.\begin{aligned} \frac{d\vec{r}}{dt} &= \vec{v} \\ \frac{d\vec{v}}{dt} &= -\nabla\Phi = \vec{F}(\vec{r},t) \end{aligned}\right\} \text{leap-frog integration}$$

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  - the potential

$$\Delta\Phi = 4\pi G\rho \quad \Big\} \text{ solve by using...} \quad \begin{cases} \circ \text{ "particle" approach} \\ \circ \text{ "grid" approach} \end{cases}$$

- collisionless system of N-bodies
  - equations-of-motion

$$\left.\begin{aligned} \frac{d\vec{r}}{dt} &= \vec{v} \\ \frac{d\vec{v}}{dt} &= -\nabla\Phi = \vec{F}(\vec{r},t) \end{aligned}\right\} \text{leap-frog integration}$$

  - the potential

$$\left.\Delta\Phi = 4\pi G\rho \quad\right\} \text{“\textit{solving for gravity}” lectures}$$

- **collisionless** system of N-bodies

  but what about collisions?

  - equations-of-motion

$$\left.\begin{aligned}\frac{d\vec{r}}{dt} &= \vec{v}\\ \frac{d\vec{v}}{dt} &= -\nabla\Phi = \vec{F}(\vec{r},t)\end{aligned}\right\}\ \text{leap-frog integration}$$

  - the potential

$$\Delta\Phi = 4\pi G\rho \quad \Big\}\ \text{“\textit{solving for gravity}” lectures}$$

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    - are unwanted when modeling collisionless systems, e.g. dark matter
    - are part of the system when modeling, for instance, gases

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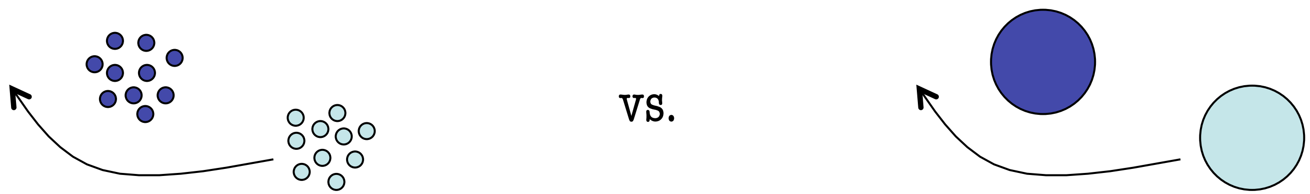
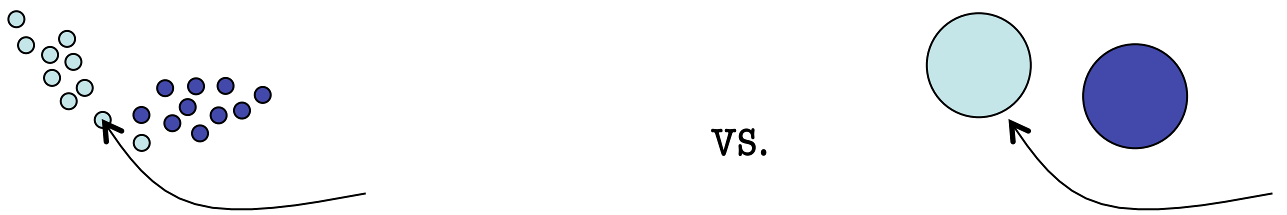
- ...may enter our experiment due to numerical problems!



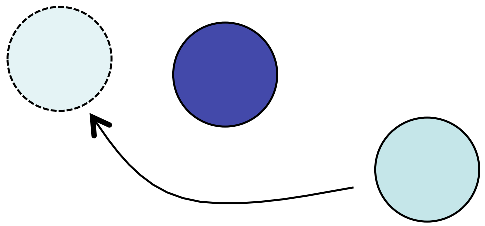
well sampled system *undersampled system*

- particle collisions...
    - are unwanted when modeling collisionless systems, e.g. dark matter
    - are part of the system when modeling, for instance, gases
- ...may enter our experiment due to numerical problems!

