Alexander Knebe, *Universidad Autonoma de Madrid*

Adaptive Mesh Refinement

- Poisson's equation

$$\Delta\Phi(\vec{x}) = 4\pi G\rho(\vec{x})$$

- Poisson's equation

$$\vec{F}(\vec{x}) = -m\nabla\Phi(\vec{x})$$

$$\Delta\Phi(\vec{x}) = 4\pi G\rho(\vec{x})$$

<u>particle approach</u>

$$\vec{F}(\vec{x}_i) = -\sum_{i\neq j} \frac{Gm_i m_j}{(x_i - x_j)^3}(\vec{x}_i - \vec{x}_j)$$

<u>grid approach</u> ($\vec{x}_{i,j,k}$ =position of centre of grid cell (i,j,k))

$$\Delta\Phi(\vec{x}_{i,j,k}) = 4\pi G\rho(\vec{x}_{i,j,k})$$

$$\vec{F}(\vec{x}_{i,j,k}) = -m\nabla\Phi(\vec{x}_{i,j,k})$$

- Poisson's equation

$$\vec{F}(\vec{x}) = -m\nabla\Phi(\vec{x})$$

$$\Delta\Phi(\vec{x}) = 4\pi G\rho(\vec{x})$$

weapon of choice: tree codes

particle approach

$$\vec{F}(\vec{x}_i) = -\sum_{i\neq j} \frac{Gm_i m_j}{(x_i - x_j)^3}(\vec{x}_i - \vec{x}_j)$$

grid approach ($\vec{x}_{i,j,k}$ =position of centre of grid cell (i,j,k))

$$\Delta\Phi(\vec{x}_{i,j,k}) = 4\pi G\rho(\vec{x}_{i,j,k})$$

$$\vec{F}(\vec{x}_{i,j,k}) = -m\nabla\Phi(\vec{x}_{i,j,k})$$

- Poisson's equation

$$\vec{F}(\vec{x}) = -m\nabla\Phi(\vec{x})$$

$$\Delta\Phi(\vec{x}) = 4\pi G\rho(\vec{x})$$

particle approach

$$\vec{F}(\vec{x}_i) = -\sum_{i\neq j}\frac{Gm_i m_j}{(x_i - x_j)^3}(\vec{x}_i - \vec{x}_j)$$

grid approach ($\vec{x}_{i,j,k}$ =position of centre of grid cell (i,j,k))

$$\Delta\Phi(\vec{x}_{i,j,k}) = 4\pi G\rho(\vec{x}_{i,j,k})$$

$$\vec{F}(\vec{x}_{i,j,k}) = -m\nabla\Phi(\vec{x}_{i,j,k})$$

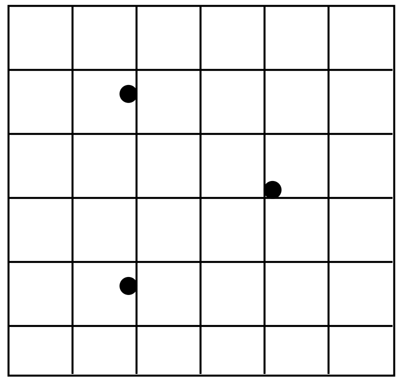
weapon of choice: AMR codes

Solving for Gravity

- Particle-Mesh (PM) method

$$\Delta\Phi(\vec{g}_{k,l,m}) = 4\pi G\rho(\vec{g}_{k,l,m})$$

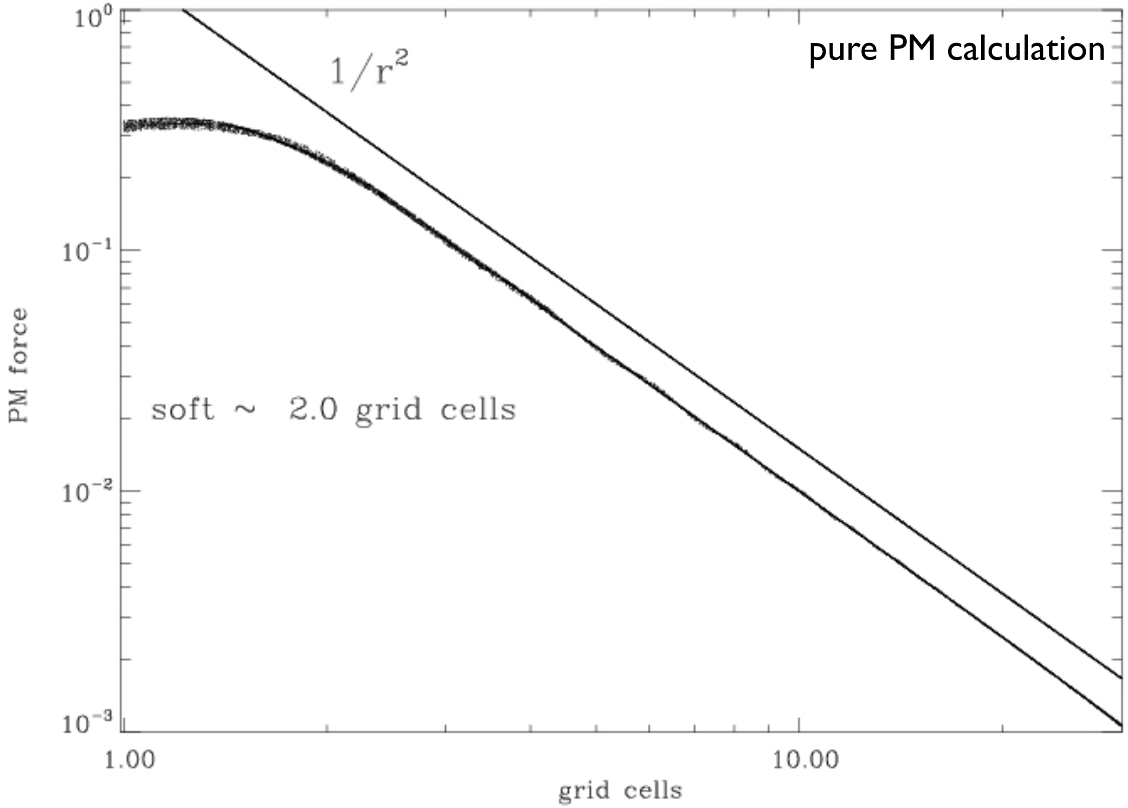
$$\vec{F}(\vec{g}_{k,l,m}) = -m\nabla\Phi(\vec{g}_{k,l,m})$$



1. calculate mass density on grid $\vec{x}_i \rightarrow \rho(\vec{g}_{k,l,m})$
2. solve Poisson's equation on grid $\Phi(\vec{g}_{k,l,m})$
3. differentiate potential to get forces $\vec{F}(\vec{g}_{k,l,m})$
4. interpolate forces back to particles $\vec{F}(\vec{g}_{k,l,m}) \rightarrow \vec{F}(\vec{x}_i)$

Solving for Gravity

- numerically integrate Poisson's equation



Solving for Gravity

- numerically integrate Poisson's equation

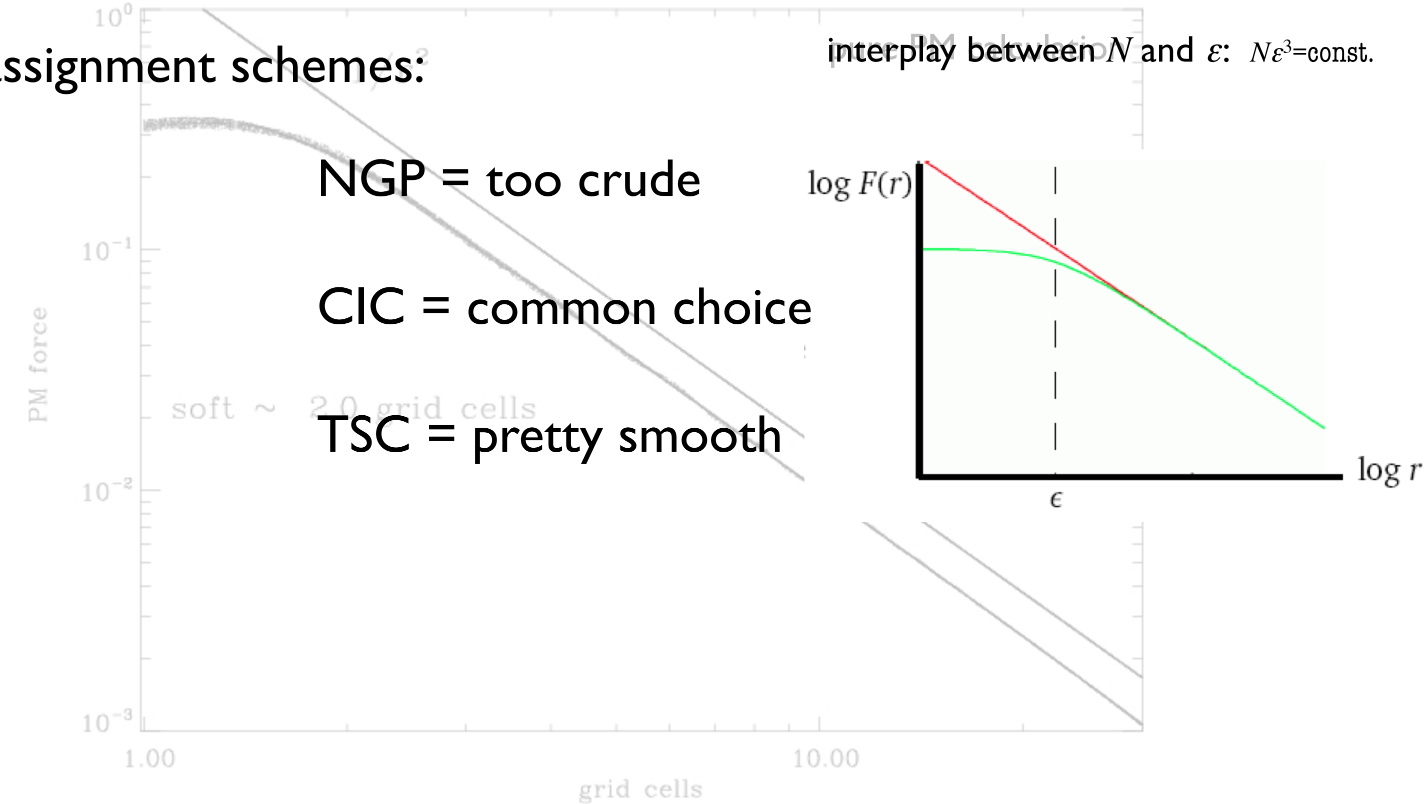
- density assignment schemes:

NGP = too crude

CIC = common choice

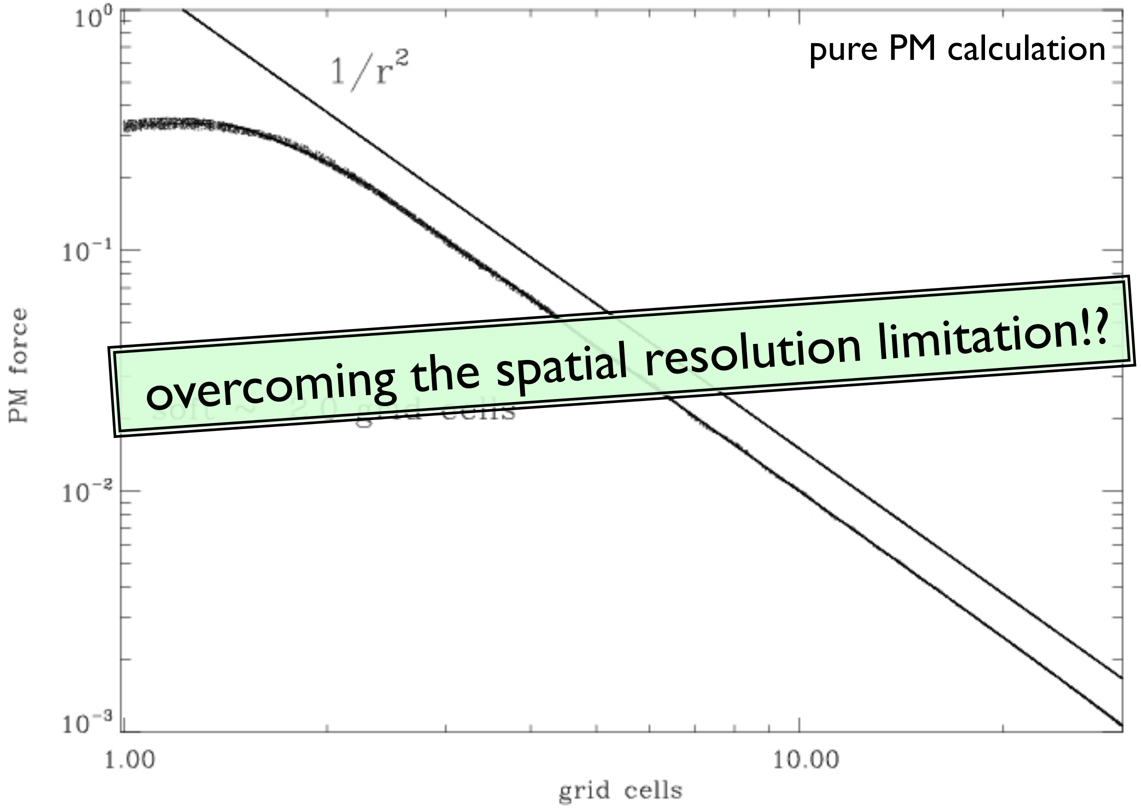
TSC = pretty smooth

interplay between N and ε: $N\varepsilon^3$=const.



Solving for Gravity

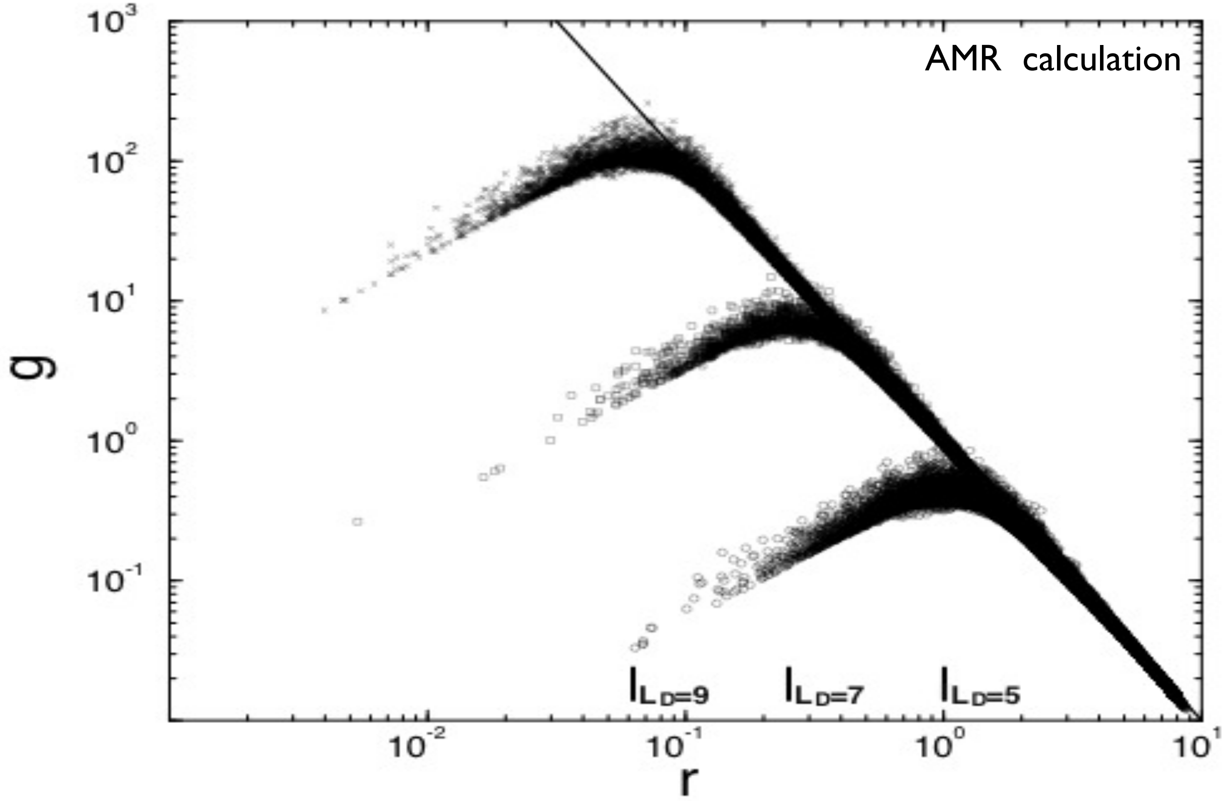
- numerically integrate Poisson's equation



overcoming the spatial resolution limitation!?

Solving for Gravity

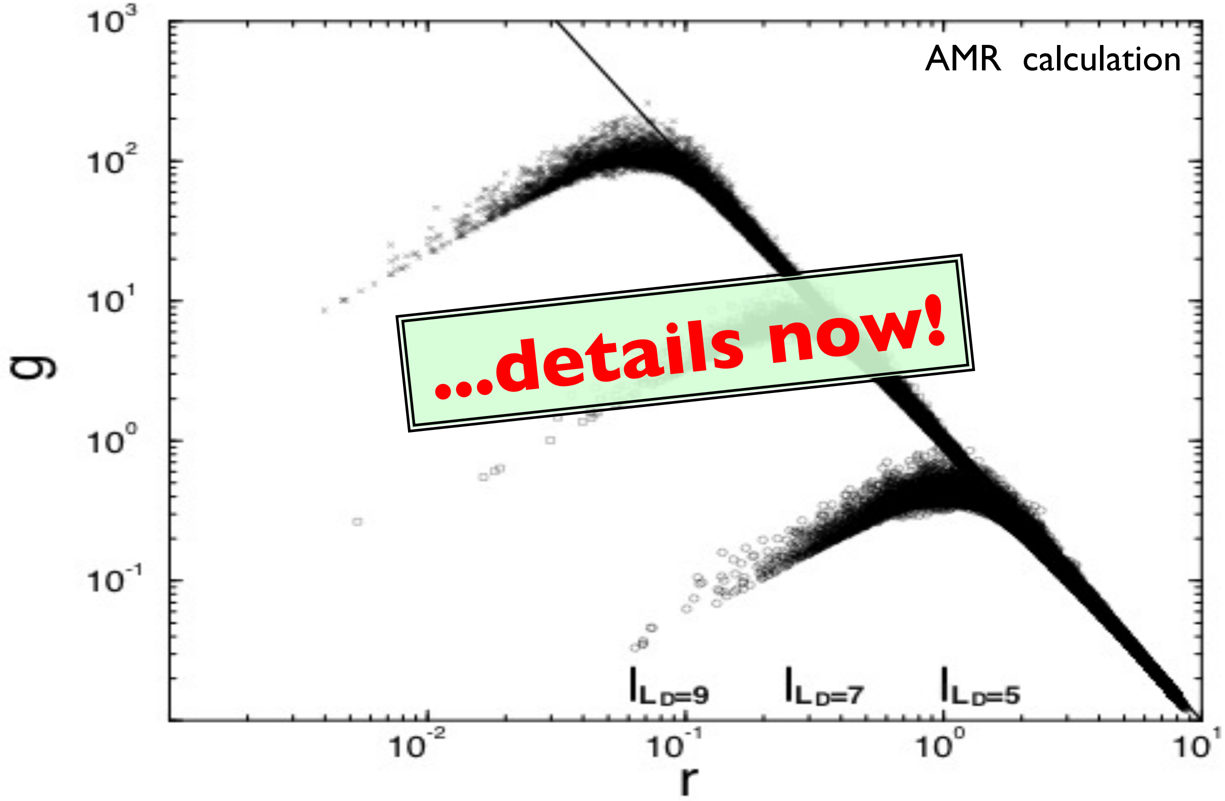
- numerically integrate Poisson's equation



Yahagi & Yoshi (2001)

Solving for Gravity

- numerically integrate Poisson's equation



Yahagi & Yoshi (2001)

- mesh refinements
- adaptive mesh refinement
- adaptive mesh refinement for N-body codes
- handling irregular grids
- adaptive leap-frog integration

- **mesh refinements**
- adaptive mesh refinement
- adaptive mesh refinement for N-body codes
- handling irregular grids
- adaptive leap-frog integration

- types of mesh refinement

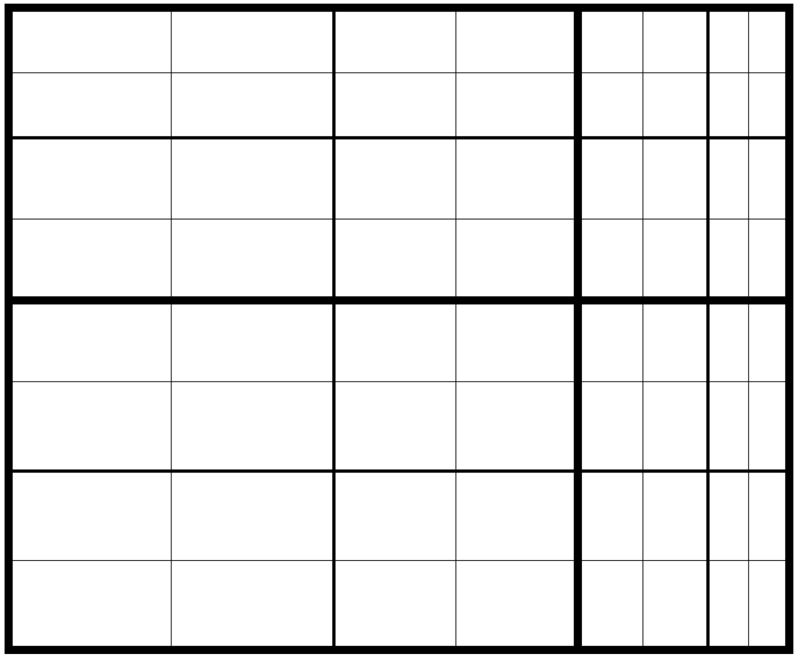
  - *r* refinement: move or stretch the mesh
  - *p* refinement: adjust the order of the method
  - *h* refinement: change the mesh spacing

- types of mesh refinement – *r* refinement

  - non-uniform mesh *(refined region is known)*

    = advantages:
      − simple to implement

    = disadvantages:
      − difference expression for non-constant zone spacing



COSMOS code (Ricker 2000)

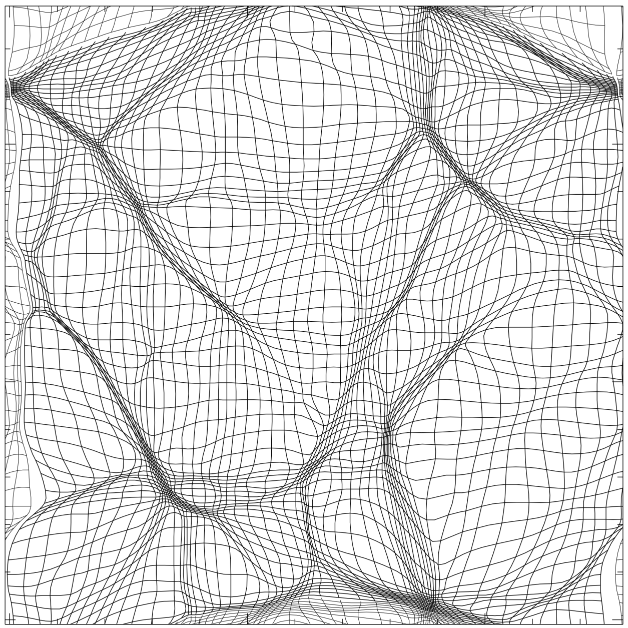
Solving for Gravity

- types of mesh refinement – *r* refinement

  - Lagragian mesh *(mesh is tied to fluid)*

    = advantages:
      - constant mass resolution
      - sharp resolution of contacts

    = disadvantages:
      - grid stretching causes numerical dissipation
      - grid tangling in rotational flows



MMH code (Pen 1998)