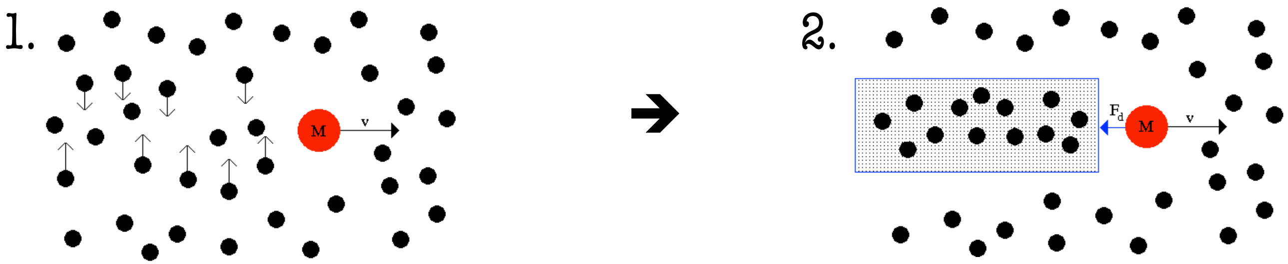


well sampled system vs. *undersampled system*

- particle collisions...
  - two-body collisions (acceleration)



  - dynamical friction (braking)



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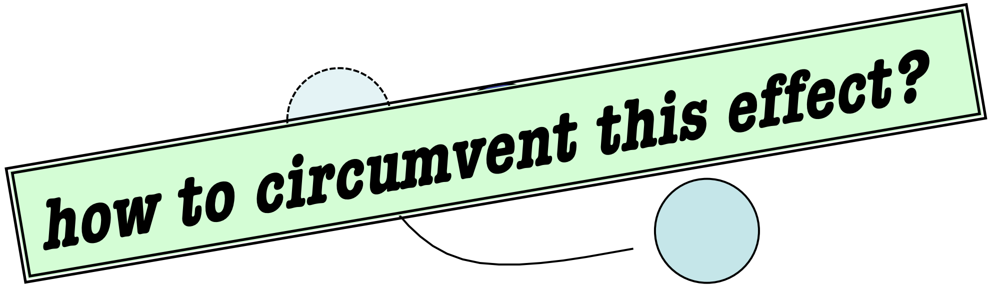

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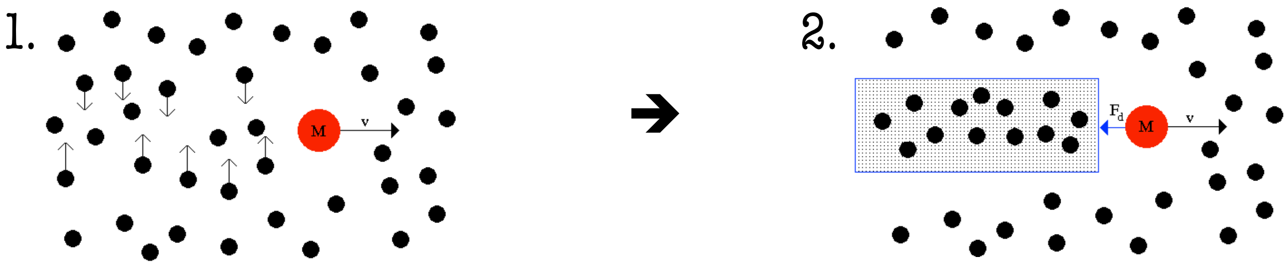
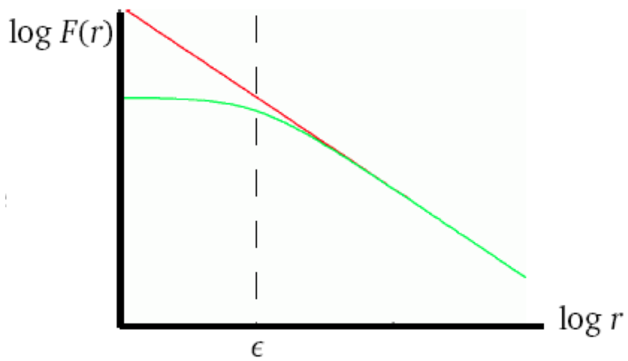


- avoiding particle collisions...
  - larger N
    - as high as the computer at your disposal allows!
  - softened force law ε
    - $F(r) = -G\frac{m_1 m_2}{r^2 + \varepsilon^2}$



  - smoothed potential
    - e.g., low-order multipole expansion for spherical systems

- avoiding particle collisions…
  - interplay between N and ε
    - N=const., ε=↘ more unphysical two-body collisions
    - N=↗, ε=const. minimising collisions, but no gain in spatial resolution
    - increase N **and** decrease ε according to $N\varepsilon^3$=const.

- avoiding particle collisions...

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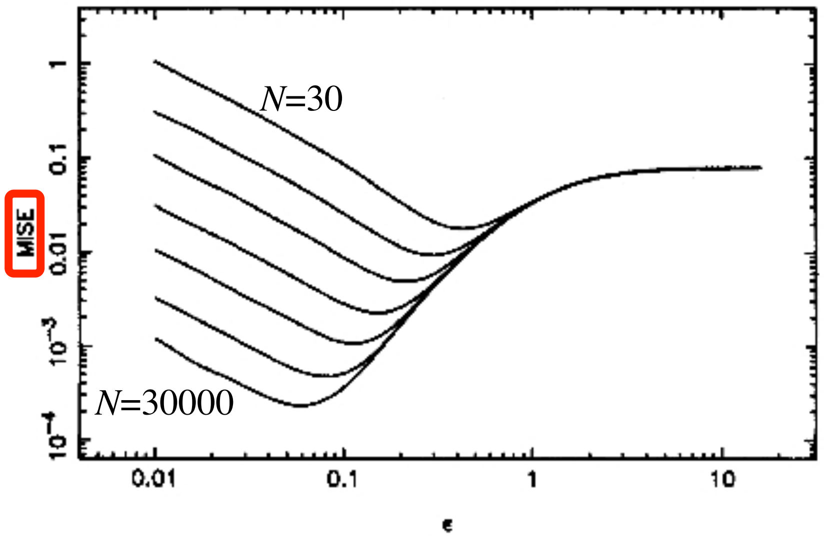
some error measurement...



Merritt, ApJ 111, 2462 (1996)

- avoiding particle collisions...

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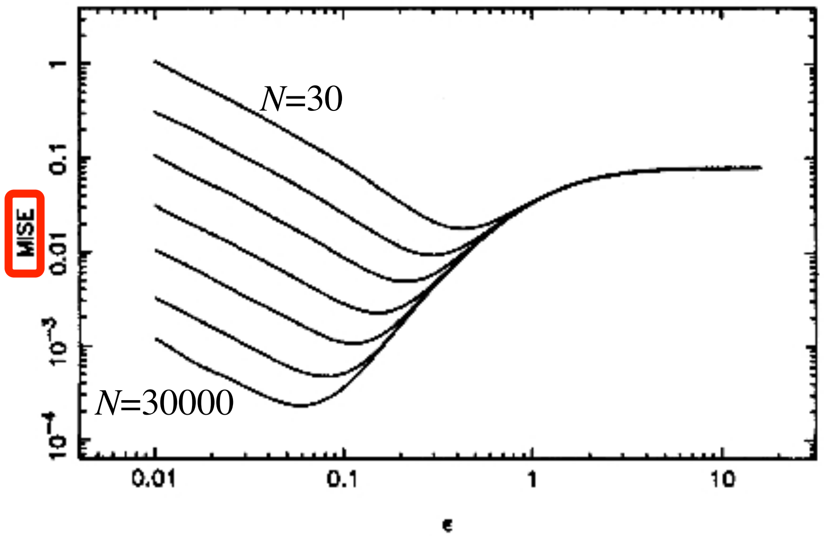

Merritt, ApJ 111, 2462 (1996)

*for cosmological simulations the const. ≈ $(B/30)^3$ where B is the size of the domain

- forward
- N-bodies
- Boltzmann equation
- collisions
- **magnetohydrodynamics**
- summary

- Boltzmann equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(v_i \frac{\partial f}{\partial r_i} - \frac{\partial \Phi}{\partial r_i}\frac{\partial f}{\partial v_i} \right) = \left(\frac{\delta f}{\delta t} \right)_c$$

- coupled with Poisson's equation

$$\Delta\Phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

- and the collisional integral

$$\left(\frac{\delta f}{\delta t} \right)_c = \int \left| \vec{v} - \vec{v}_2 \right| \sigma(\Omega) \left[f(\vec{p}_2')f(\vec{p}') - f(\vec{p}_2)f(\vec{p}) \right] d\Omega d^3 p_2$$

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moments ➔ MHD equations

- coupled with Poisson's equation

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- moments of the distribution function

$$n \equiv \int f(x,v) d^3 v \qquad \langle \chi \rangle \equiv \frac{1}{n} \int \chi f(x,v) d^3 v$$

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$$\Downarrow \quad \int \left[\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(v_i \frac{\partial f}{\partial r_i} - \frac{\partial \Phi}{\partial r_i}\frac{\partial f}{\partial v_i} \right) = \left(\frac{\delta f}{\delta t} \right)_c \right] \chi \, d^3v$$

$$\frac{\partial}{\partial t}\left(n\langle \chi \rangle\right) + \nabla_x \cdot \left(n\langle v\chi \rangle\right) + n\nabla_x\Phi \cdot \langle \nabla_v \chi \rangle = 0$$