Solving for Gravity

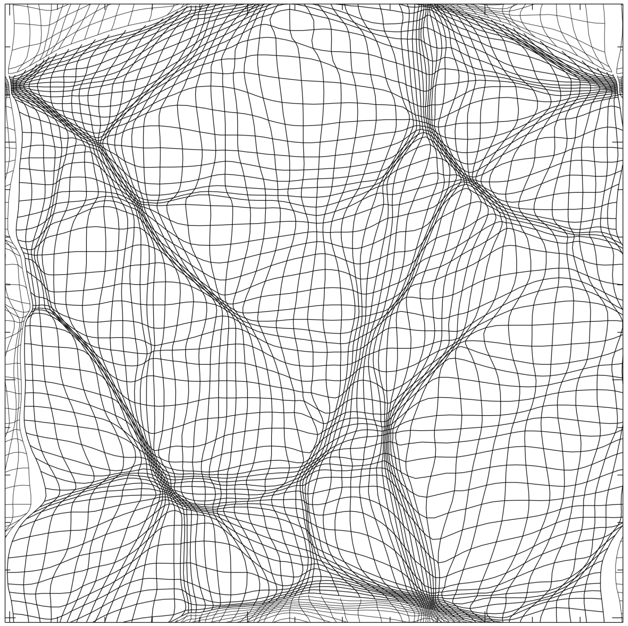
- types of mesh refinement – *r* refinement

  - Lagragian mesh *(mesh is tied to fluid)*

    = advantages:
    − constant mass resolution
    − sharp resolution of contacts

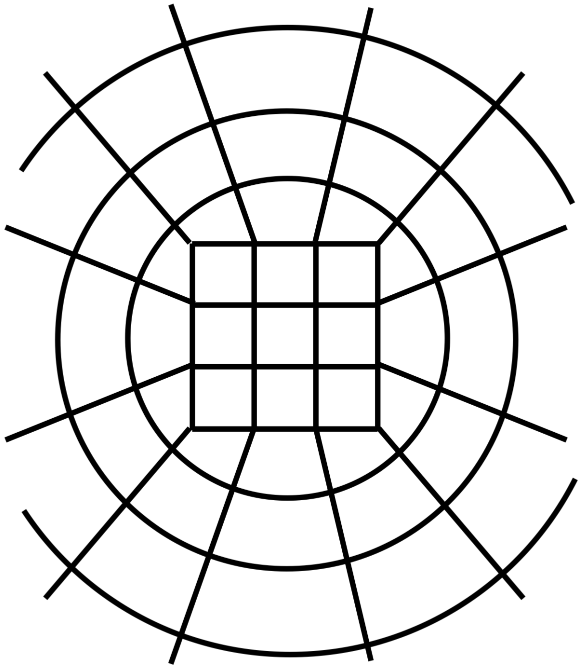
    = disadvantages:
    − grid stretching causes numerical dissipation
    − grid tangling in rotational flows

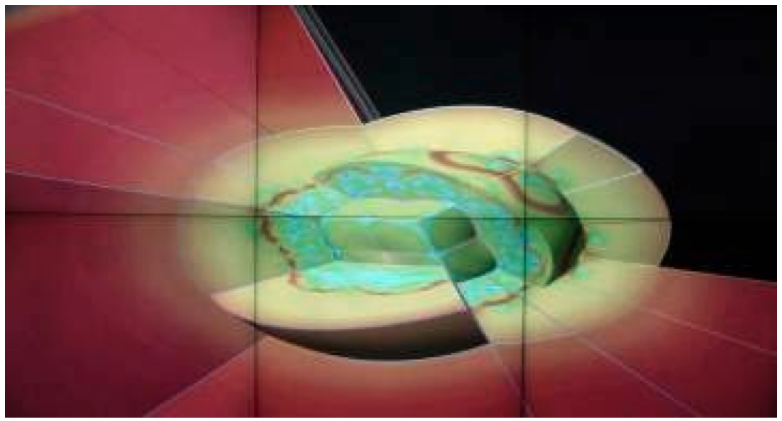
usually used only in 1D (e.g. stellar evolution codes)



MMH code (Pen 1998)

- types of mesh refinement – *r* refinement
  - arbitrary Lagrangian-Eulerion mesh *(mesh moves arbitrarily fluid)*
    - = advantages:
      - − Lagrangian mesh where flow is irrotational
      - − Eulerian where mesh distortion is problematic
    - = disadvantages:
      - − difficult to handle...





DJEHUTY code (Dearborn et al. 2002)

Solving for Gravity

- types of mesh refinement – *p* refinement

not in this course...

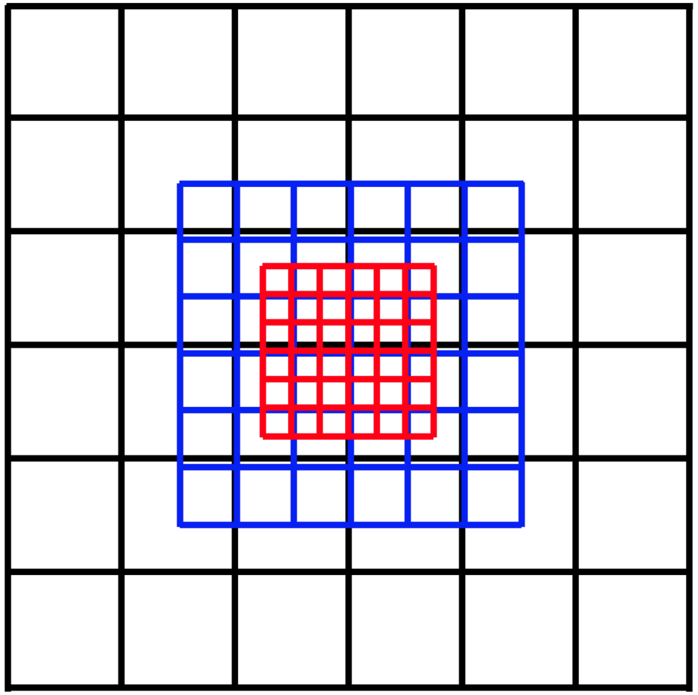
Solving for Gravity

- types of mesh refinement – *h* refinement

  - nested grids *(static meshes with different resolutions)*

    = advantages:
      – easy to handle boundaries between meshes

    = disadvantages:
      – refined region should not move



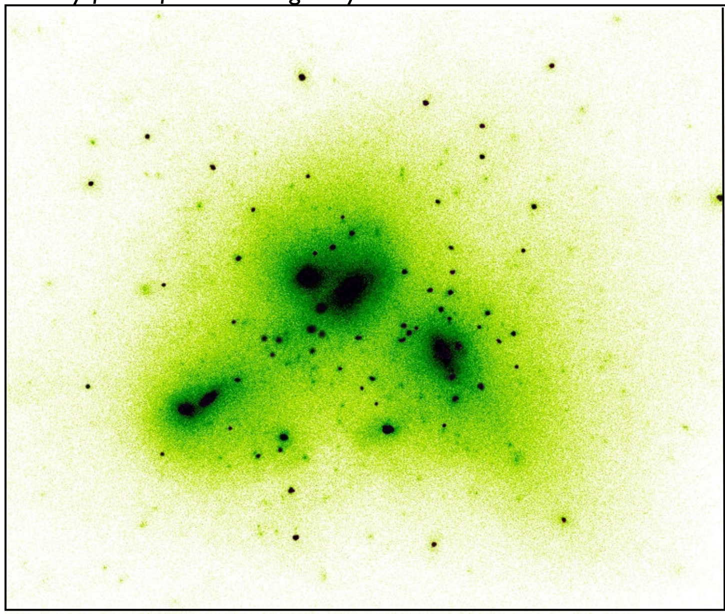
- types of mesh refinement – *h* refinement

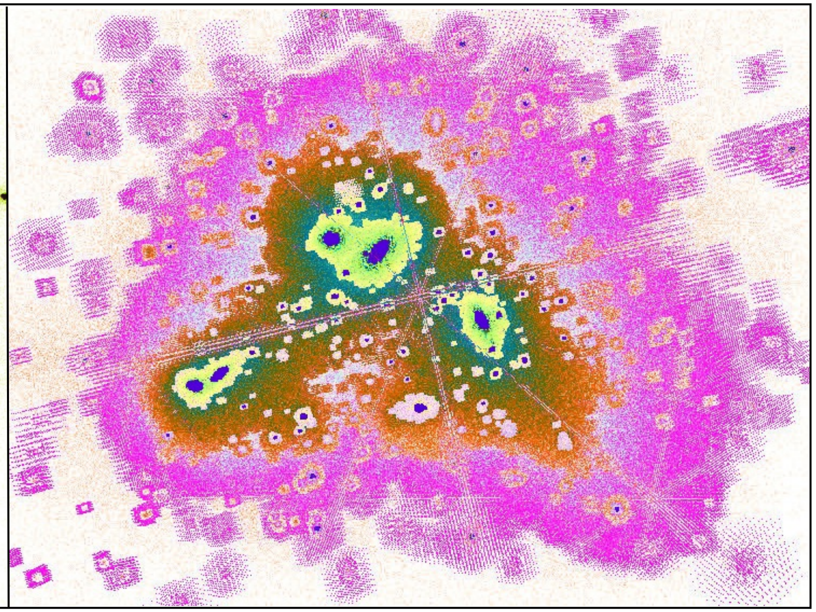
  - adaptive mesh refinement *(refined patches are created and destroyed as needed)*

    = advantages:
    - fully flexible to problem

    = disadvantages:
    - serious book-keeping for grid hierarchy

density field of simulated galaxy cluster





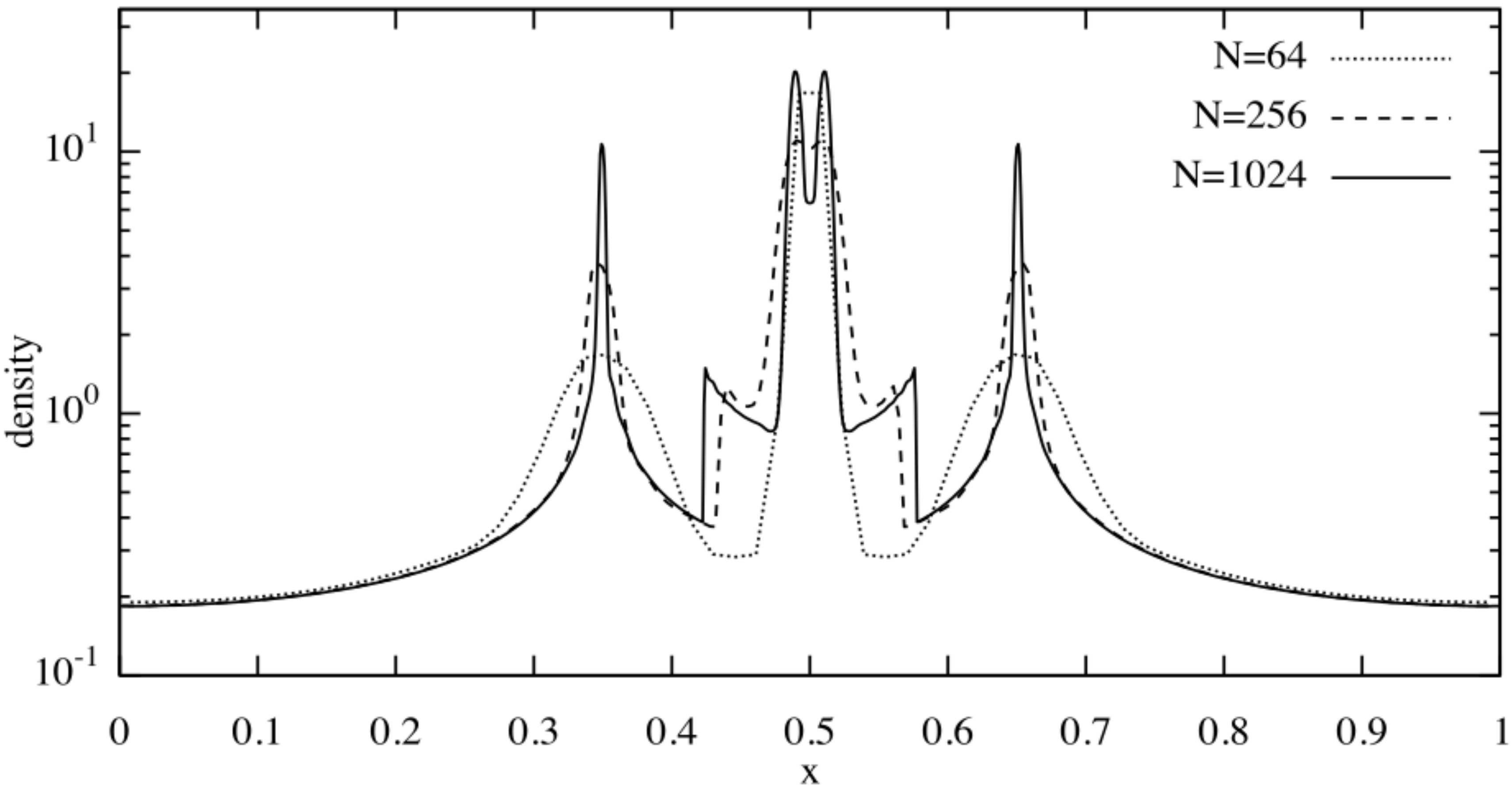
AMIGA code (Doumler & Knebe 2010)

adaptive grid hierarchy

- mesh refinements
- **adaptive mesh refinement**
- adaptive mesh refinement for N-body codes
- handling irregular grids
- adaptive leap-frog integration

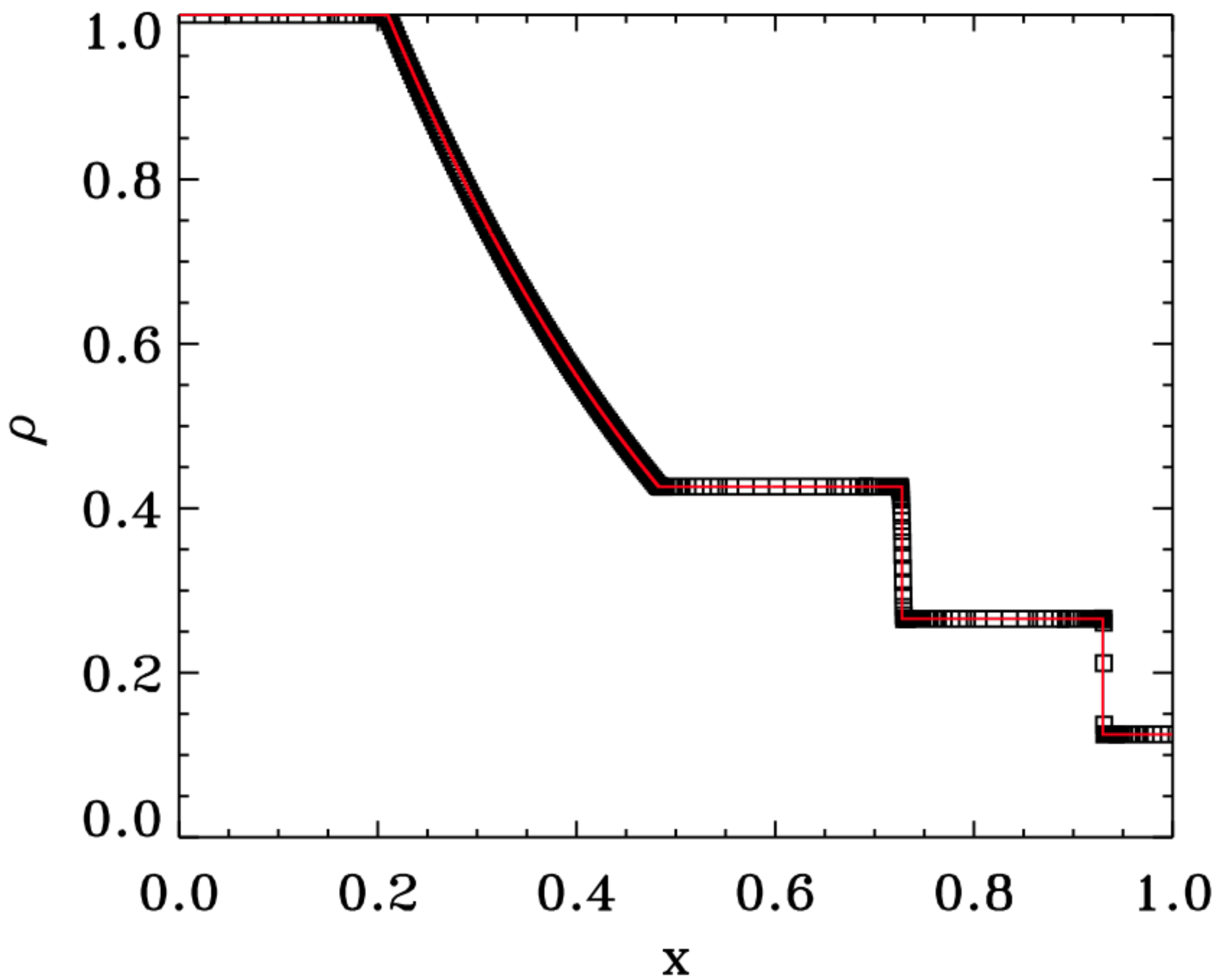
Solving for Gravity

- adaptive mesh refinement – improvements using finer grids



double pancake test (Doumler & Knebe 2010)

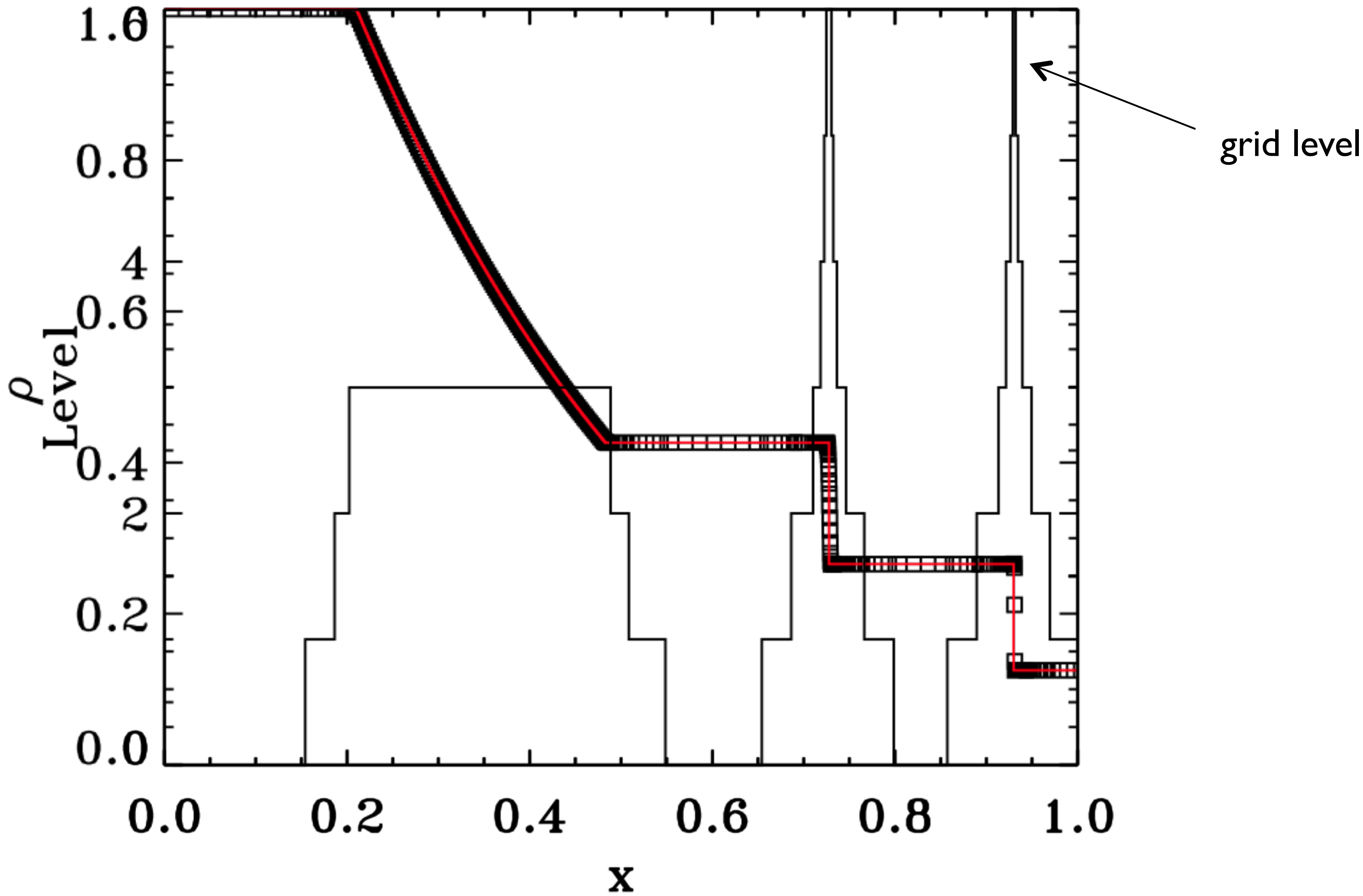
- adaptive mesh refinement – improvements using finer grids



shock tube test (Teyssier 2002)

Solving for Gravity

- adaptive mesh refinement – improvements using finer grids



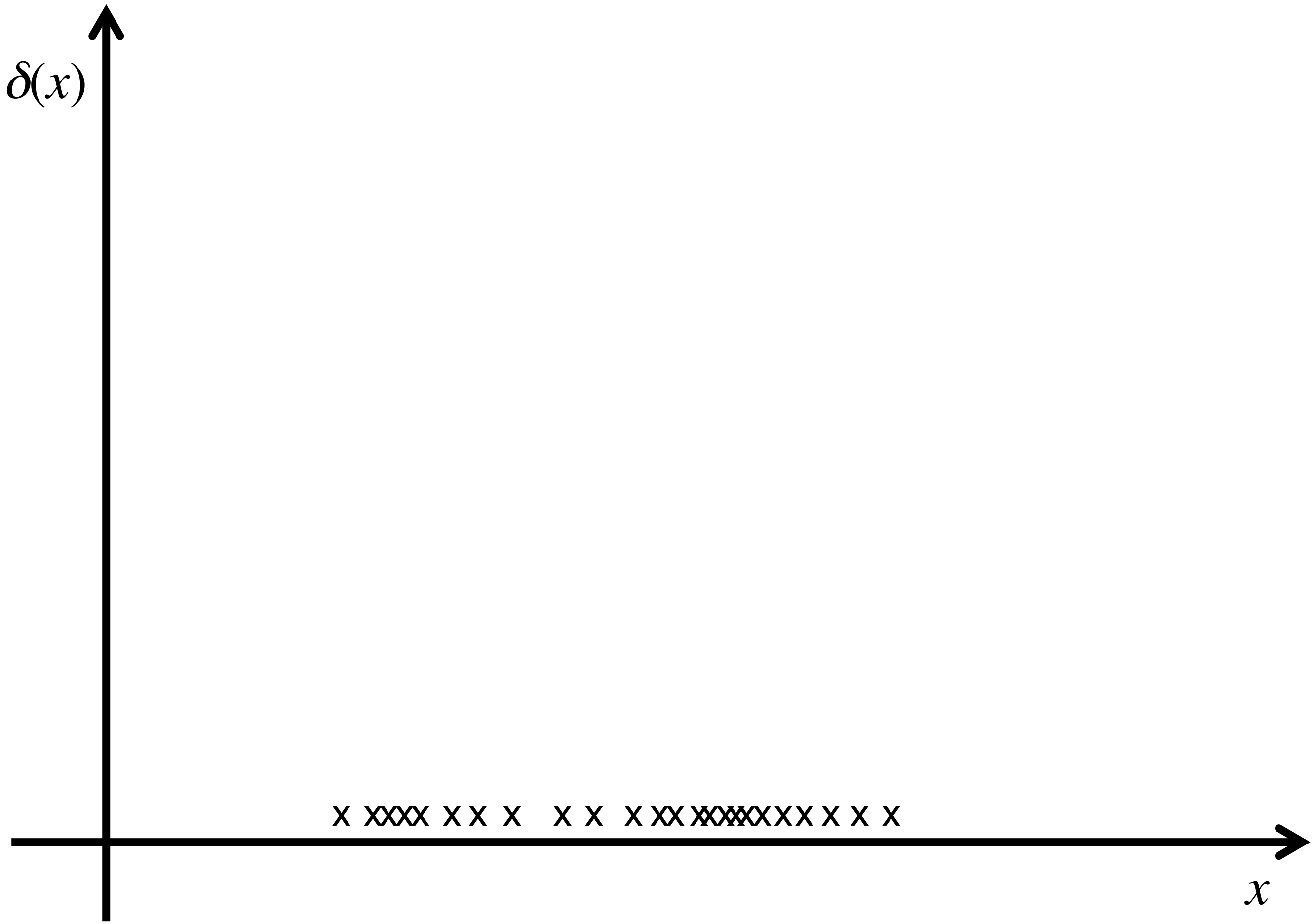
shock tube test (Teyssier 2002)