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$$n \equiv \int f(x,v)\,d^3v \qquad \langle \chi \rangle \equiv \frac{1}{n}\int \chi\, f(x,v)\,d^3v$$

$$\Downarrow$$

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$$\Downarrow$$

- $\chi = m$: $$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\vec{v}\right) = 0$$

- $\chi = mv$: $$\frac{\partial\left(\rho\vec{v}\right)}{\partial t} + \nabla\cdot\left(\rho\vec{v}\otimes\vec{v} + p\vec{1}\right) = \rho\left(-\nabla\Phi\right)$$

- $\chi = mv^2/2$: $$\frac{\partial\left(\rho E\right)}{\partial t} + \nabla\cdot\left(\left[\rho E + p\right]\vec{v}\right) = \rho\vec{v}\cdot\left(-\nabla\phi\right) + \left(\Gamma - L\right)$$

- full set of MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial (\rho \vec{v})}{\partial t} + \nabla \cdot \left(\rho \vec{v} \otimes \vec{v} + \left(p + \frac{1}{2\mu} B^2 \right) \overleftrightarrow{1} - \frac{1}{\mu} \vec{B} \otimes \vec{B} \right) = \rho \ (-\nabla \phi)$$

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(+ Poisson's & Maxwell's equations...)

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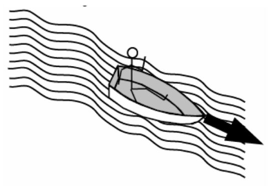
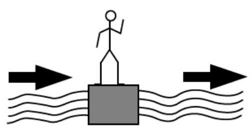
- Lagrangian viewpoint:  (particle approach)

- Eulerian viewpoint:  (grid approach)

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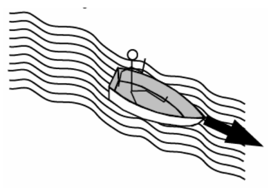
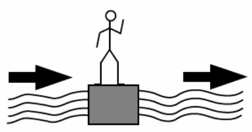
(+ Poisson's & Maxwell's equations...)

popular when MHD is coupled to a collisionless component, e.g. dark matter physics

• Lagrangian viewpoint:  (**particle approach**)

• Eulerian viewpoint:  (grid approach)

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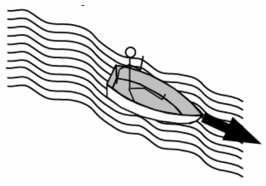
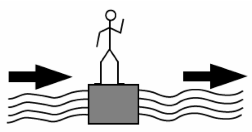
• Lagrangian viewpoint:  (particle approach)

standard approach for decades across all disciplines...

• Eulerian viewpoint:  (**grid approach**)

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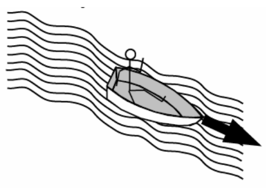
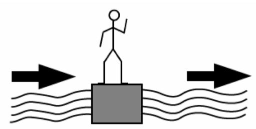
- Lagrangian viewpoint:  (**particle approach**)

both have their advantages and drawbacks...

- Eulerian viewpoint:  (**grid approach**)

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(+ Poisson's & Maxwell's equations...)

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(+ Poisson's equations...)

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(+ Poisson's equations...)

more unknowns than equations!

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(+ Poisson's equations...)

- "closure" equation(s)

$$p = (\gamma - 1)\rho \varepsilon \qquad \rho \varepsilon = \rho E - \frac{1}{2}\rho v^2$$

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(+ Poisson's equations...)

still a lot of physics to model and include!

- "closure" equation(s)

$$p = (\gamma - 1)\rho\varepsilon \qquad \rho\varepsilon = \rho E - \frac{1}{2}\rho v^2$$

- forward
- N-bodies
- Boltzmann equation
- collisions
- magnetohydrodynamics
- **summary**

- Boltzmann equation

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- Boltzmann equation

casting the Boltzmann equation into integrateable form for collisionless and collisional matter...