Solving for Gravity

- adaptive mesh refinement – refinement criterion
    - density
    - truncation error
    - physics

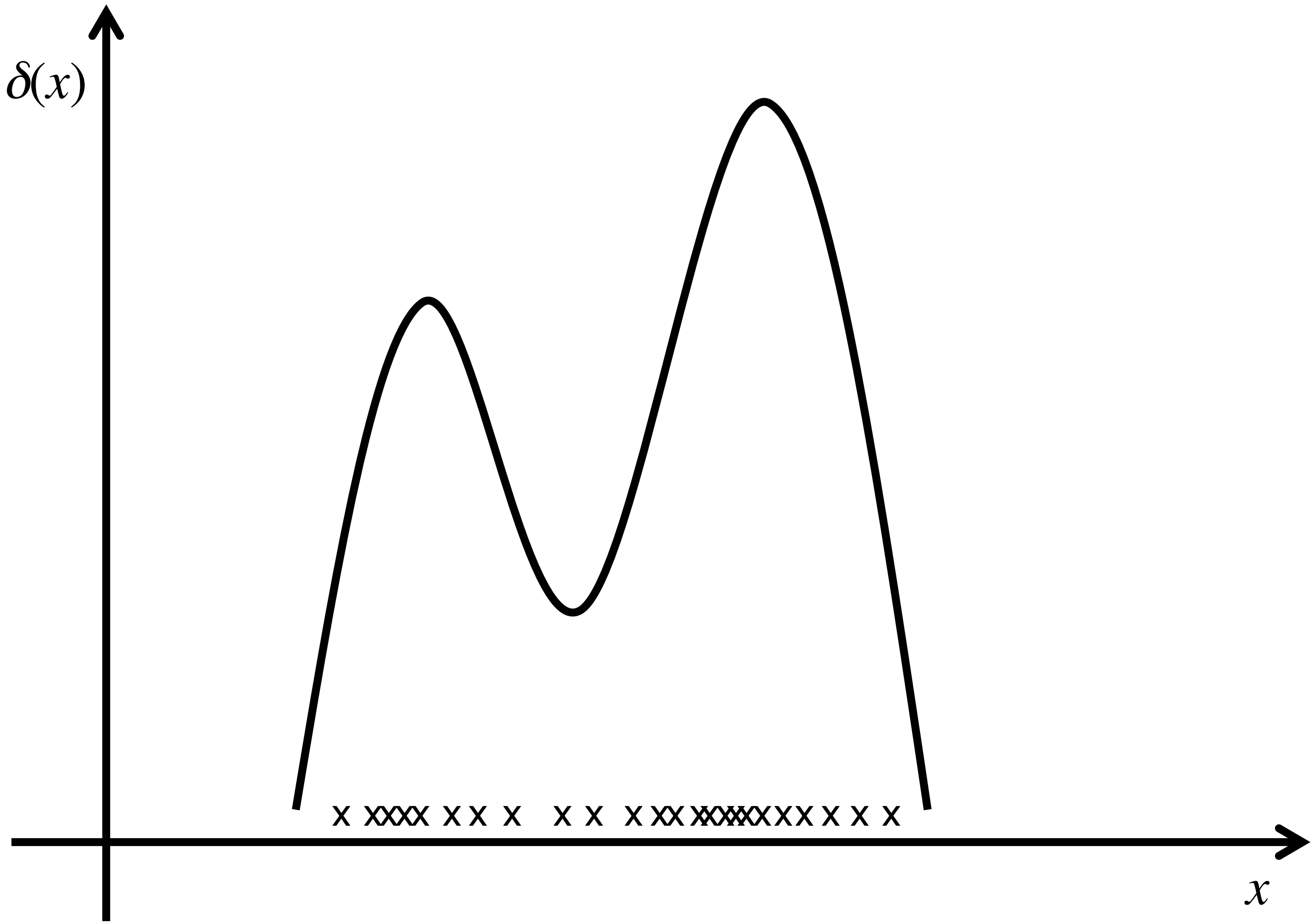
Solving for Gravity

- adaptive mesh refinement – refinement criterion
    - density – 1D density distribution

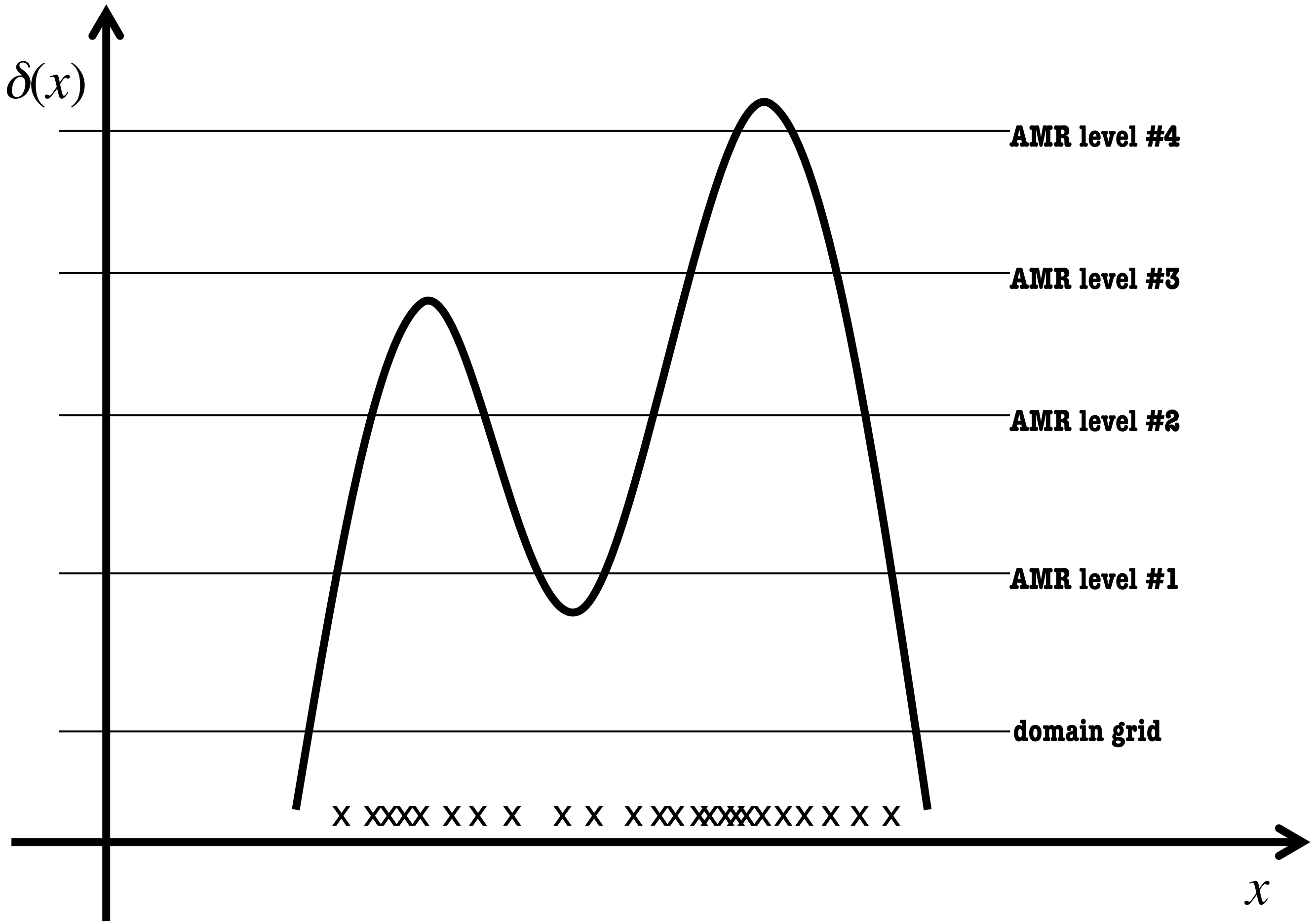


Solving for Gravity

- adaptive mesh refinement – refinement criterion
    - density – 1D density distribution

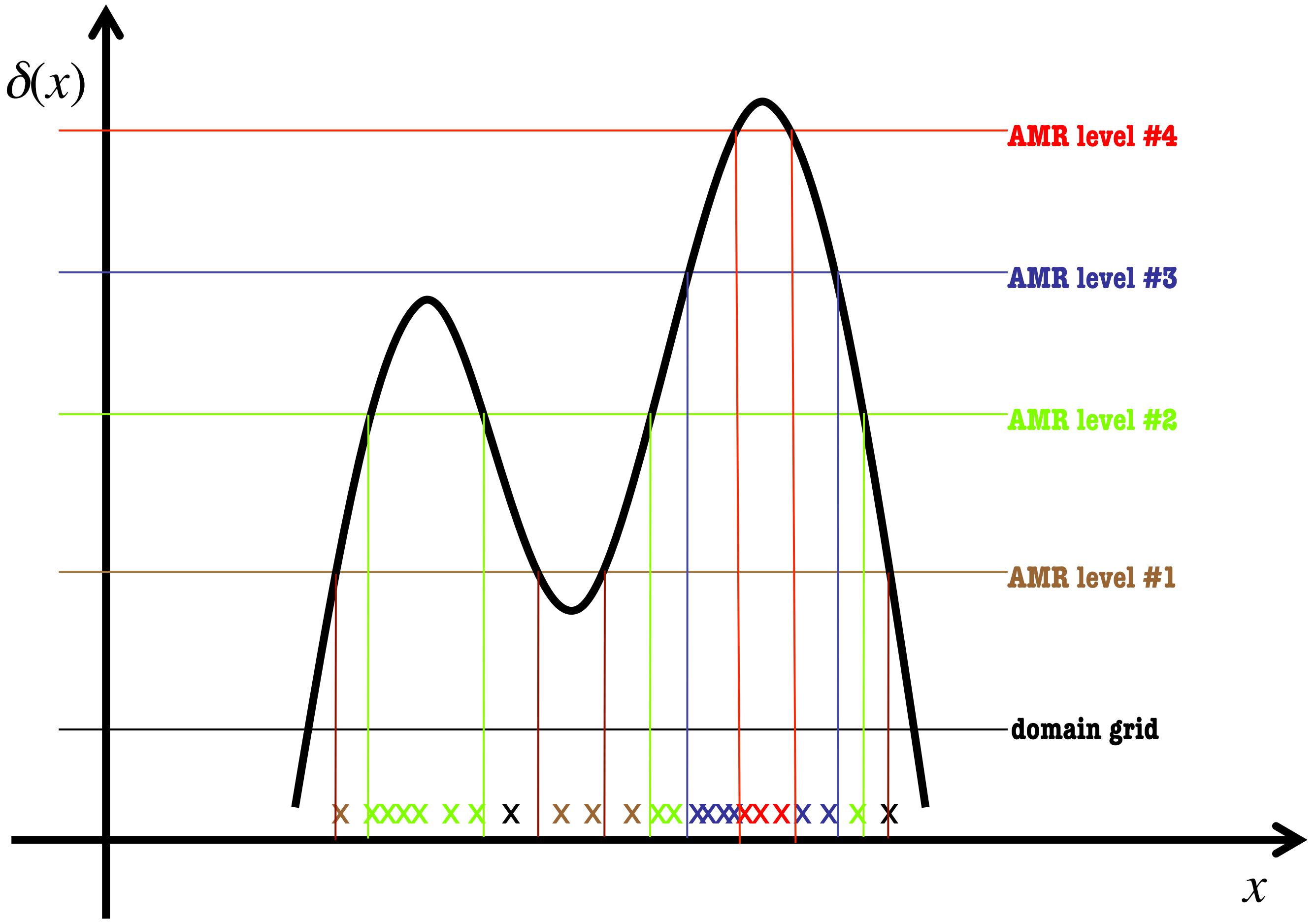


- adaptive mesh refinement – refinement criterion
    - density – 1D density distribution



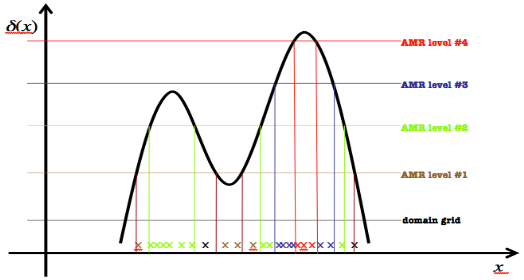
Solving for Gravity

- adaptive mesh refinement – refinement criterion
  - density – 1D density distribution



Solving for Gravity

- adaptive mesh refinement – refinement criterion
  - density:
    - – refine regions of high density



  - truncation error:

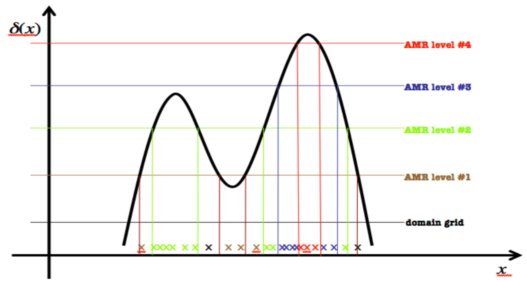
  - physics:

Solving for Gravity

- adaptive mesh refinement – refinement criterion

  - density:

    - refine regions of high density



  - truncation error:

    - refine regions of large truncation errors

$$R^{i}_{k,l,m} = \Delta\Phi^{i}_{k,l,m} - \rho_{k,l,m} \leq \varepsilon \mathrm{T}_{k,l,m} \quad \text{with} \quad \mathrm{T}_{k,l,m} = \mathcal{P}\left[\Delta\left(\mathcal{R}\Phi^{i}_{k,l,m}\right)\right] - \left(\Delta\Phi^{i}_{k,l,m}\right)$$

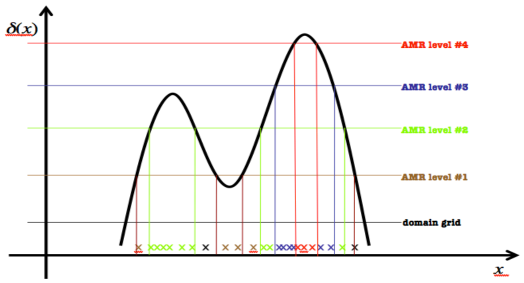
  - physics:

Solving for Gravity

- adaptive mesh refinement – refinement criterion

  - density:

    - refine regions of high density



  - truncation error:

    - refine regions of large truncation errors

$$R^i_{k,l,m} = \Delta\Phi^i_{k,l,m} - \rho_{k,l,m} \le \varepsilon \mathrm{T}_{k,l,m} \quad \text{with} \quad \mathrm{T}_{k,l,m} = \mathcal{P}\left[\Delta\left(\mathcal{R}\Phi^i_{k,l,m}\right)\right] - \left(\Delta\Phi^i_{k,l,m}\right)$$

  - physics:

    - compare grid spacing against local critical wavelength

$$\Delta x < \varepsilon\,\lambda \quad \text{with} \quad \lambda = c_s\sqrt{\frac{\pi}{G\rho}}$$

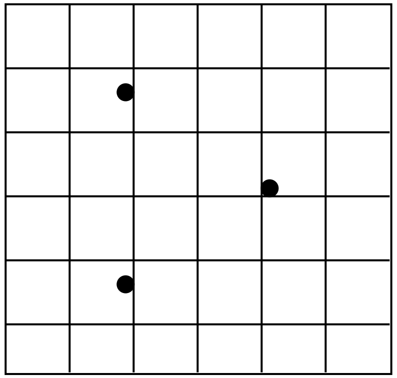
- mesh refinements
- adaptive mesh refinement
- **adaptive mesh refinement for N-body codes**
- handling irregular grids
- adaptive leap-frog integration

- mesh refinements
- adaptive mesh refinement
- **adaptive mesh refinement for N-body codes**
  - gravity
  - generating refinements
  - density assignment
  - solving Poisson’s equation
- handling irregular grids
- adaptive leap-frog integration

- mesh refinements
- adaptive mesh refinement
- **adaptive mesh refinement for N-body codes**
  - ***gravity***
  - generating refinements
  - density assignment
  - solving Poisson's equation
- handling irregular grids
- adaptive leap-frog integration

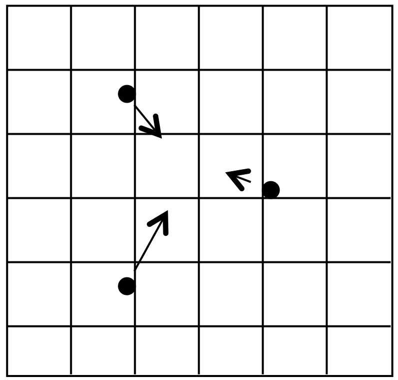
Solving for Gravity

- gravity tends to clump matter together...



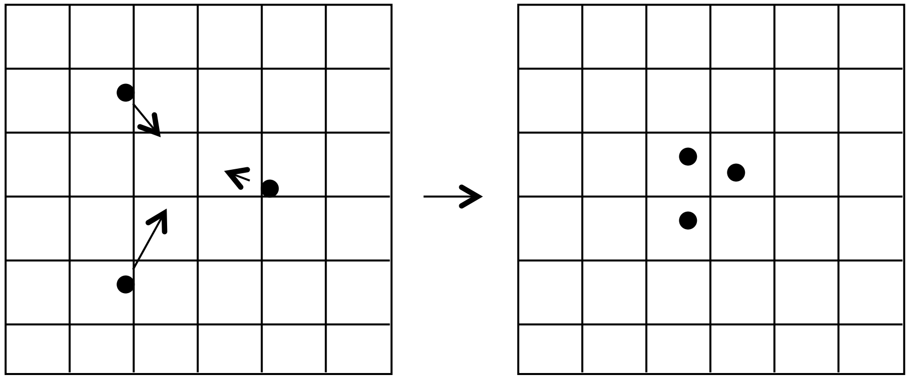
Solving for Gravity

- gravity tends to clump matter together...



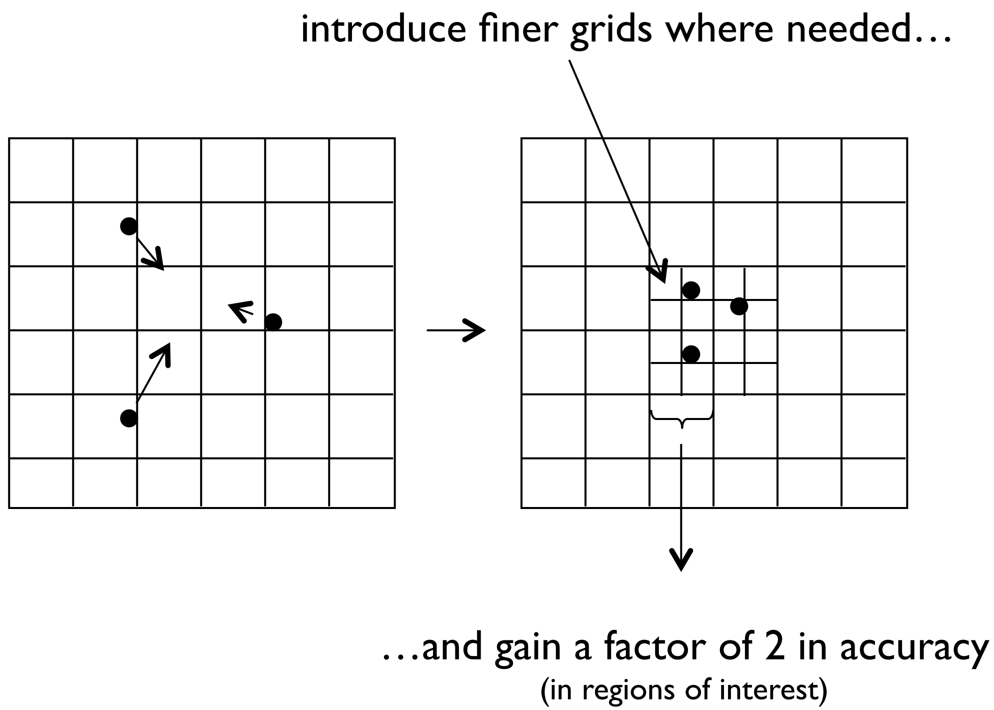
Solving for Gravity

- gravity tends to clump matter together...

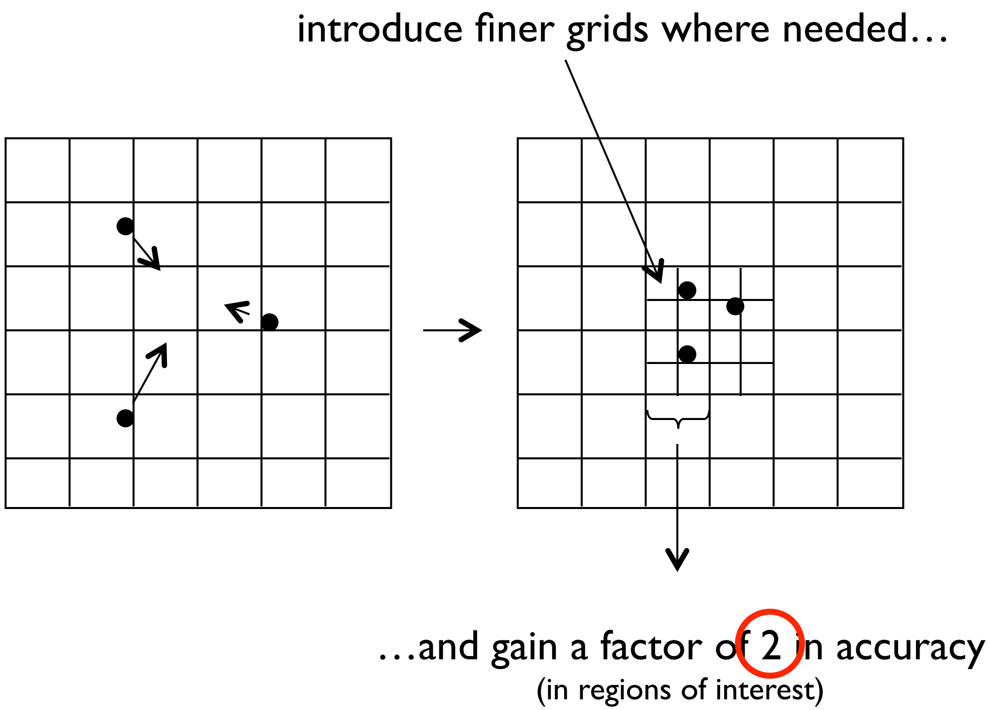


Solving for Gravity

- gravity tends to clump matter together...



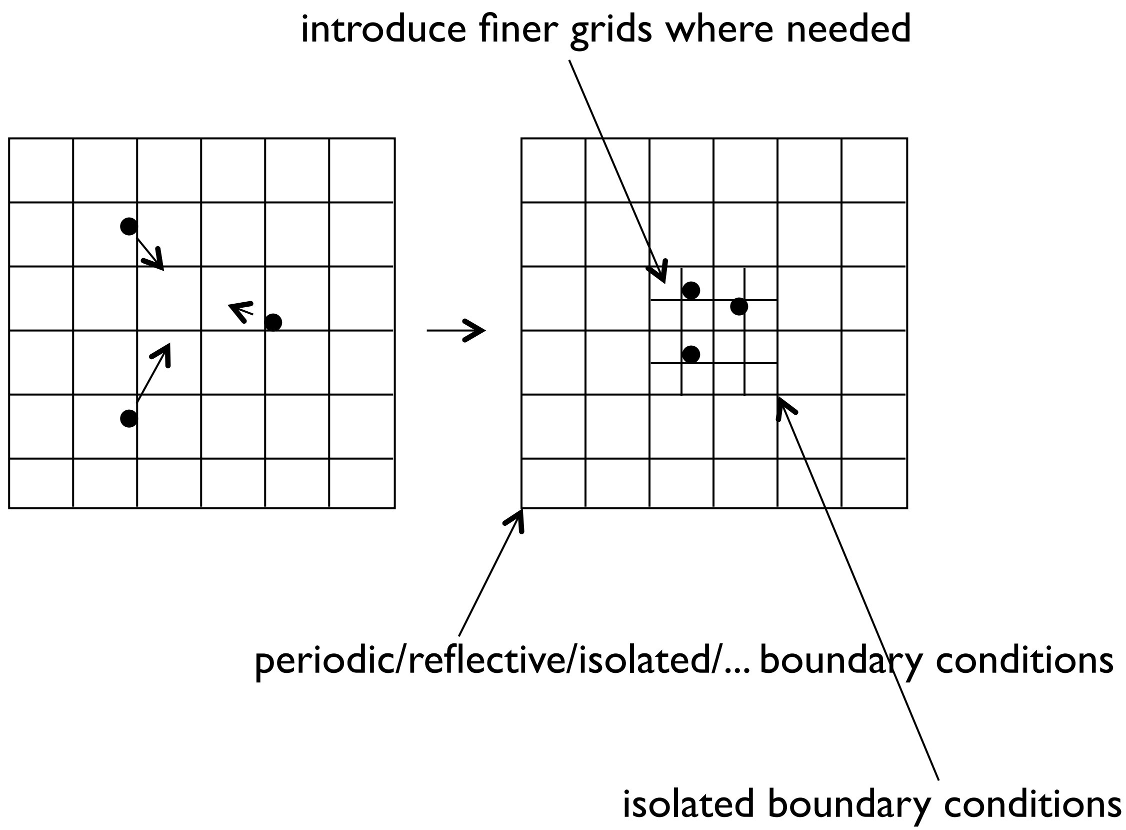
- gravity tends to clump matter together...



factor 2 not mandatory, but most common choice...

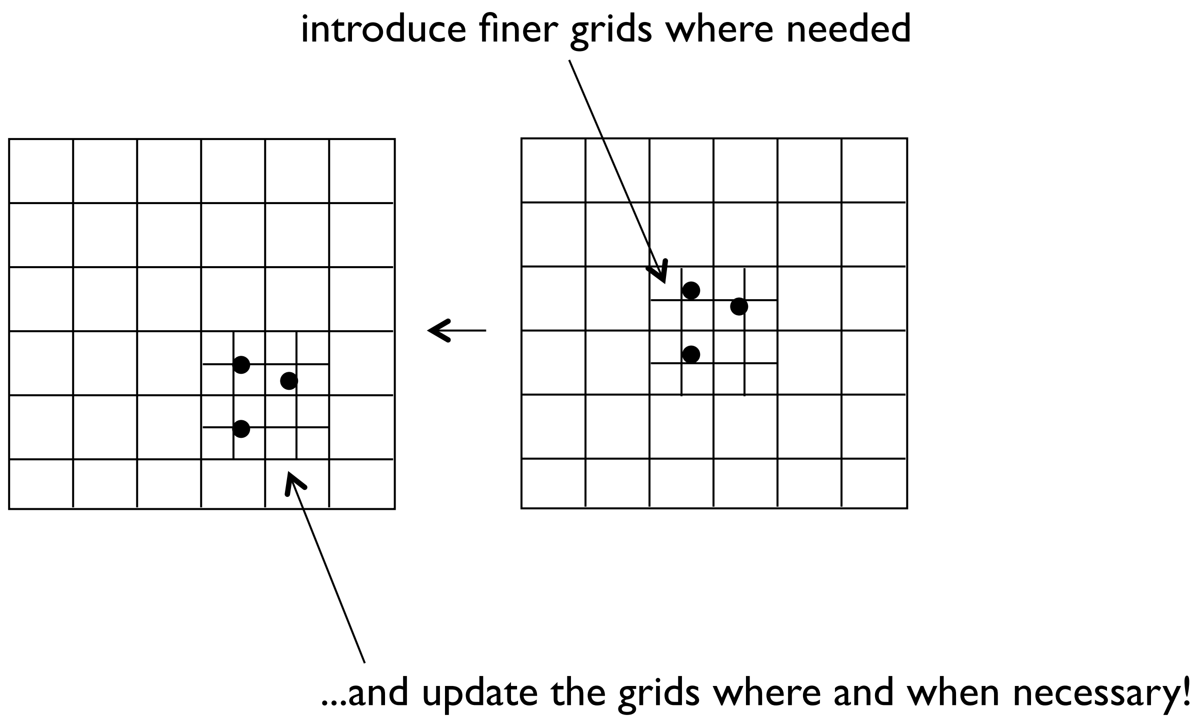
Solving for Gravity

- gravity tends to clump matter together...



Solving for Gravity

- gravity tends to clump matter together...

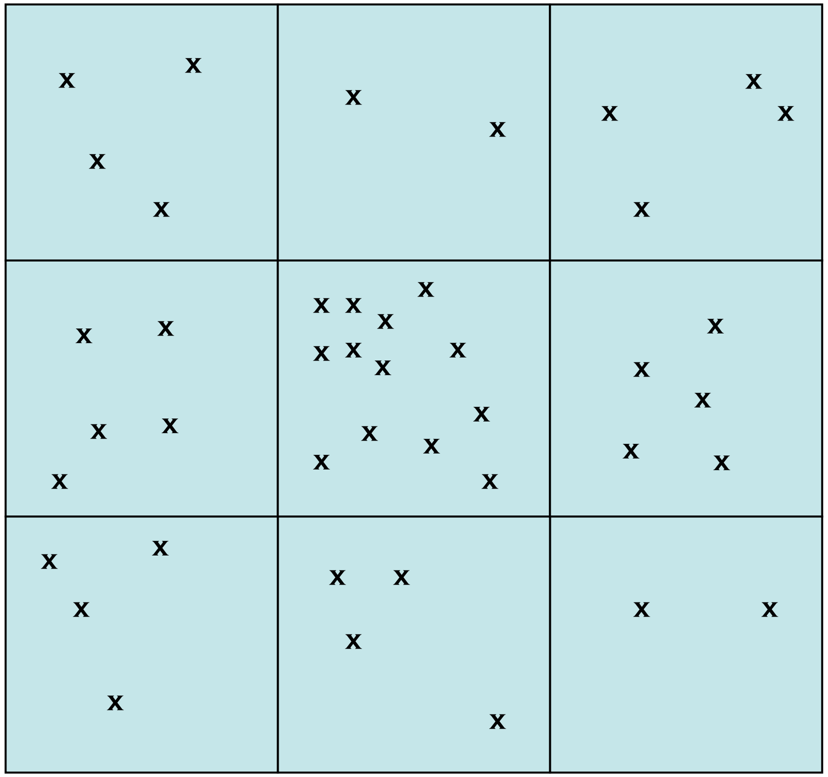


- mesh refinements
- adaptive mesh refinement
- **adaptive mesh refinement for N-body codes**
  - gravity
  - ***generating refinements***
  - density assignment
  - solving Poisson's equation
- handling irregular grids
- adaptive leap-frog integration

Solving for Gravity

- generating refinements
  - N-body simulations:

number of particles per cell



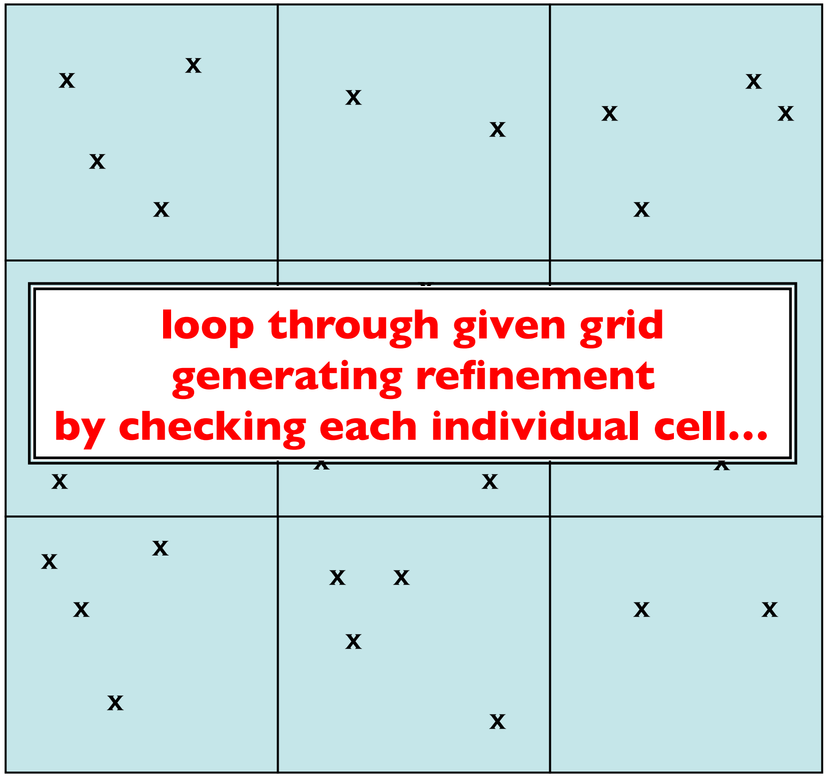
refinement criterion: 6 particles/cell

Solving for Gravity

- generating refinements

  - N-body simulations:

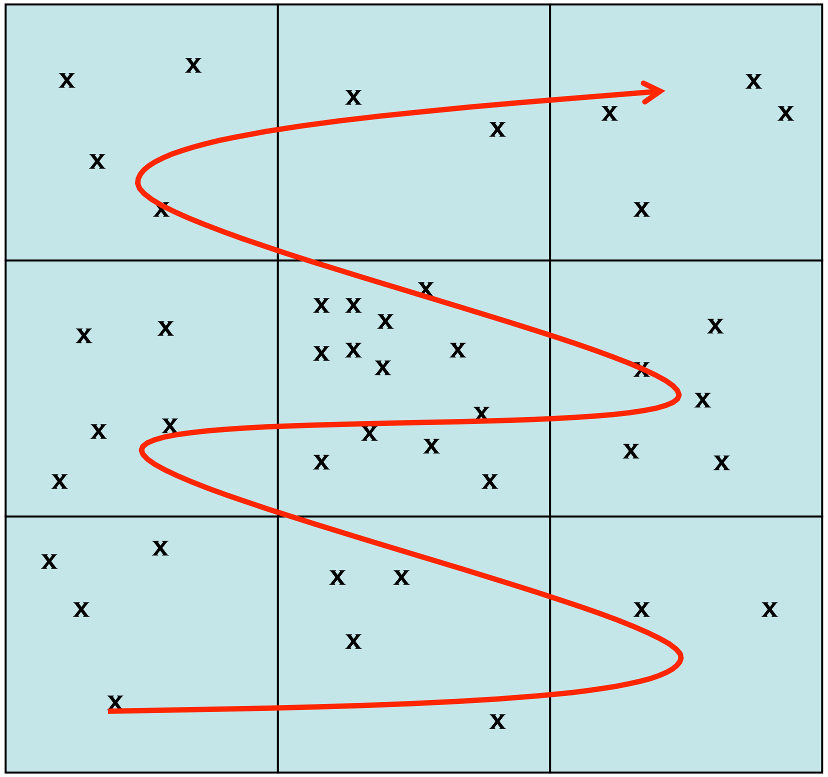
number of particles per cell



refinement criterion: 6 particles/cell

- generating refinements
  - N-body simulations:

number of particles per cell

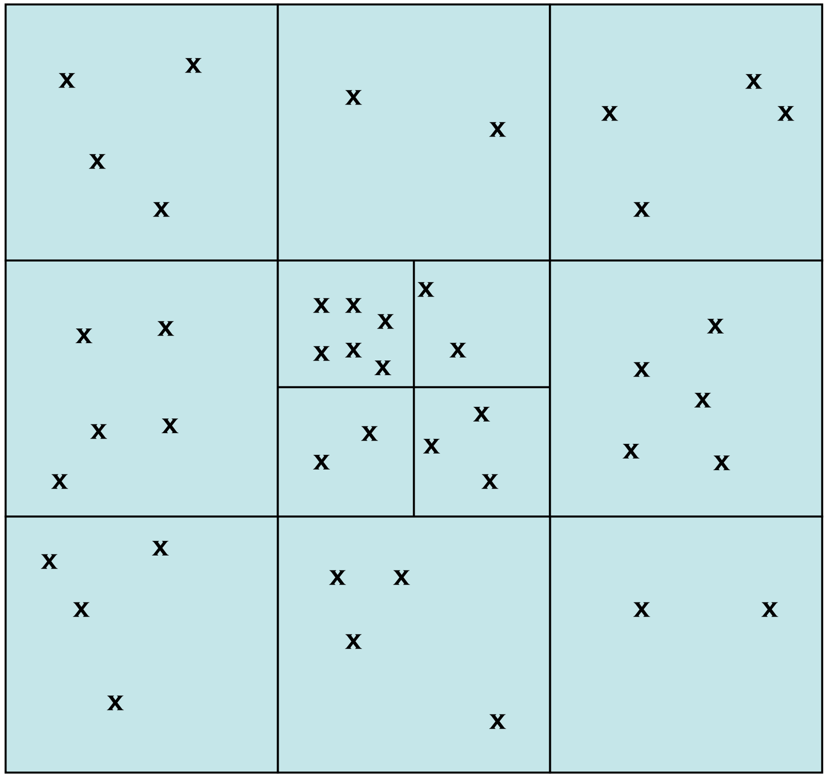


refinement criterion: 6 particles/cell

- generating refinements

    - N-body simulations:

number of particles per cell

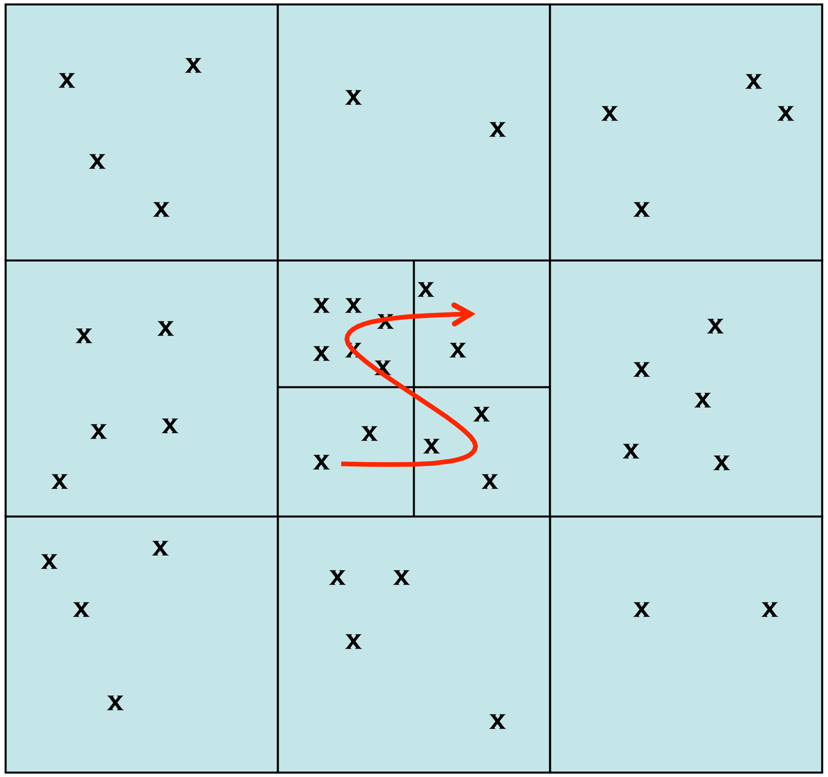


refinement criterion: 6 particles/cell

Solving for Gravity

- generating refinements
  - N-body simulations:

number of particles per cell



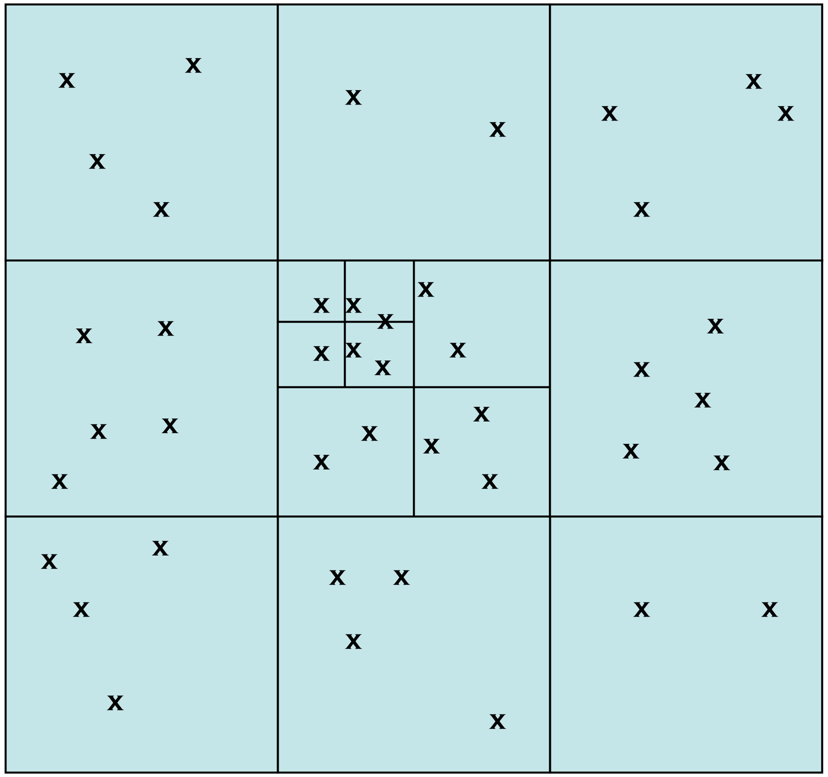
refinement criterion: 6 particles/cell

Solving for Gravity

- generating refinements

  - *N*-body simulations:

number of particles per cell

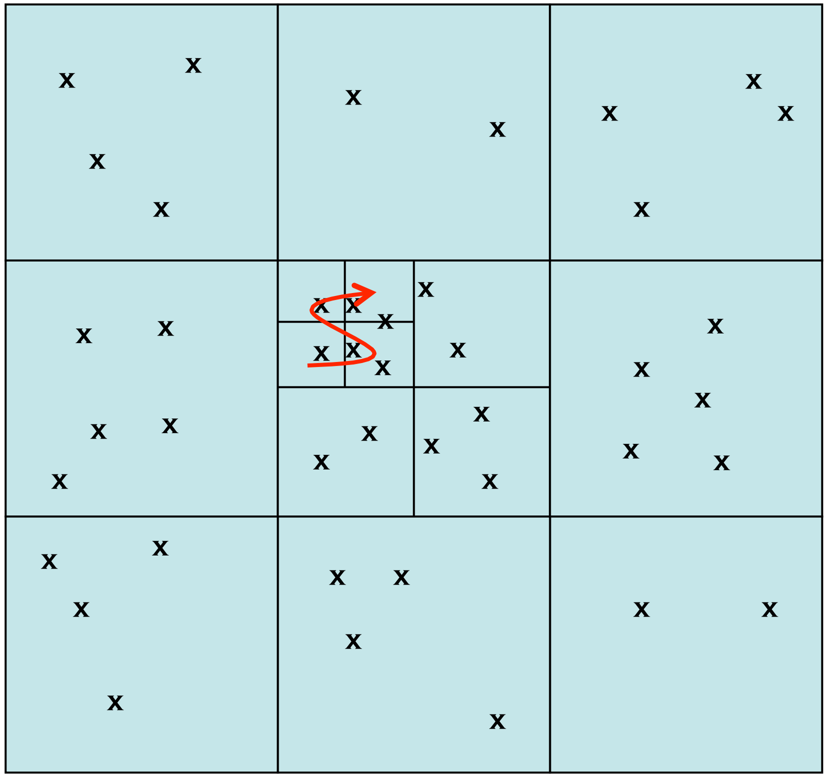


refinement criterion: 6 particles/cell

Solving for Gravity

- generating refinements
  - N-body simulations:

number of particles per cell

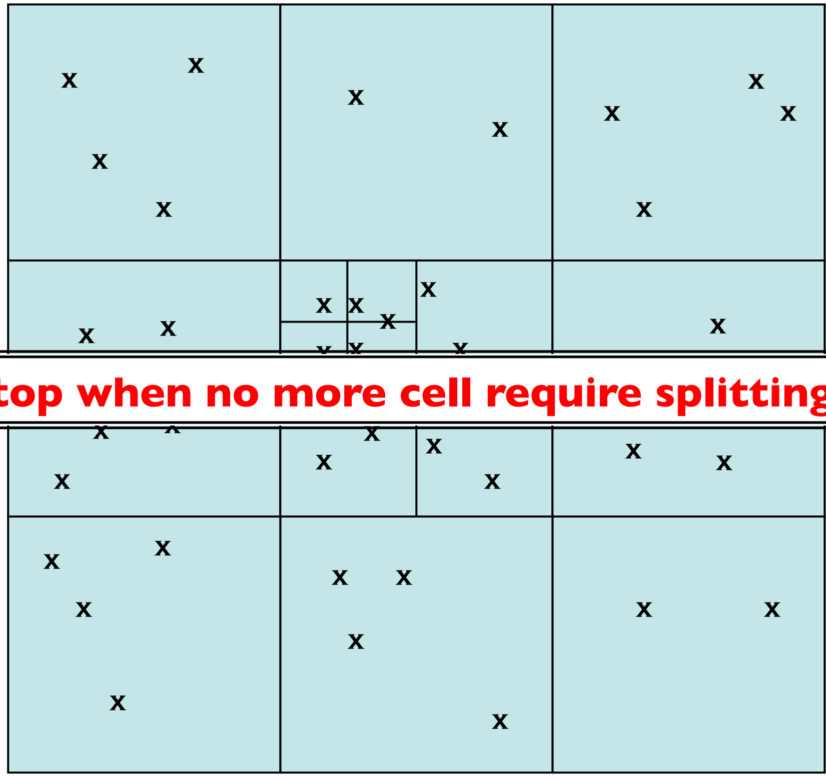


refinement criterion: 6 particles/cell

Solving for Gravity

- generating refinements
  - N-body simulations:

number of particles per cell



stop when no more cell require splitting...

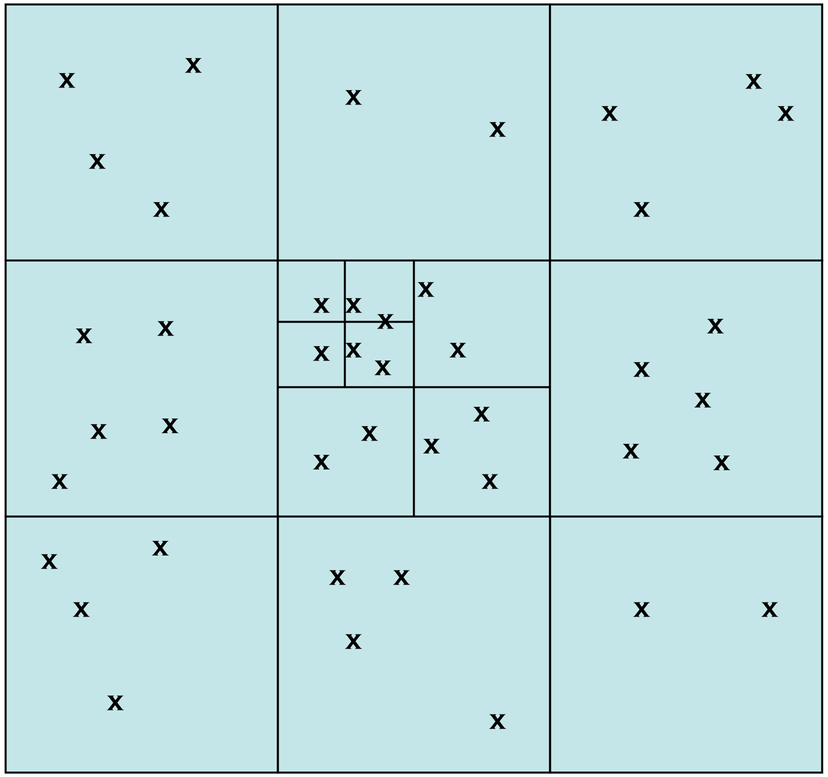
refinement criterion: 6 particles/cell

Solving for Gravity

- generating refinements

  - N-body simulations:

number of particles per cell



refinement criterion: 6 particles/cell

Note:
in this scheme we split the volume of a coarse cell into eight equal sub-cells…

=> non-cospatial scheme!