- coupled with Poisson's equation

$$\Delta\Phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

- and the collisional integral

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- full set of equations

  - collisionless matter (e.g. dark matter)

$$\frac{d\vec{x}_{DM}}{dt} = \vec{v}_{DM}$$
$$\frac{d\vec{v}_{DM}}{dt} = -\nabla\phi$$

$$\Delta\phi = 4\pi G\rho_{tot}$$

  - collisional matter (e.g. gas)

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \left(\vec{v} \times \vec{B}\right)$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\vec{v}\right) = 0$$

$$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot\left(\rho\vec{v}\otimes\vec{v} + \left(p + \frac{1}{2\mu}B^2\right)\vec{\vec{1}} - \frac{1}{\mu}\vec{B}\otimes\vec{B}\right) = \rho\ \left(-\nabla\phi\right)$$

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- full set of equations

  - collisionless matter (e.g. dark matter)

$$\frac{d\vec{x}_{DM}}{dt} = \vec{v}_{DM}$$
$$\frac{d\vec{v}_{DM}}{dt} = -\nabla\phi$$

applying the "method of characteristics" to the collisionless Boltzmann equation

$$\Delta\phi = 4\pi G\rho_{tot}$$

  - collisional matter (e.g. gas)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\vec{v}\right) = 0$$

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$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\left(\vec{v}\times\vec{B}\right)$$

taking moments of the collisional Boltzmann equation

$$p = (\gamma - 1)\rho\varepsilon$$

$$\rho\varepsilon = \rho E - \frac{1}{2}\rho v^2$$

- full set of equations

  - collisionless matter (e.g. dark matter)

$$\frac{d\vec{x}_{DM}}{dt} = \vec{v}_{DM}$$

$$\frac{d\vec{v}_{DM}}{dt} = -\nabla\phi$$

$$\Delta\phi = 4\pi G\rho_{tot}$$

  - collisional matt

and how to solve all these coupled equations?

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho \quad\quad = 0\right.$$

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\left(\vec{v}\times\vec{B}\right)$$

$$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot\left(\rho\vec{v}\otimes\vec{v} + \left(p + \frac{1}{2\mu}B^2\right)\vec{\vec{1}} - \frac{1}{\mu}\vec{B}\otimes\vec{B}\right) = \rho\ (-\nabla\phi)$$

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- full set of equations

  - collisionless matter (e.g. dark matter)

$$\frac{d\vec{x}_{DM}}{dt} = \vec{v}_{DM}$$
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leap-frog integration using N-body approach (cf. *Review of Numerical Methods* Lecture)

$$\Delta\phi = 4\pi G\rho_{tot}$$

  - collisional matter (e.g. gas)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times(\vec{v}\times\vec{B})$$

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$$\rho\varepsilon = \rho E - \frac{1}{2}\rho v^2$$

either "particle" or "grid" approach (Lagrangian vs. Eulerian viewpoint, cf. *Gas Dynamics* Lecture)

- full set of equations

  - collisionless matter (e.g. dark matter)

$$\frac{d\vec{x}_{DM}}{dt} = \vec{v}_{DM}$$
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either "particle" or "grid" approach
(cf. *Solving for Gravity* Lectures)

$$\Delta\phi = 4\pi G\rho_{tot}$$

  - collisional matter (e.g. gas)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

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- full set of equations

  - collisionless matter (e.g. dark matter)

$$\frac{d\vec{x}_{DM}}{dt} = \vec{v}_{DM}$$
$$\frac{d\vec{v}_{DM}}{dt} = -\nabla\phi$$

$$\Delta\phi = 4\pi G\rho_{tot}$$

  - collisional matter (e.g. gas)

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\left(\vec{v}\times\vec{B}\right)$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\vec{v}\right) = 0$$

$$\frac{\partial\left(\rho\vec{v}\right)}{\partial t} + \nabla\cdot\left(\rho\vec{v}\otimes\vec{v} + \left(p + \frac{1}{2\mu}B^2\right)\vec{1} - \frac{1}{\mu}\vec{B}\otimes\vec{B}\right) = \rho\ \left(-\nabla\phi\right)$$

either "particle" or "grid" approach (cf. *Magnetohydrodynamics* Lecture)

$$\frac{\partial\left(\rho E\right)}{\partial t} + \nabla\cdot\left(\left[\rho E + p + \frac{1}{2\mu}B^2\right]\vec{v} - \frac{1}{\mu}\left[\vec{v}\cdot\vec{B}\right]\vec{B}\right) = \rho\vec{v}\cdot\left(-\nabla\phi\right) + \left(\Gamma - L\right)$$

$$p = \left(\gamma - 1\right)\rho\varepsilon$$

$$\rho\varepsilon = \rho E - \frac{1}{2}\rho v^2$$

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