Solving for Gravity

- generating refinements
  - interpolation between grids:

$$f(x_i) = F(x_i) + F'(x_i)\Delta x$$

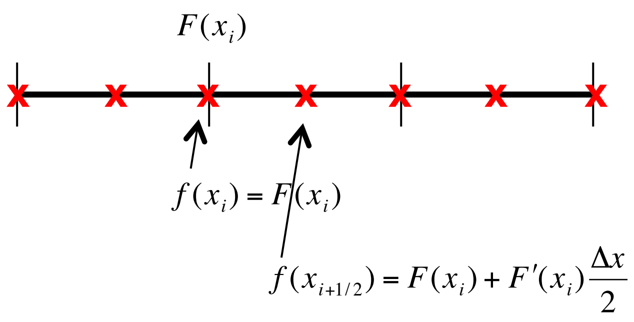
F = value on coarse grid
f = value on fine grid

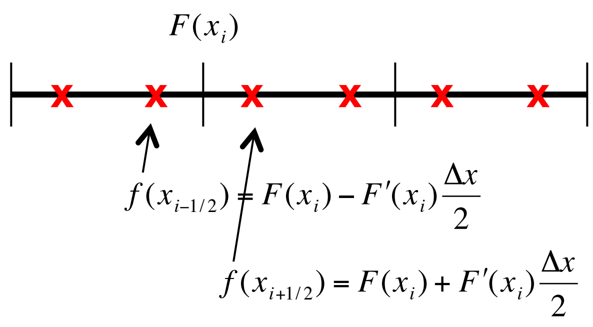
- generating refinements

  - interpolation between grids:

$$f(x_i) = F(x_i) + F'(x_i)\Delta x$$

co-spatial vs. non-cospatial

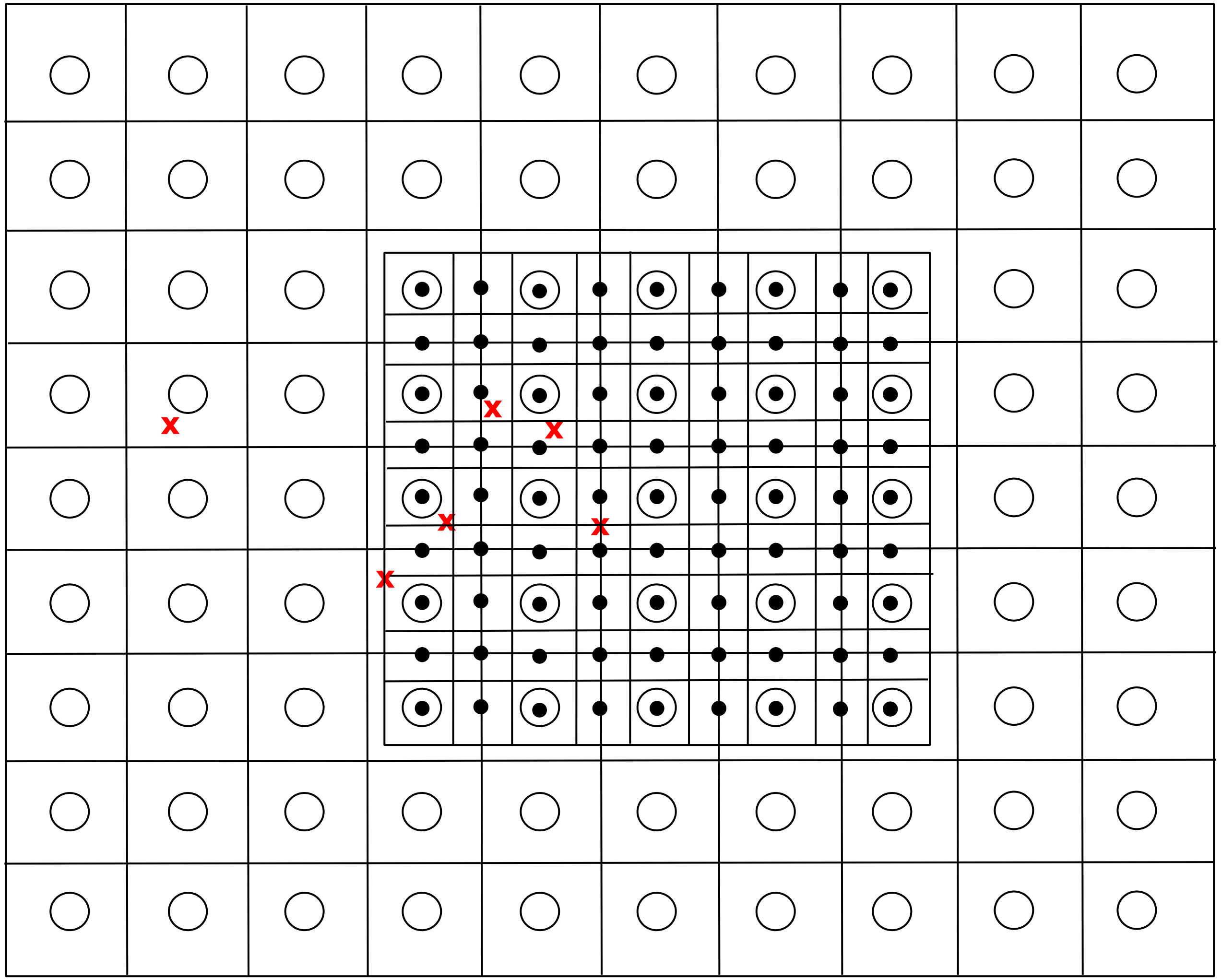




- mesh refinements
- adaptive mesh refinement
- **adaptive mesh refinement for N-body codes**
  - gravity
  - generating refinements
  - ***density assignment***
  - solving Poisson's equation
- handling irregular grids
- adaptive leap-frog integration

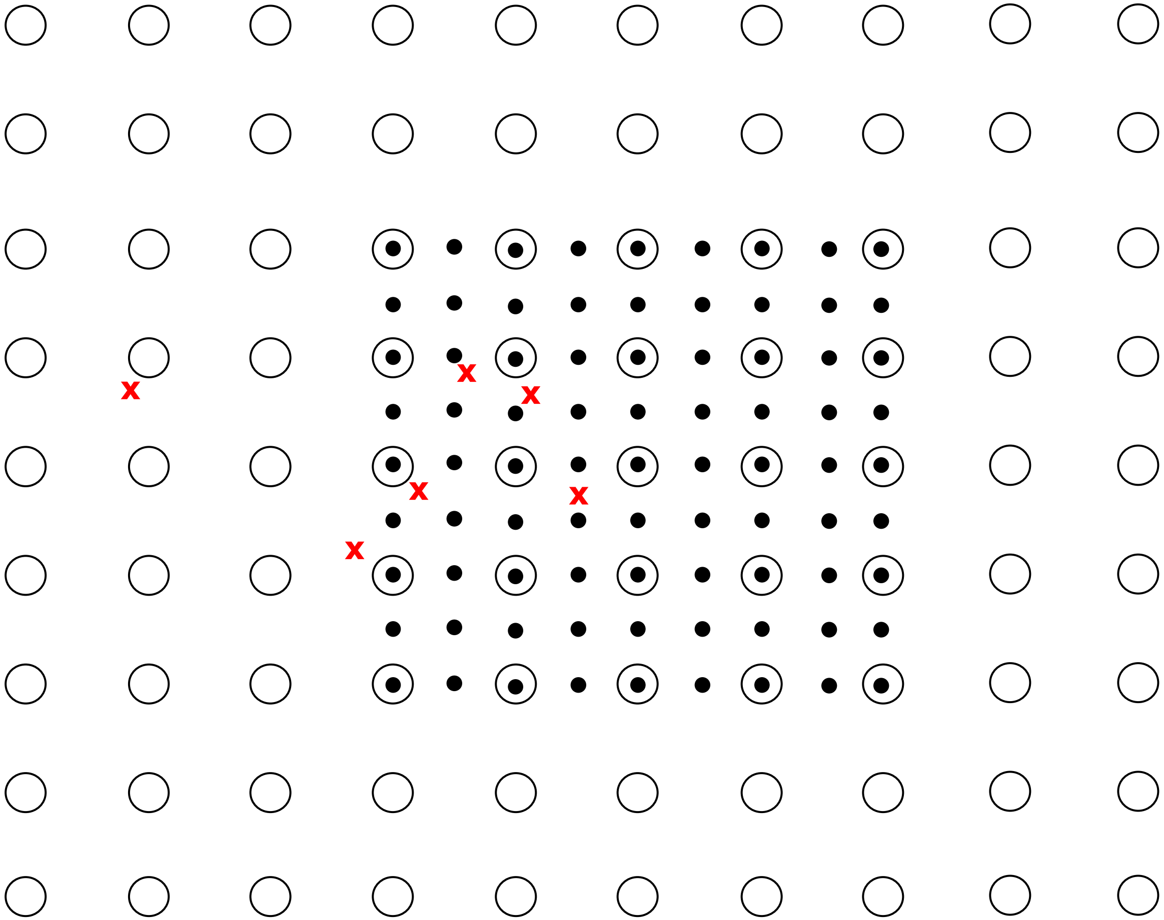
Solving for Gravity

- density assignment (co-spatial scheme)



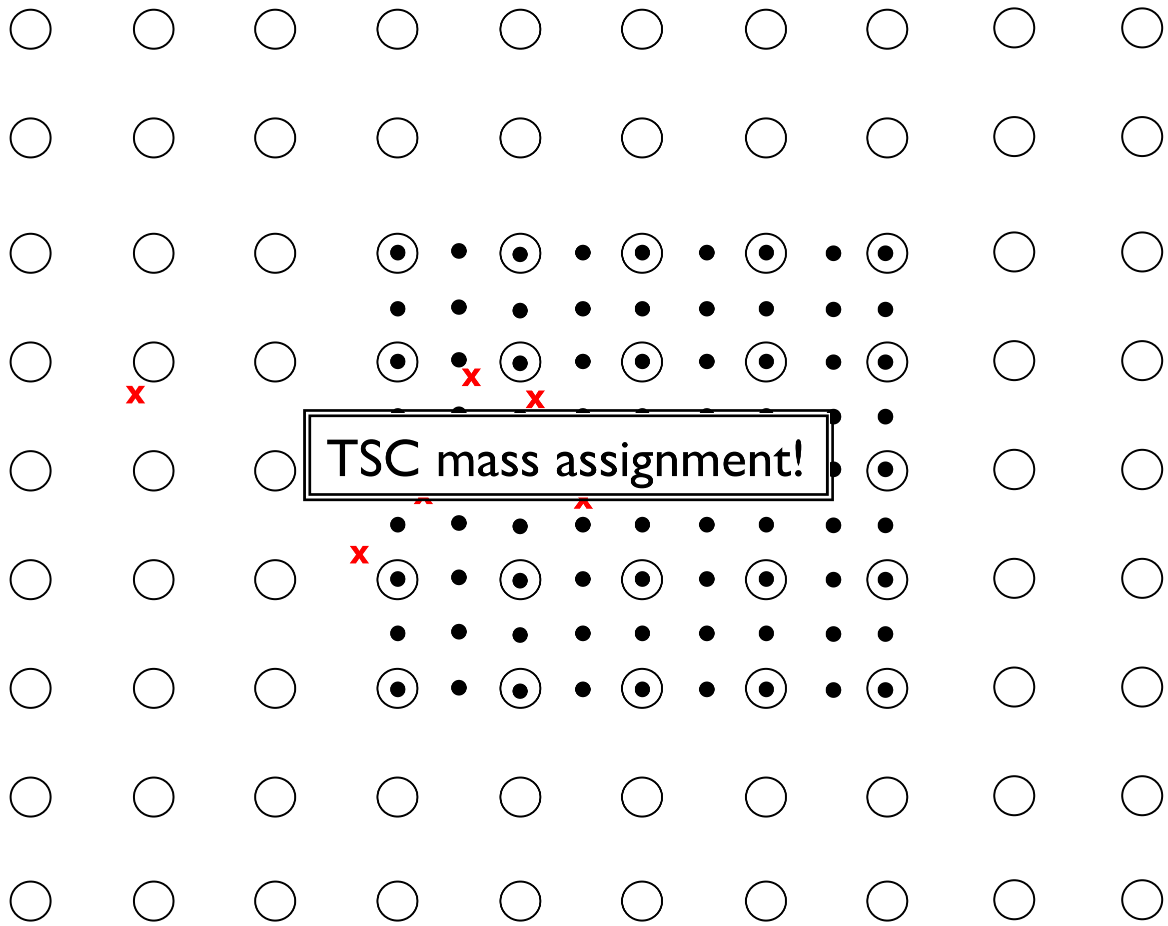
Solving for Gravity

- density assignment (co-spatial scheme)



Solving for Gravity

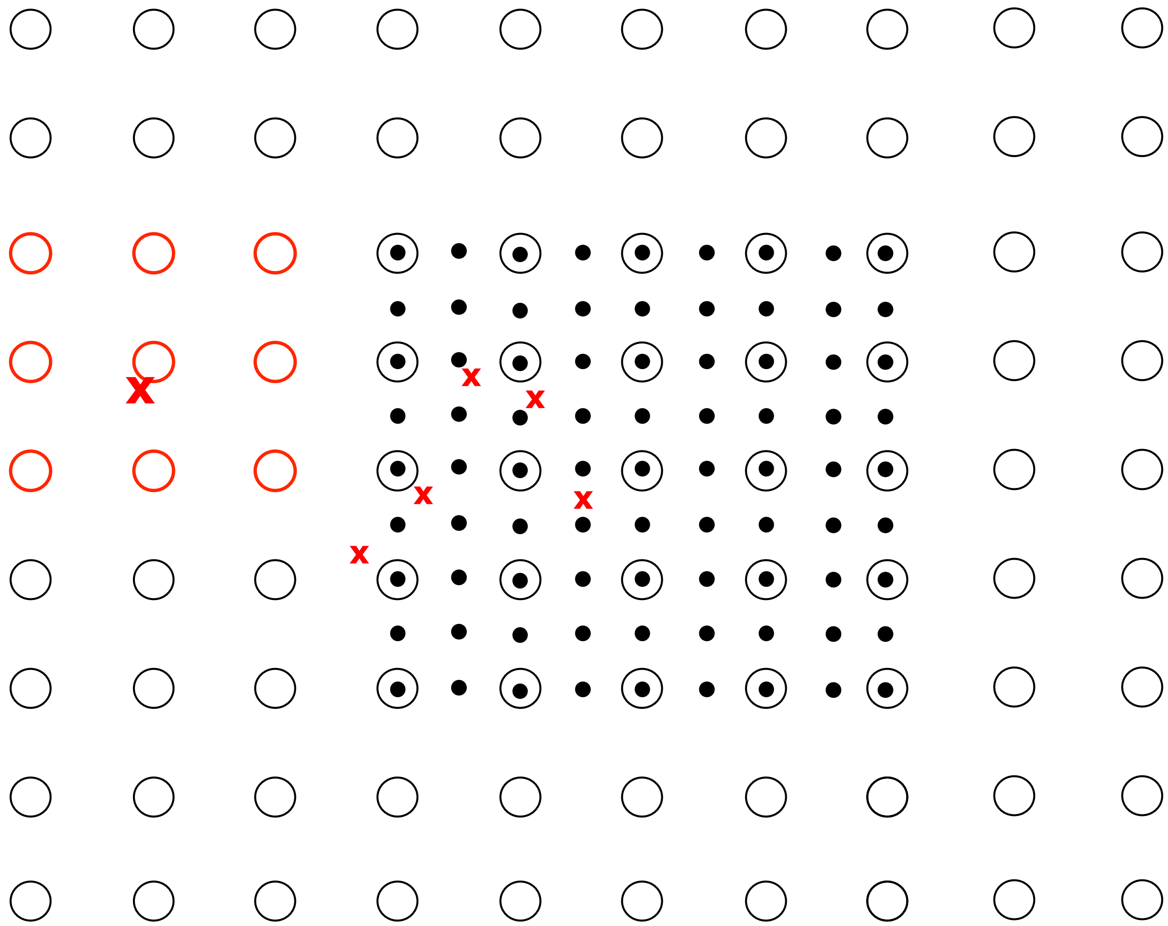
- density assignment (co-spatial scheme)



Solving for Gravity

- density assignment (co-spatial scheme)

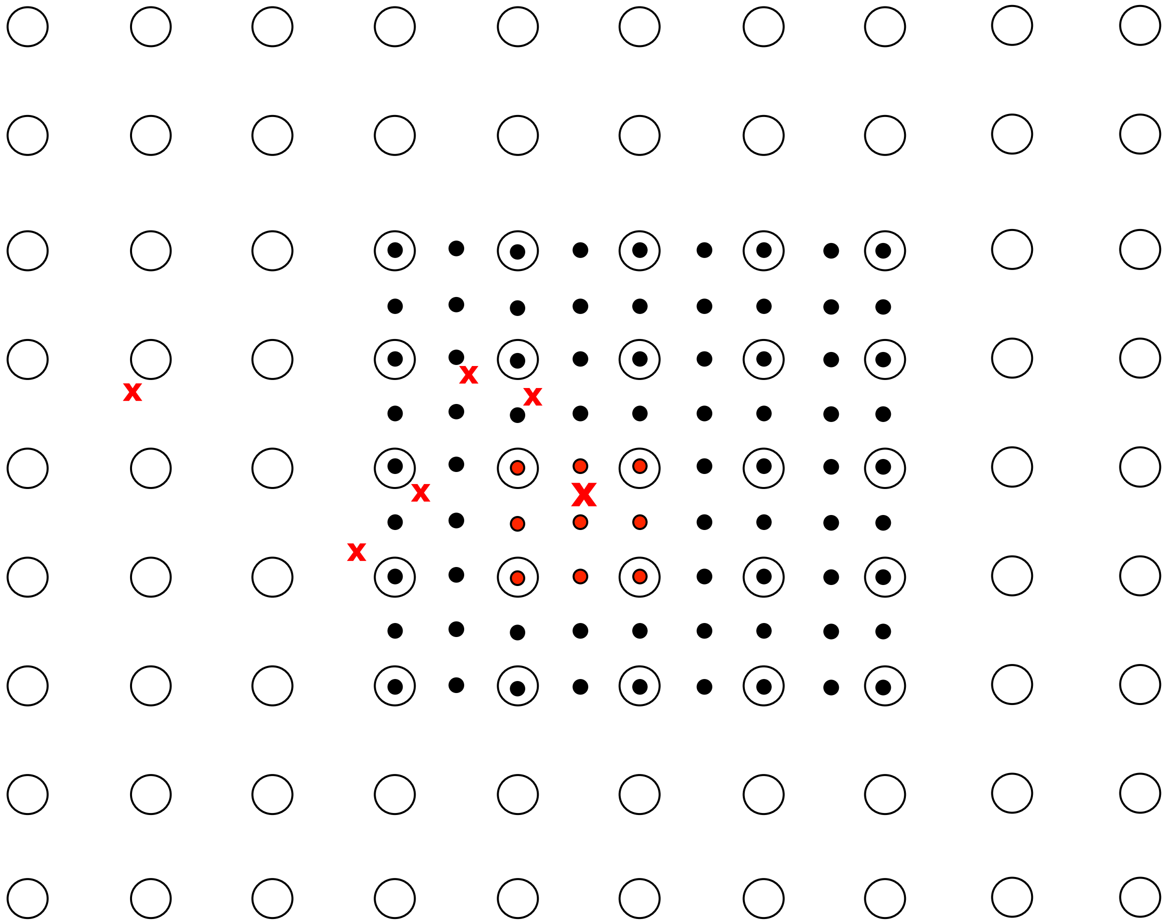
unproblematic:



Solving for Gravity

- density assignment (co-spatial scheme)

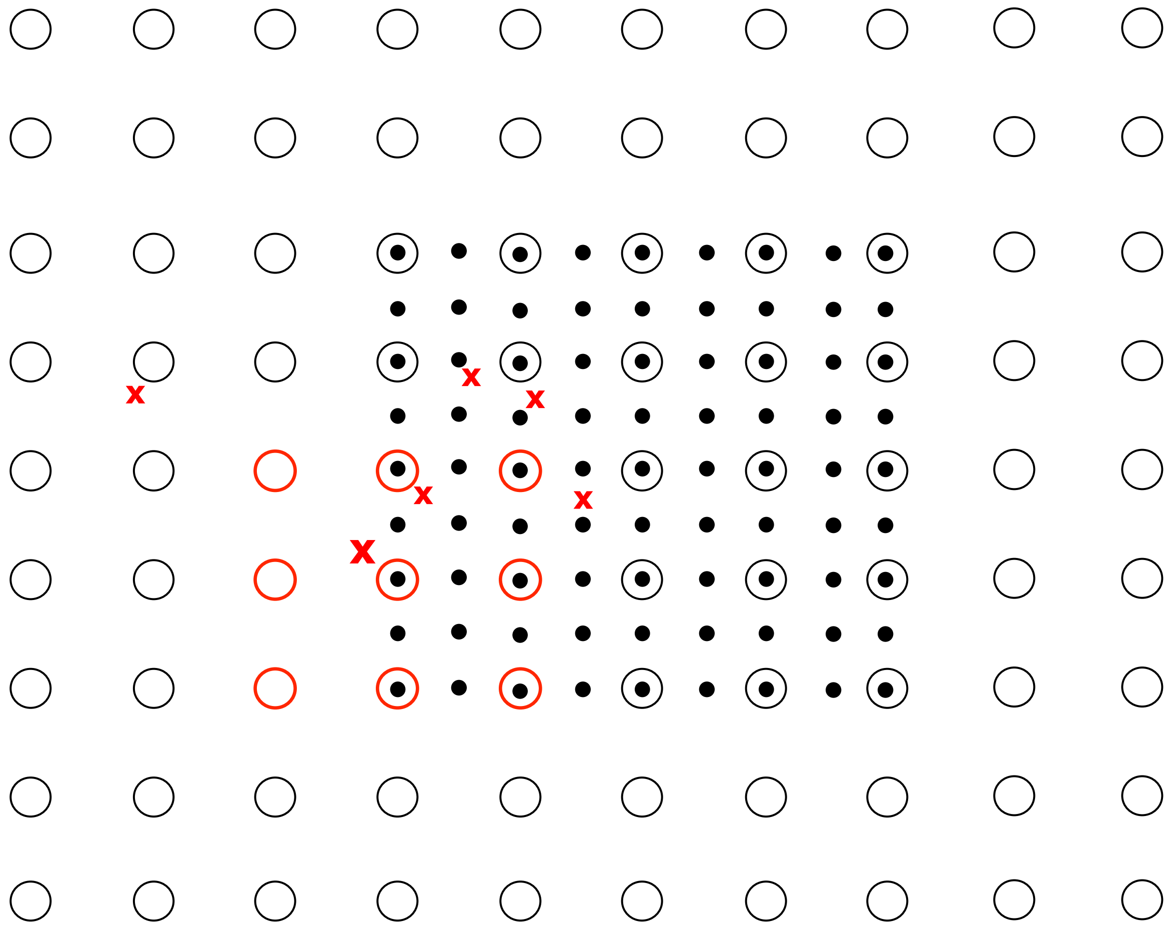
unproblematic:



Solving for Gravity

- density assignment (co-spatial scheme)

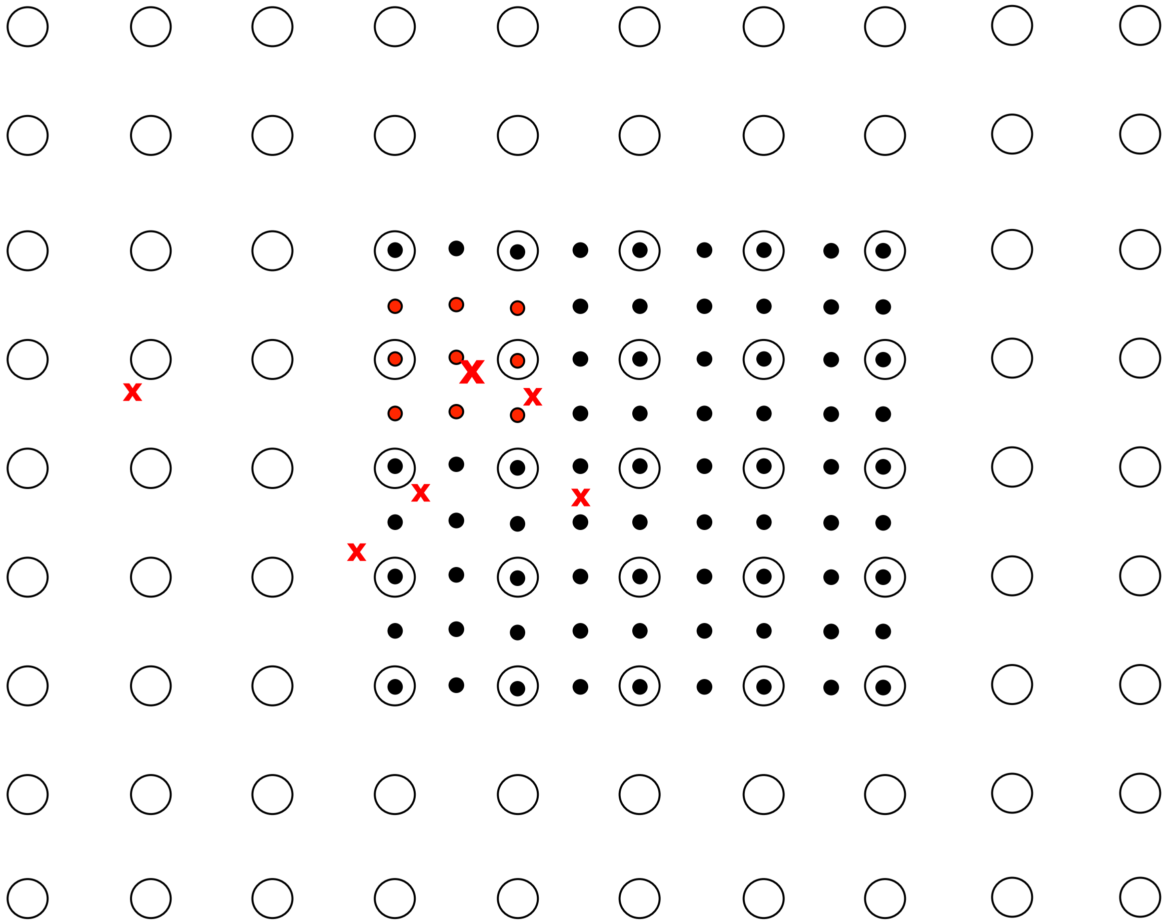
problematic:



Solving for Gravity

- density assignment (co-spatial scheme)

problematic:



Solving for Gravity

- density assignment (co-spatial scheme)

  - steps required to get density correct on both coarse and fine grid…

1. transfer particles from coarse to fine grid
2. assign "coarse" particles to coarse grid
3. assign "fine" particles to refinement grid
4. temporarily store "borderline" density
5. inject refinement density to coarse grid
6. add "borderline" density to refinement

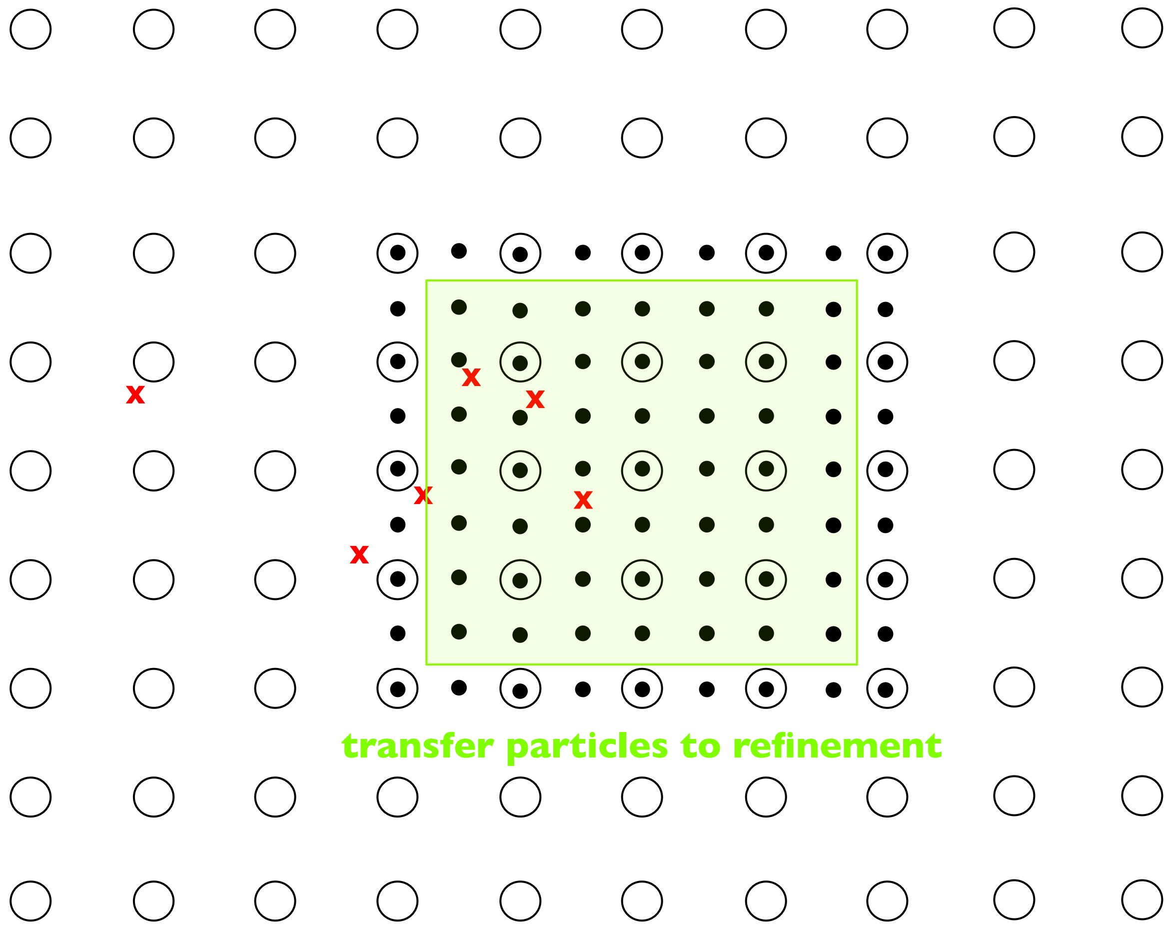
- density assignment (co-spatial scheme)

  - steps required to get density correct on both coarse and fine grid…

1. **transfer particles from coarse to fine grid**
2. assign "coarse" particles to coarse grid
3. assign "fine" particles to refinement grid
4. temporarily store "borderline" density
5. inject refinement density to coarse grid
6. add "borderline" density to refinement

Solving for Gravity

- density assignment (co-spatial scheme)



Solving for Gravity

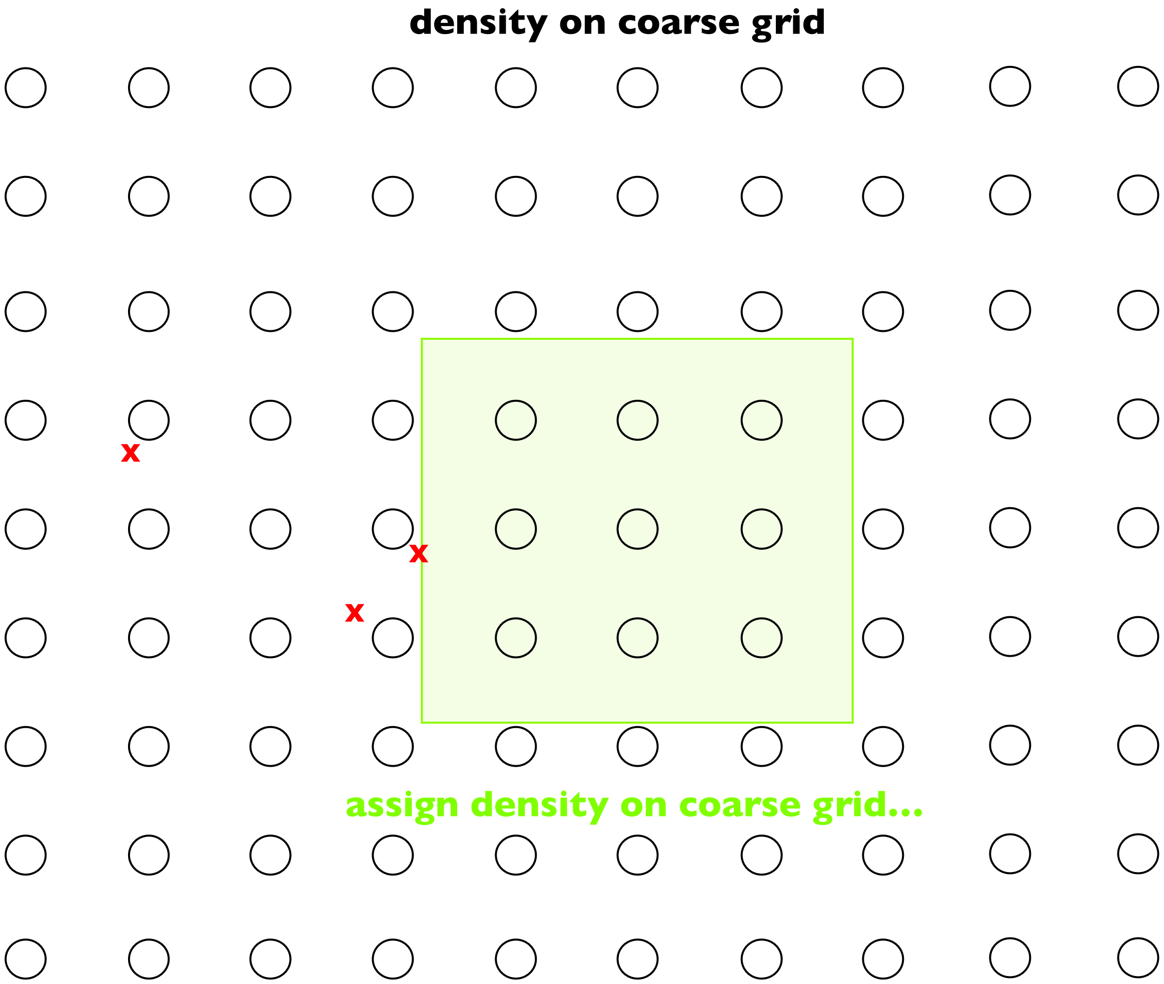
- density assignment (co-spatial scheme)

  • steps required to get density correct on both coarse and fine grid…

1. transfer particles from coarse to fine grid
2. **assign "coarse" particles to coarse grid**
3. assign "fine" particles to refinement grid
4. temporarily store "borderline" density
5. inject refinement density to coarse grid
6. add "borderline" density to refinement

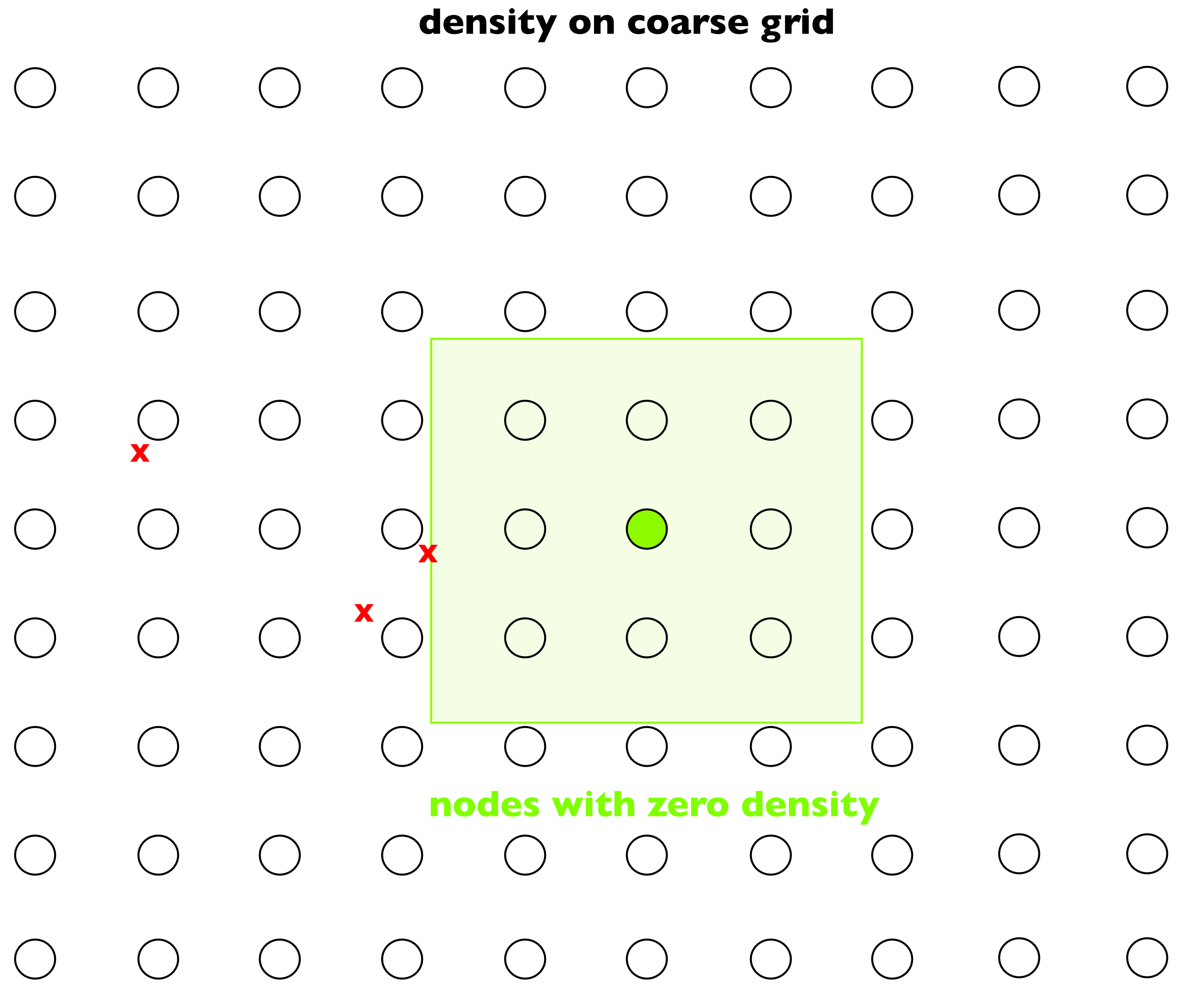
Solving for Gravity

- density assignment (co-spatial scheme)



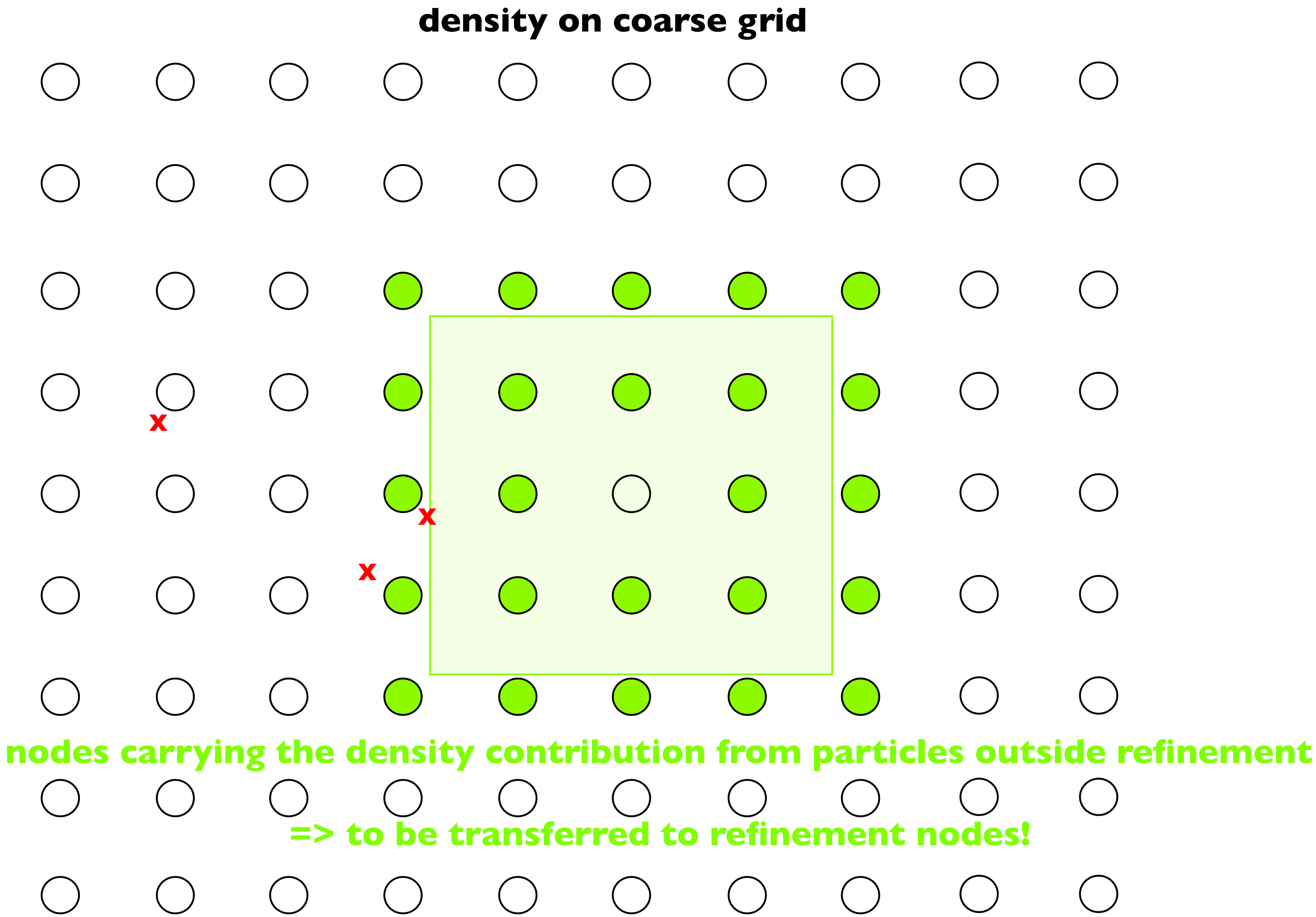
Solving for Gravity

- density assignment (co-spatial scheme)



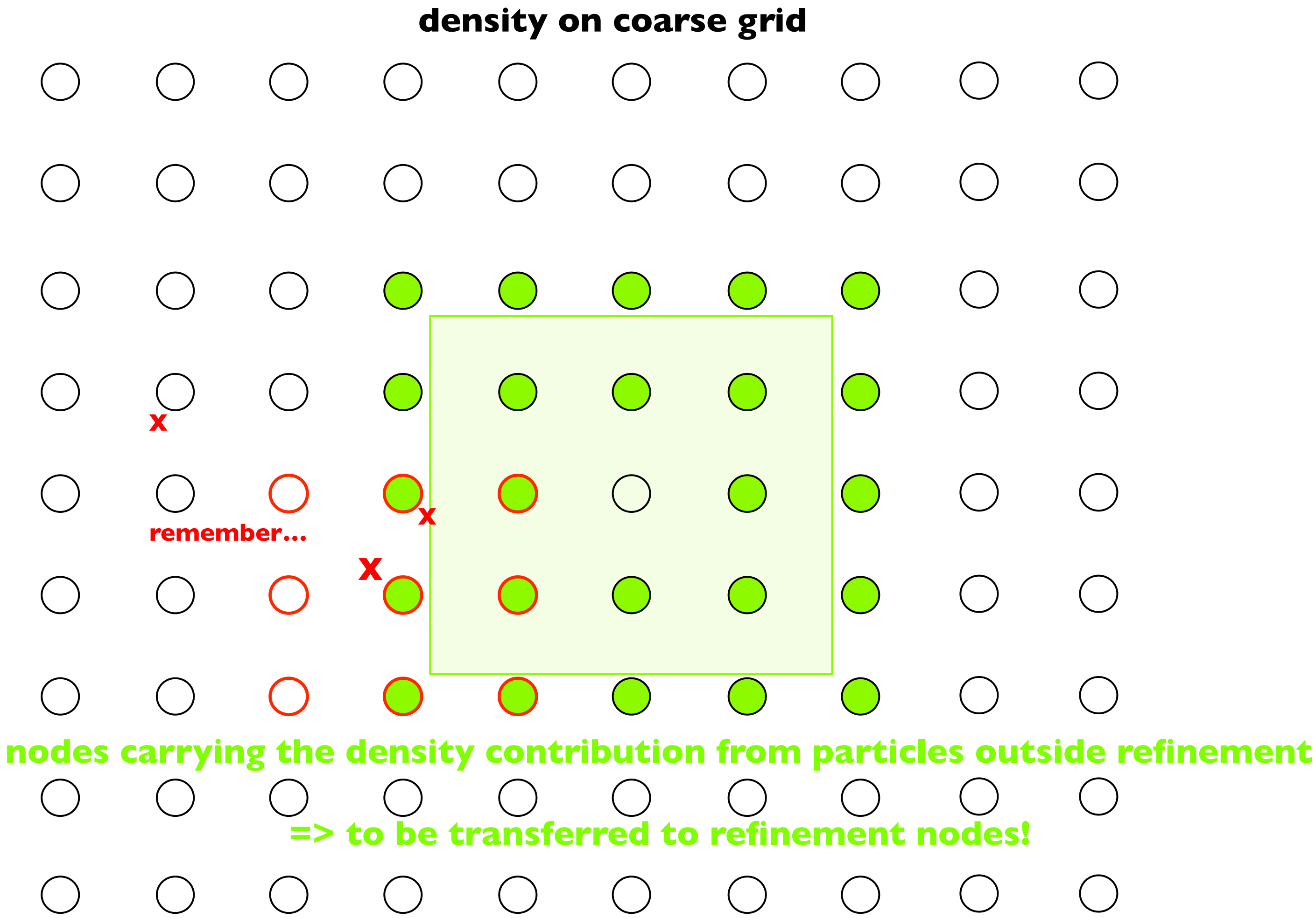
Solving for Gravity

- density assignment (co-spatial scheme)



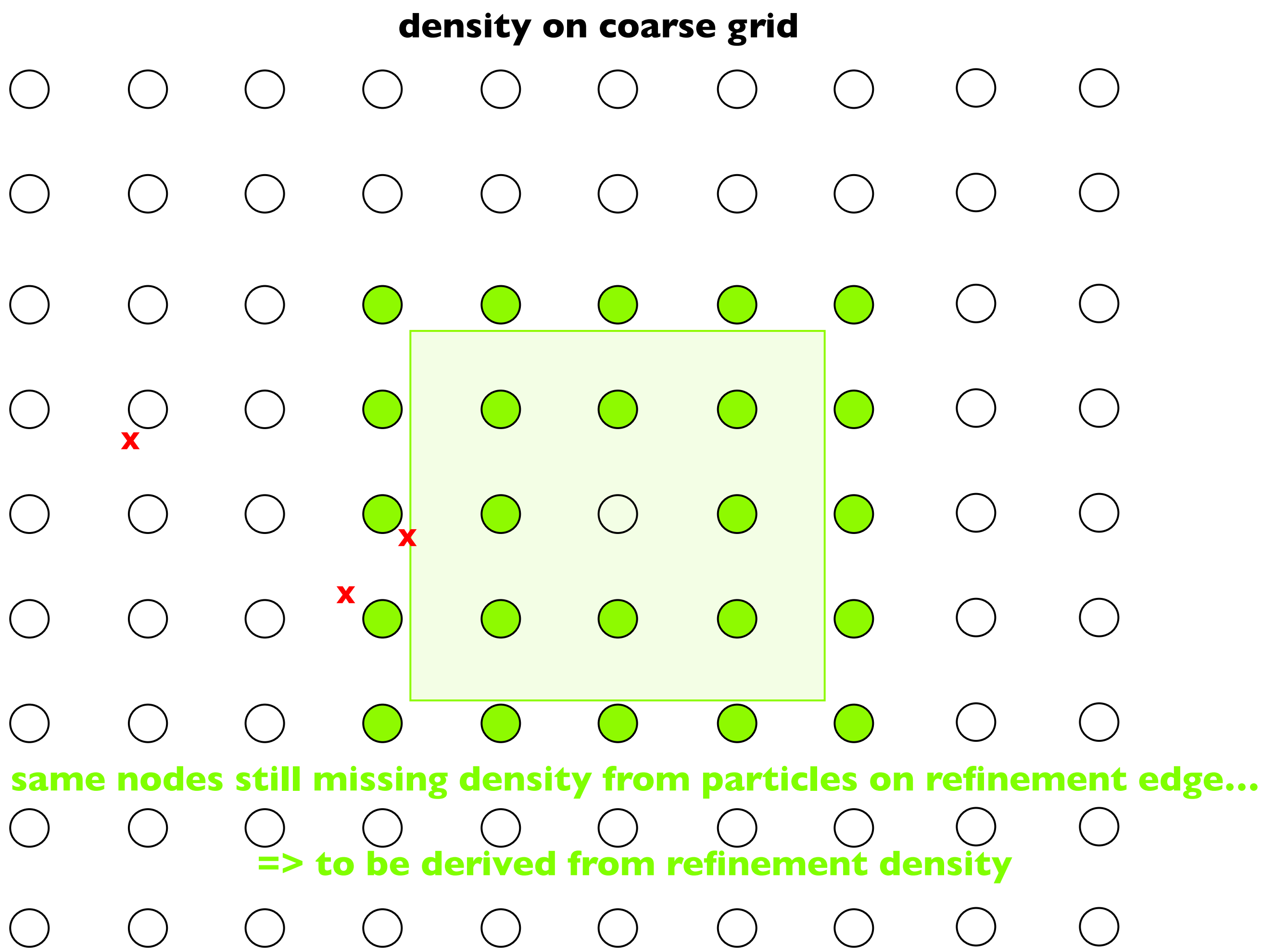
Solving for Gravity

- density assignment (co-spatial scheme)



Solving for Gravity

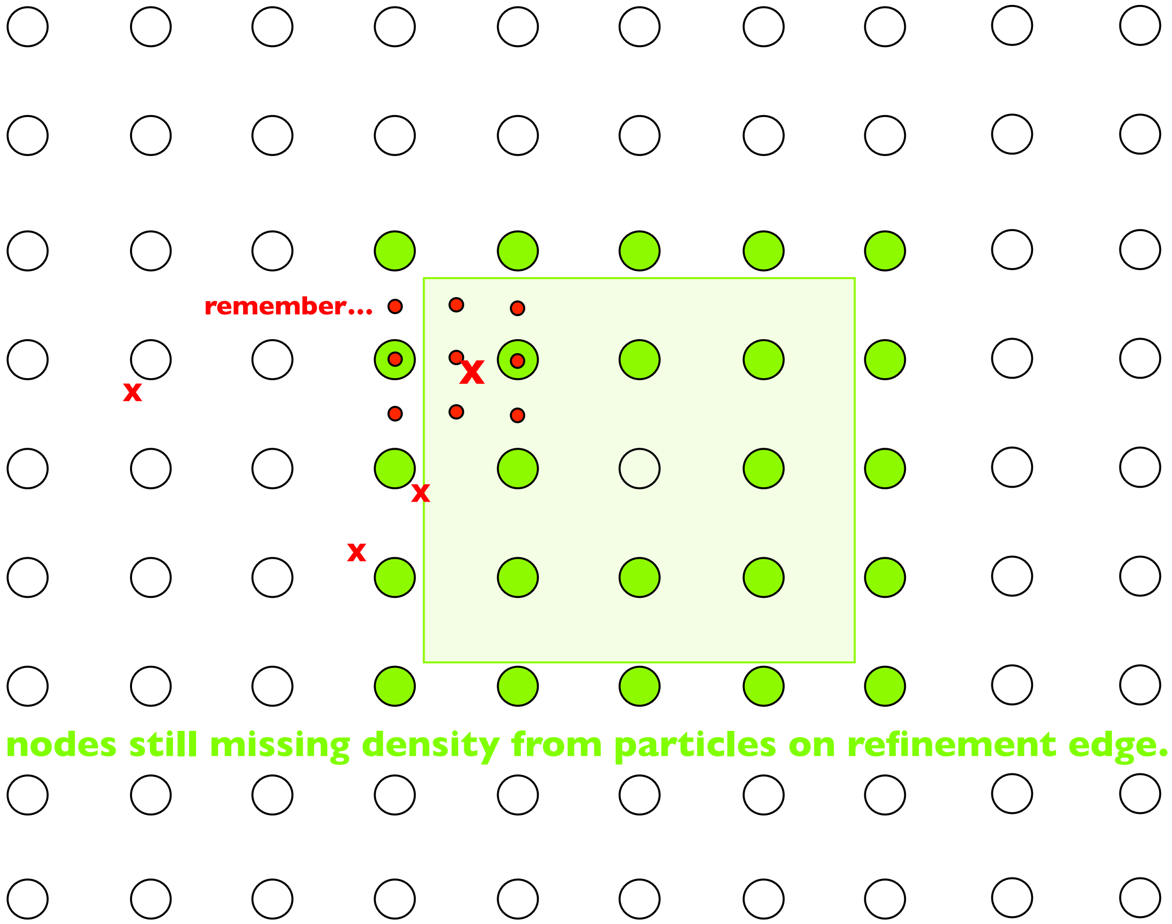
- density assignment (co-spatial scheme)



Solving for Gravity

- density assignment (co-spatial scheme)

density on coarse grid



nodes still missing density from particles on refinement edge…

Solving for Gravity

- density assignment (co-spatial scheme)

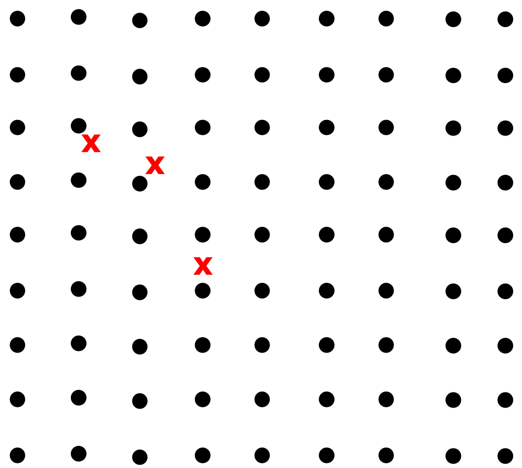
  • steps required to get density correct on both coarse and fine grid…

1. transfer particles from coarse to fine grid
2. assign "coarse" particles to coarse grid
3. **assign "fine" particles to refinement grid**
4. temporarily store "borderline" density
5. inject refinement density to coarse grid
6. add "borderline" density to refinement

Solving for Gravity

- density assignment (co-spatial scheme)

density on refinement grid

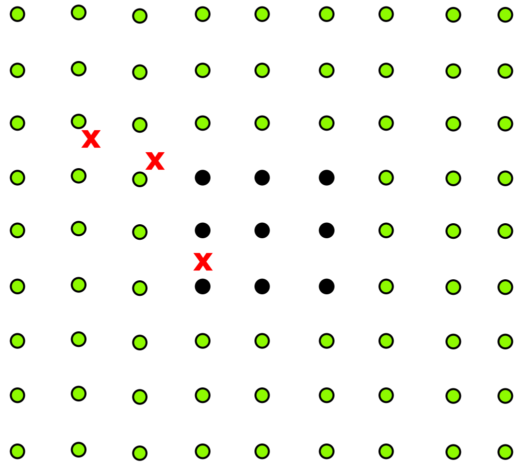


assign density on refinement grid…

Solving for Gravity

- density assignment (co-spatial scheme)

density on refinement grid

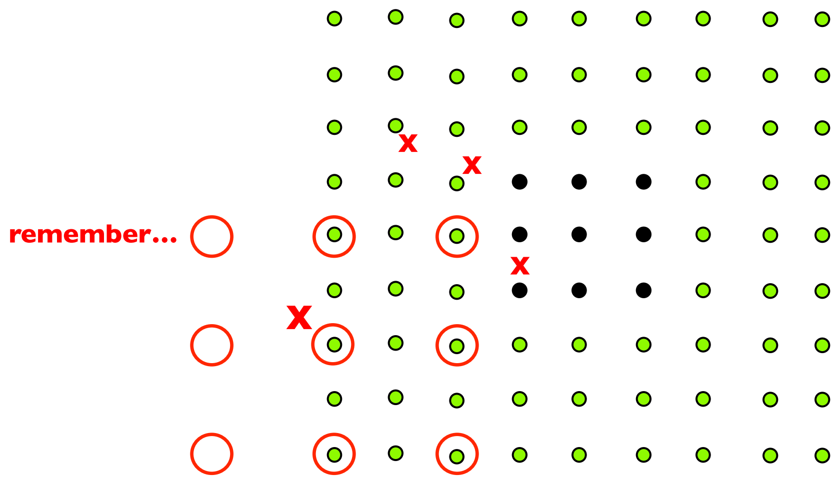


refinement nodes still missing the density contribution from particles outside refinement

Solving for Gravity

- density assignment (co-spatial scheme)

density on refinement grid

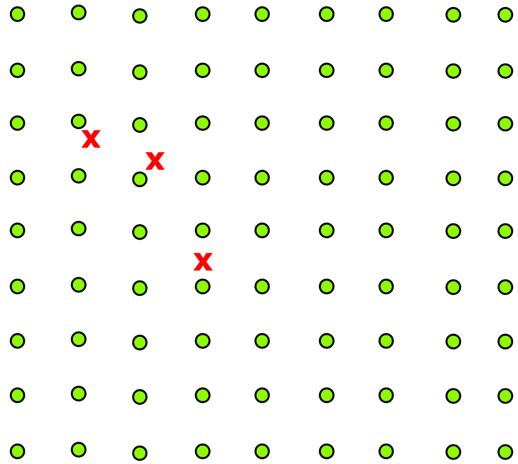


refinement nodes still missing the density contribution from particles outside refinement

Solving for Gravity

- density assignment (co-spatial scheme)

density on refinement grid



all refinement nodes carry information required by coarse nodes...

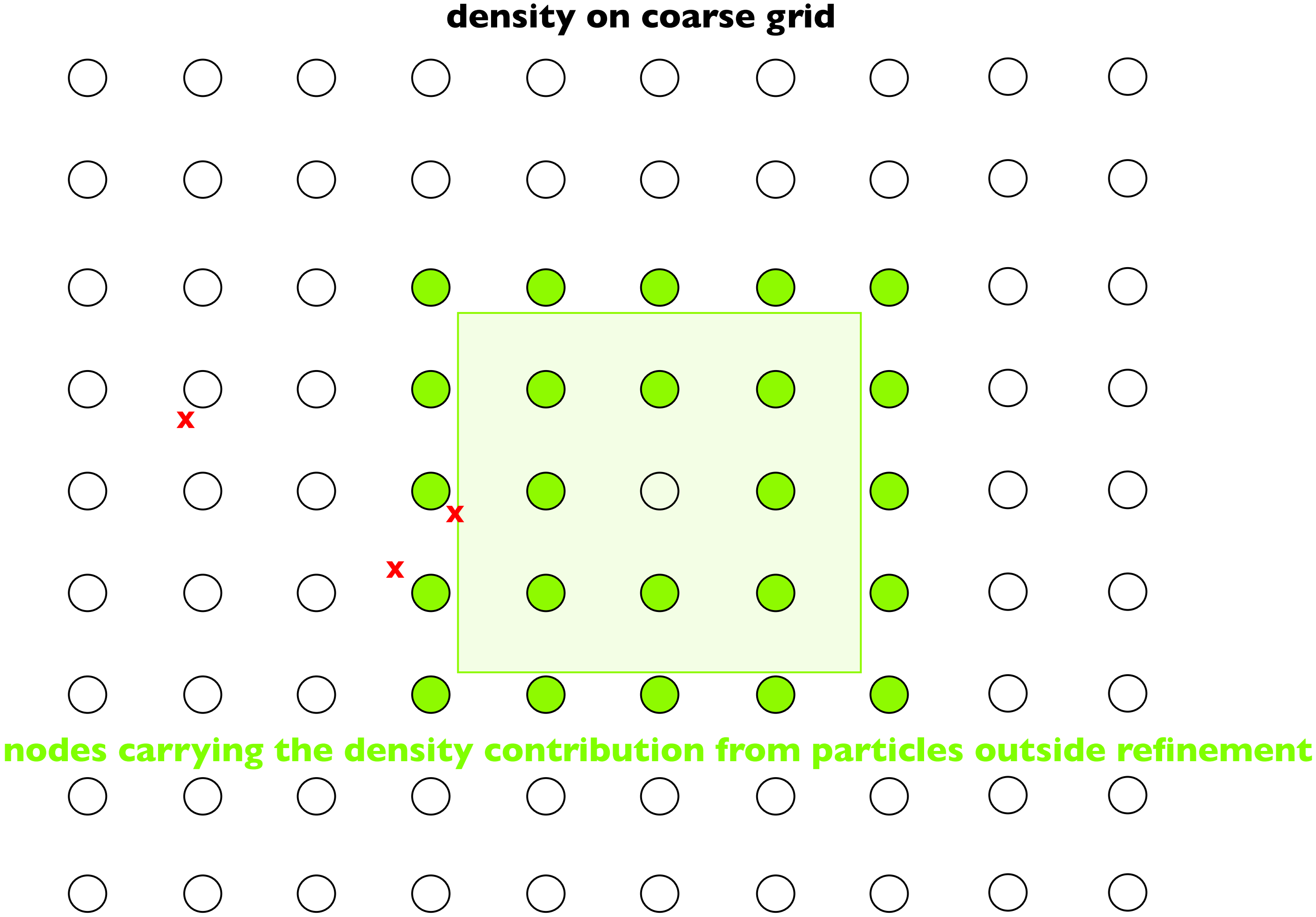
- density assignment (co-spatial scheme)

  • steps required to get density correct on both coarse and fine grid…

1. transfer particles from coarse to fine grid
2. assign "coarse" particles to coarse grid
3. assign "fine" particles to refinement grid
4. **temporarily store "borderline" density**
5. inject refinement density to coarse grid
6. add "borderline" density to refinement

Solving for Gravity

- density assignment (co-spatial scheme)



Solving for Gravity

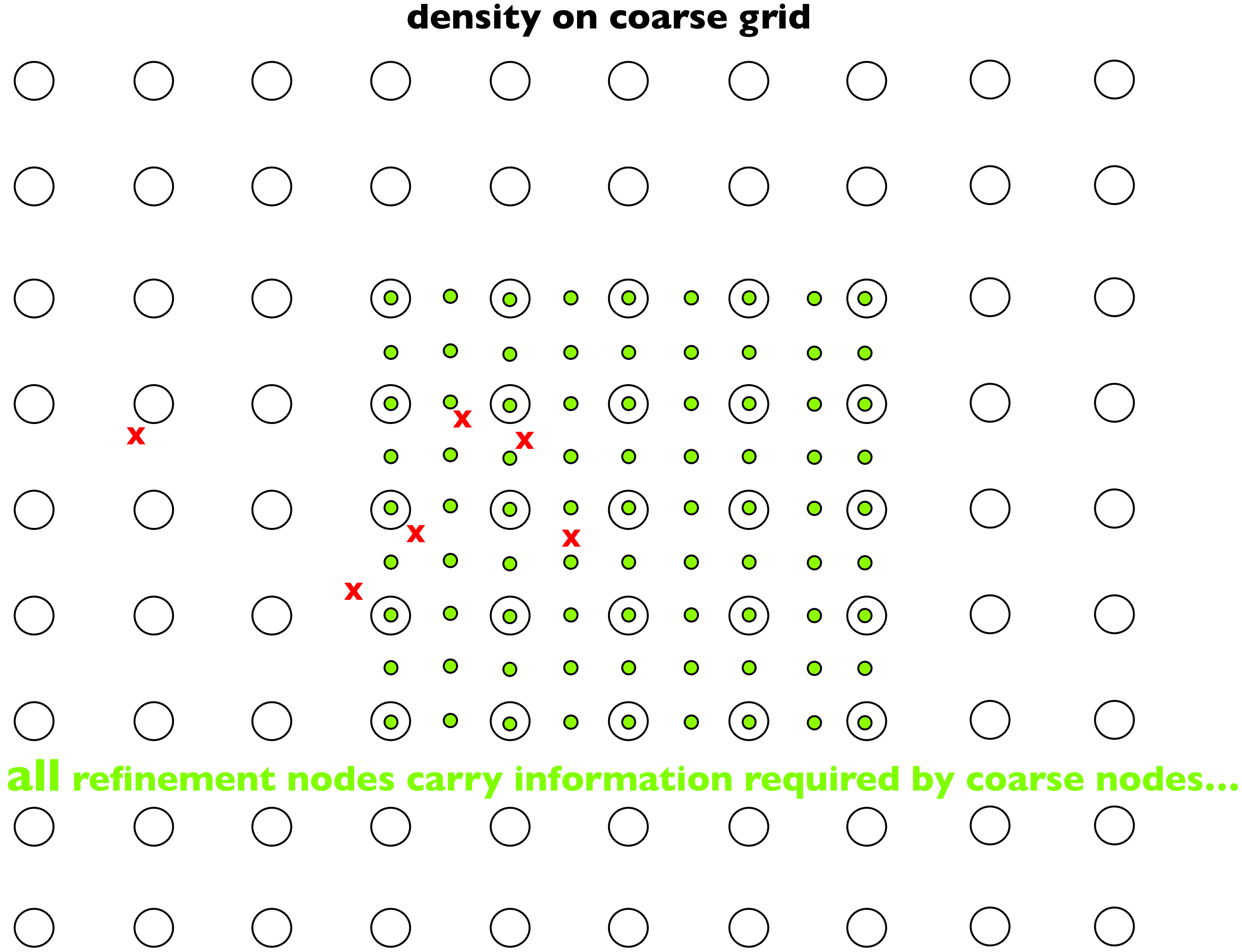
- density assignment (co-spatial scheme)

  - steps required to get density correct on both coarse and fine grid…

1. transfer particles from coarse to fine grid
2. assign "coarse" particles to coarse grid
3. assign "fine" particles to refinement grid
4. temporarily store "borderline" density
5. **inject refinement density to coarse grid**
6. add "borderline" density to refinement

Solving for Gravity

- density assignment (co-spatial scheme)



Solving for Gravity

- density assignment (co-spatial scheme)

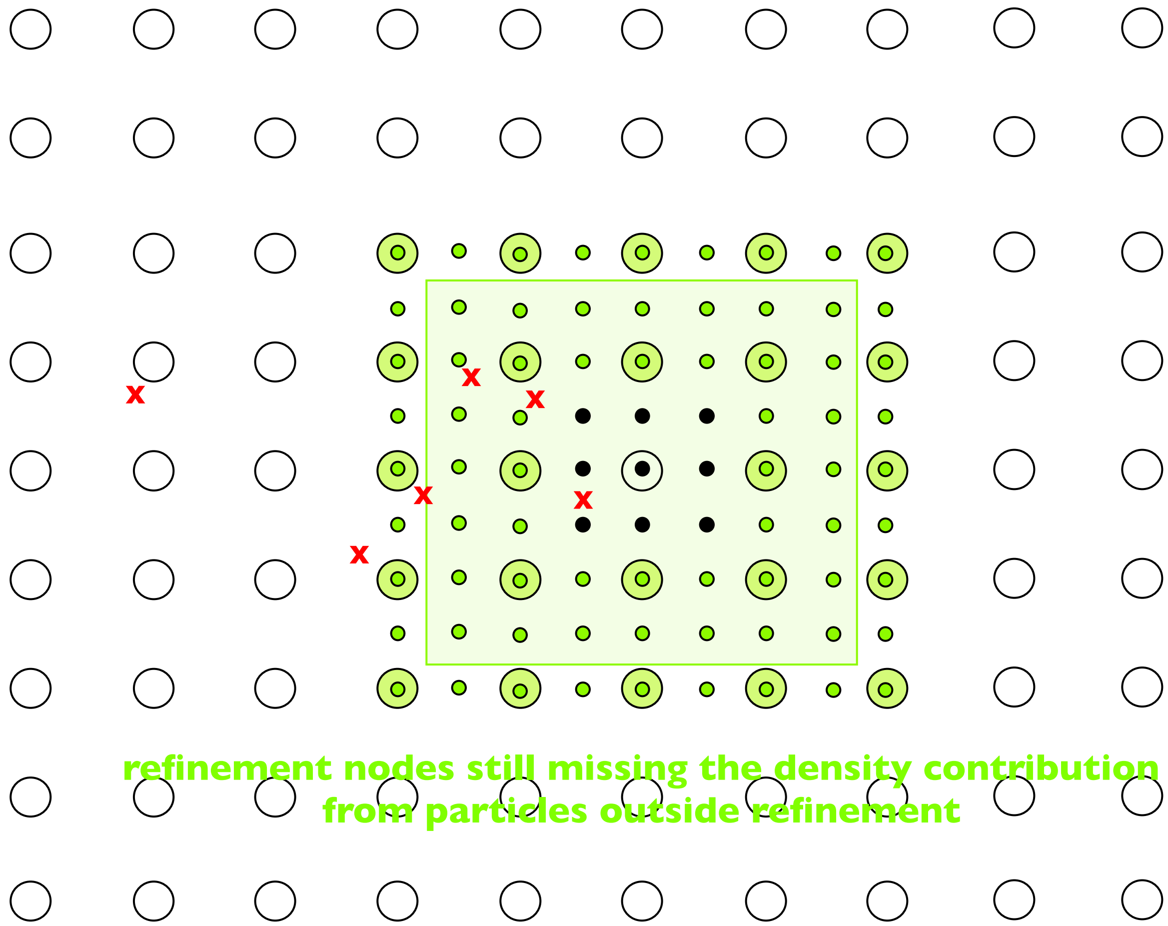
  - steps required to get density correct on both coarse and fine grid…

1. transfer particles from coarse to fine grid
2. assign "coarse" particles to coarse grid
3. assign "fine" particles to refinement grid
4. temporarily store "borderline" density
5. inject refinement density to coarse grid
6. **add "borderline" density to refinement**

Solving for Gravity

- density assignment (co-spatial scheme)

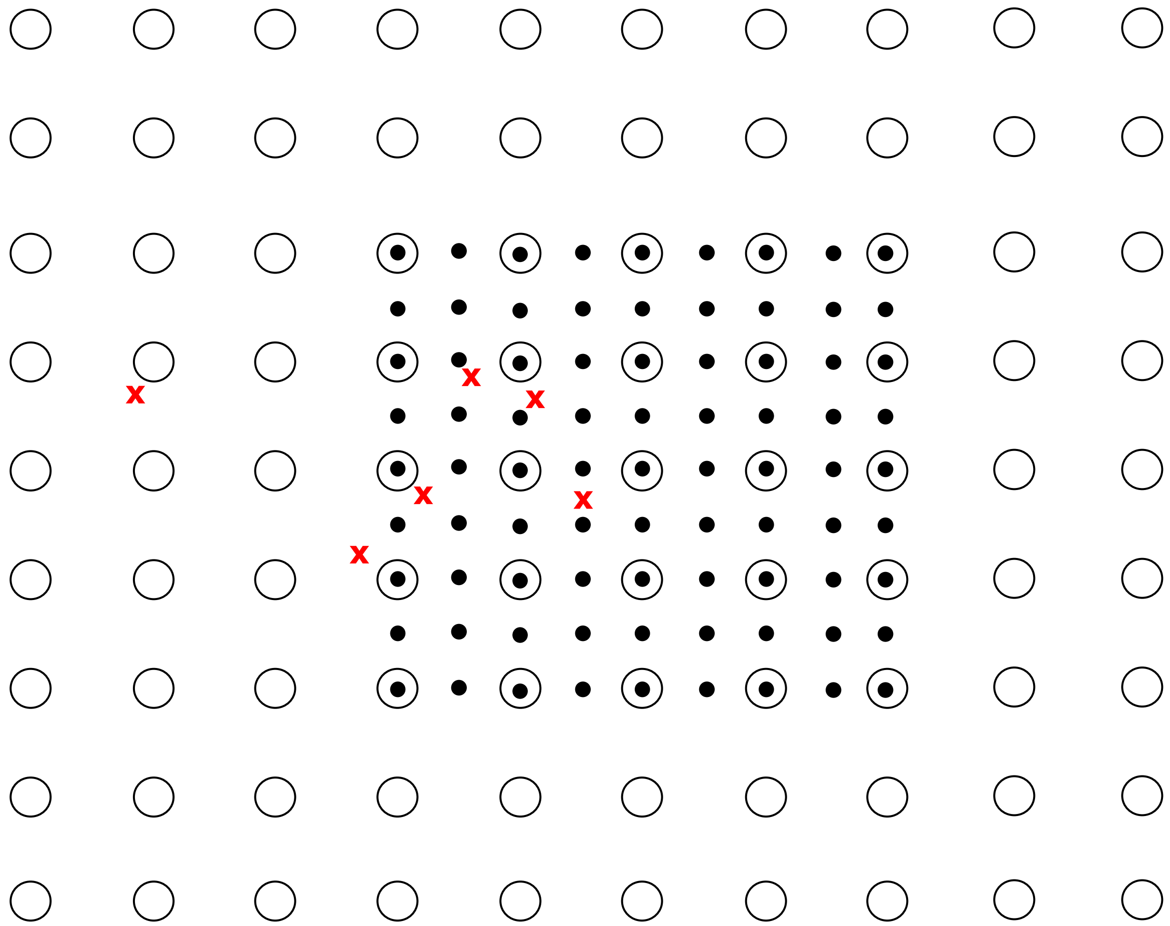
density on refinement grid



Solving for Gravity

- density assignment (co-spatial scheme)

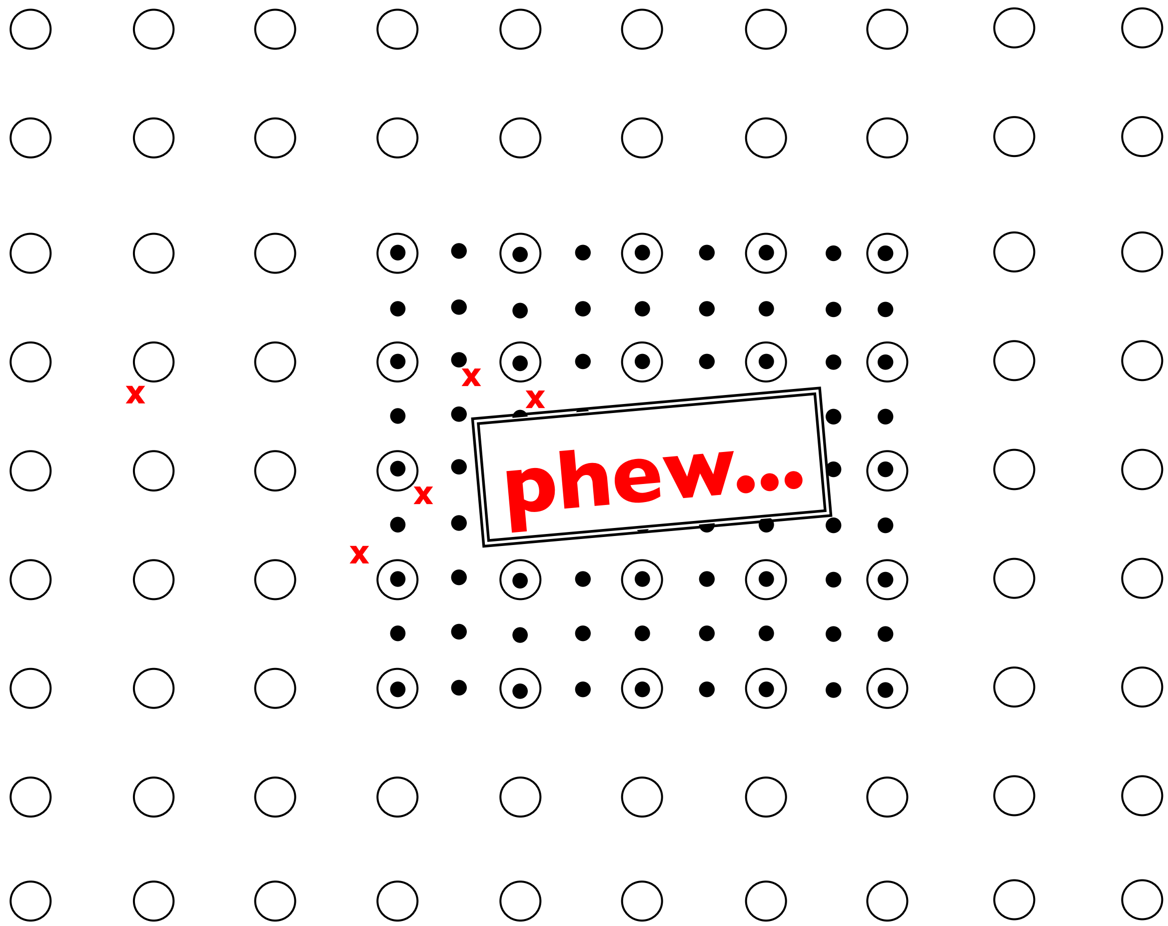
density finally correct on both levels...



Solving for Gravity

- density assignment (co-spatial scheme)

density finally correct on both levels…

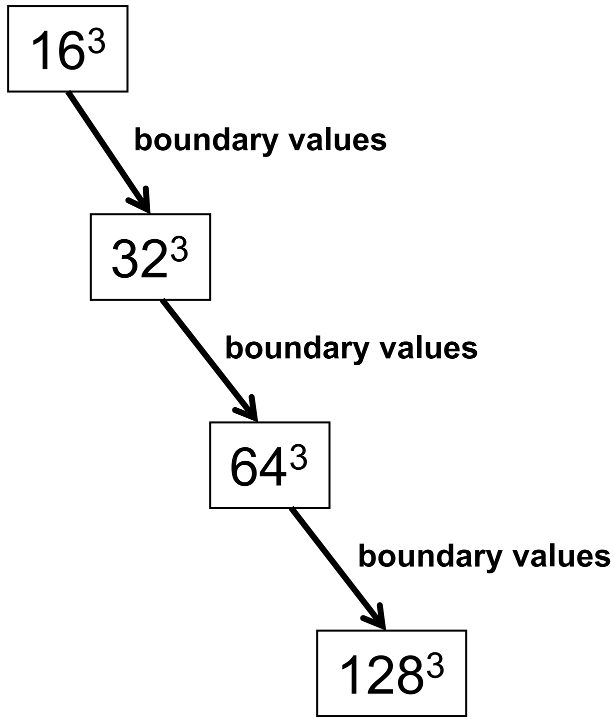


- mesh refinements
- adaptive mesh refinement
- **adaptive mesh refinement for N-body codes**
  - gravity
  - generating refinements
  - density assignment
  - ***solving Poisson's equation***
- handling irregular grids
- adaptive leap-frog integration

Solving for Gravity

- solving Poisson's equation

1. the domain grid:
   - relaxation, FFT, …
2. the refinement grids:
   - brute force relaxation!



Solving for Gravity

- adaptive mesh refinement
    - cover simulation with regular domain grid
    - create AMR hierarchy:
        - generate fine grid by comparing each node against some refinement criterion…

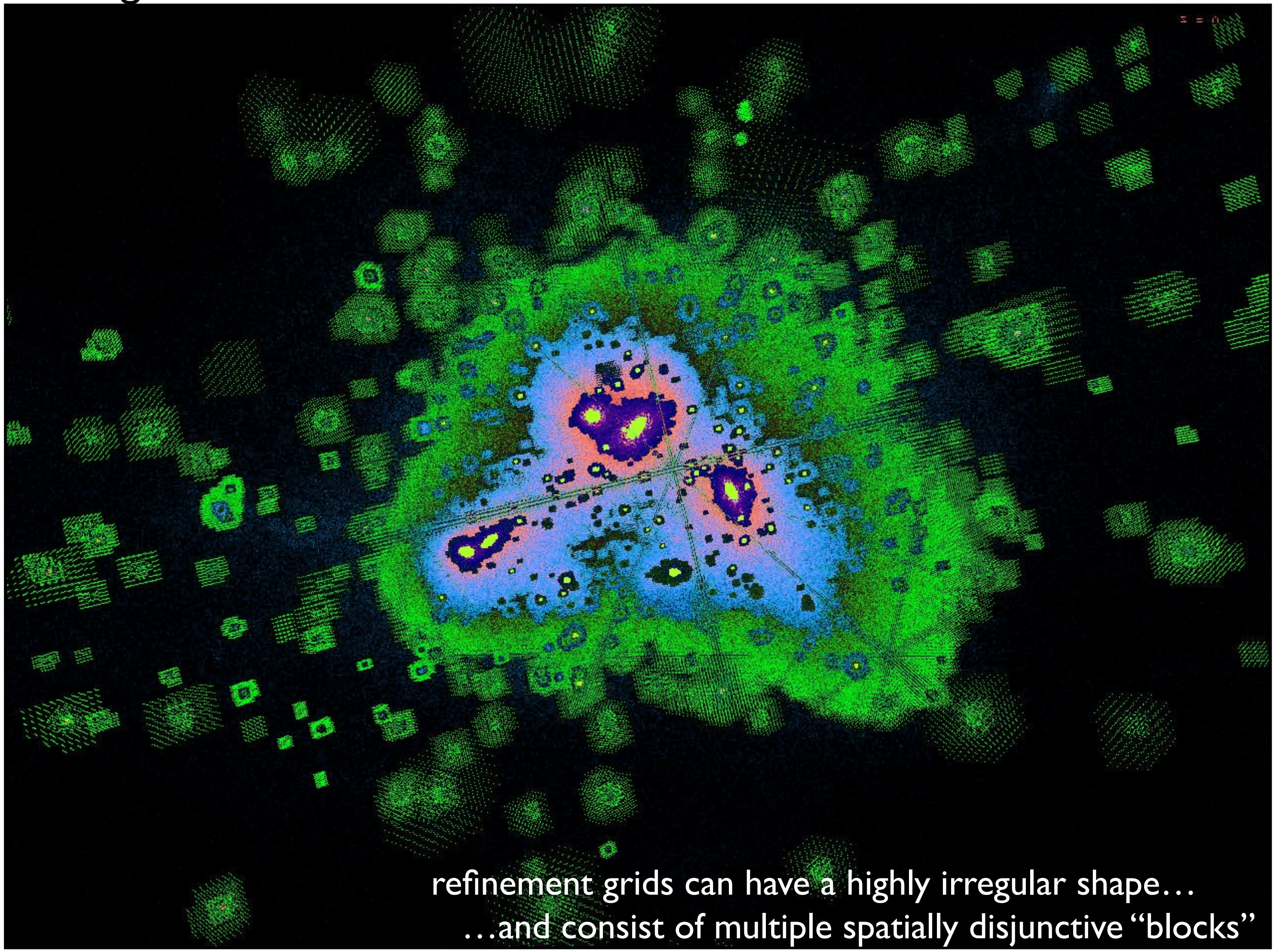
          ➔ recursive procedure!
    - assign density on all grids
    - solve Poisson's equation on regular domain grid (FFT is fastest…)
    - loop over all refinement levels:
        - interpolate potential down from parent level
        - relax potential until converged (keeping boundary values fixed)

          ➔ this will give the correct potential on all (refinement) grids

- mesh refinements
- adaptive mesh refinement
- adaptive mesh refinement for N-body codes
- **handling irregular grids**
- adaptive leap-frog integration

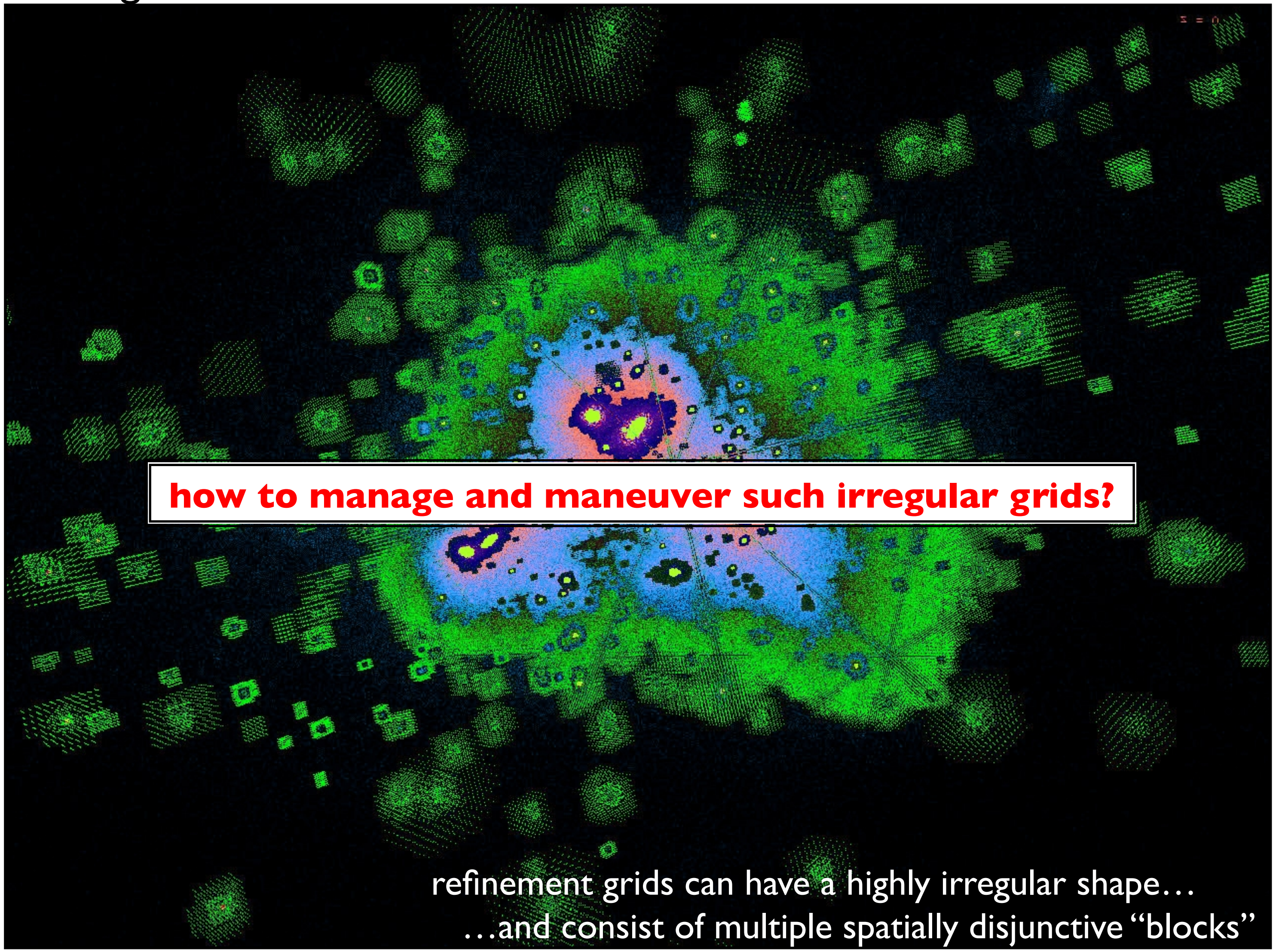
Solving for Gravity

- handling refinements



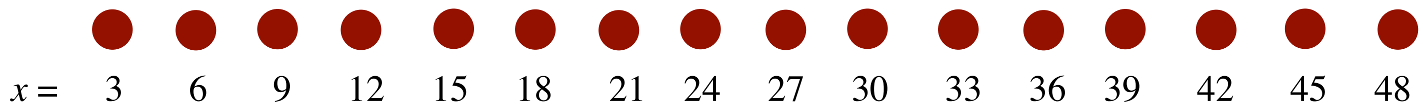
Solving for Gravity

- handling refinements



Solving for Gravity

- handling regular grids (1D)



N = 16

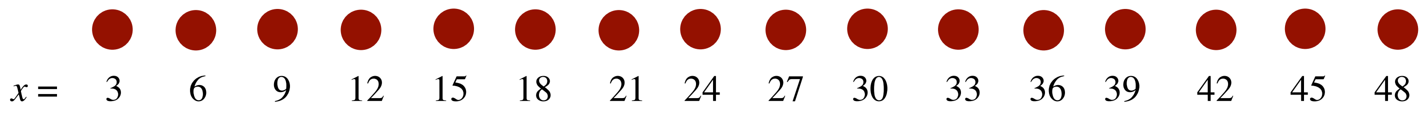
node[0].rho, x=3
node[1].rho, x=6
…,
node[N-1].rho, x=48

```
struct {

 float rho;
 …
} node;
```

Solving for Gravity

- handling regular grids (1D)



N = 16

node[0].rho, $x=3$
node[1].rho, $x=6$
…,
node[N-1].rho, $x=48$

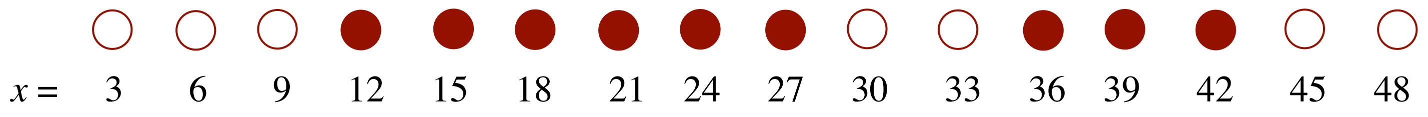
```
struct {

 float rho;
 …
} node;
```

unique mapping between array index *i* and spatial position *x* possible

Solving for Gravity

- handling irregular grids (1D)

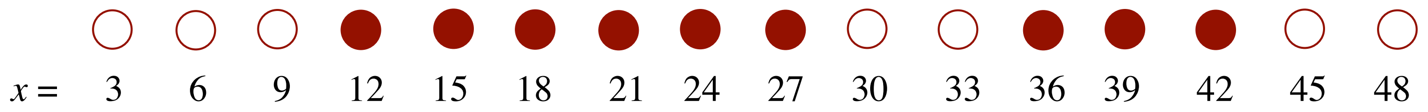


N = 9

```
struct {
 float rho;
 …
} node;
```

Solving for Gravity

- handling irregular grids (1D)



N = 9

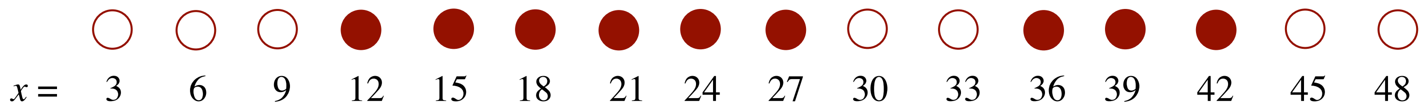
node[0].x = 12
node[1].x = 15
node[2].x = 18
node[3].x = 21
node[4].x = 24
node[5].x = 27

node[6].x = 36
node[7].x = 39
node[8].x = 42

```
struct {
 long  x;
 float rho;
 …
} node;
```

Solving for Gravity

- handling irregular grids (1D)



N = 9

node[0].x = 12
node[1].x = 15
node[2].x = 18
node[3].x = 21
node[4].x = 24
node[5].x = 27

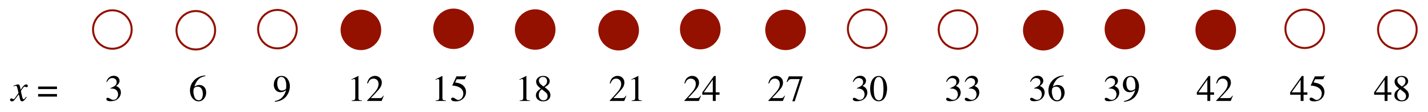
node[6].x = 36
node[7].x = 39
node[8].x = 42

```
struct {
 long  x;
 float rho;
 ...
} node;
```

no unique mapping between array index *i* and spatial position *x* possible

Solving for Gravity

- handling irregular grids (1D)



N = 9

node[0].x = 12
node[1].x = 15
node[2].x = 18
node[3].x = 21
node[4].x = 24
node[5].x = 27

node[6].x = 36
node[7].x = 39
node[8].x = 42

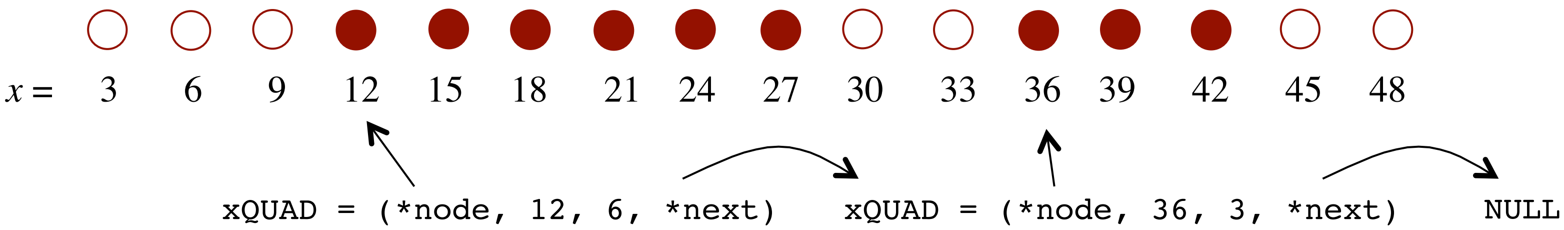
```
struct {
 long  x;
 float rho;
 …
} node;
```

no … x **possible**

generate a meta-structure storing the geometry of the grid

Solving for Gravity

- handling irregular grids (1D) ***quad's***



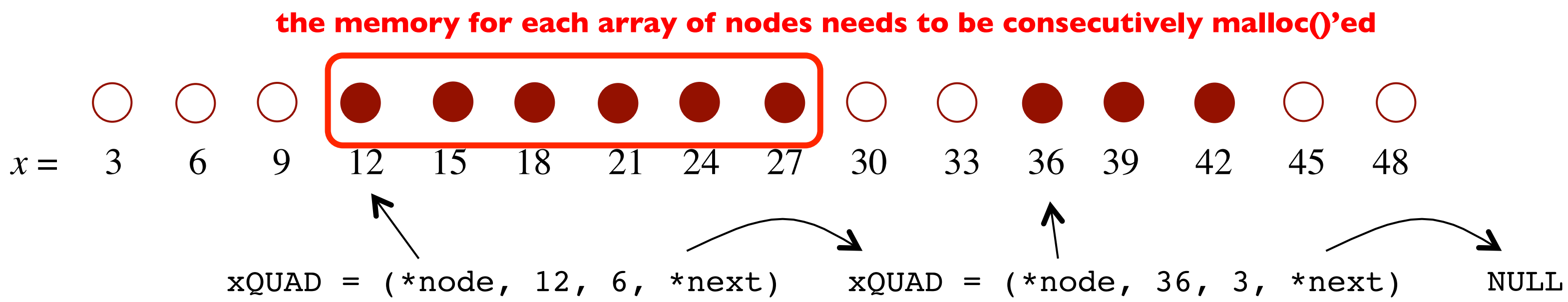

```
struct {

 float rho;
 …
} node;
```

Solving for Gravity

- handling irregular grids (1D)

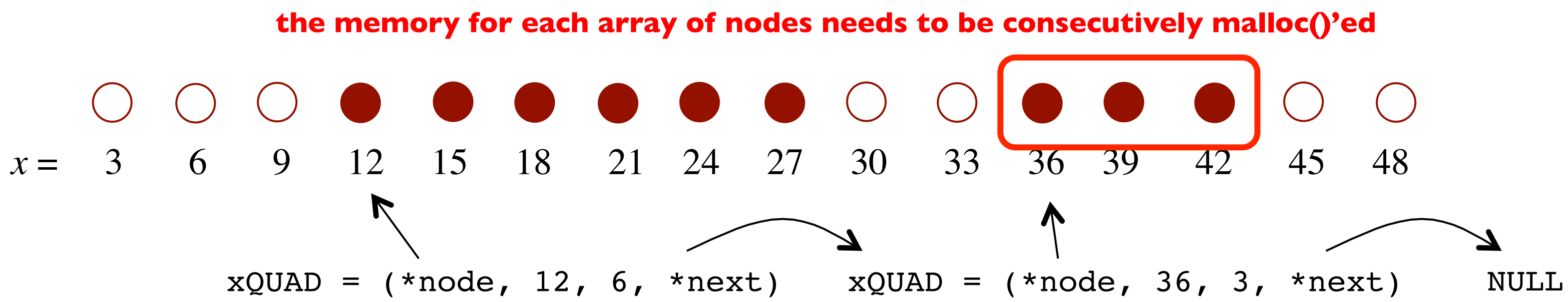
quad's




```
struct {
 float rho;
 …
} node;
```

Solving for Gravity

- handling irregular grids (1D) ***quad's***

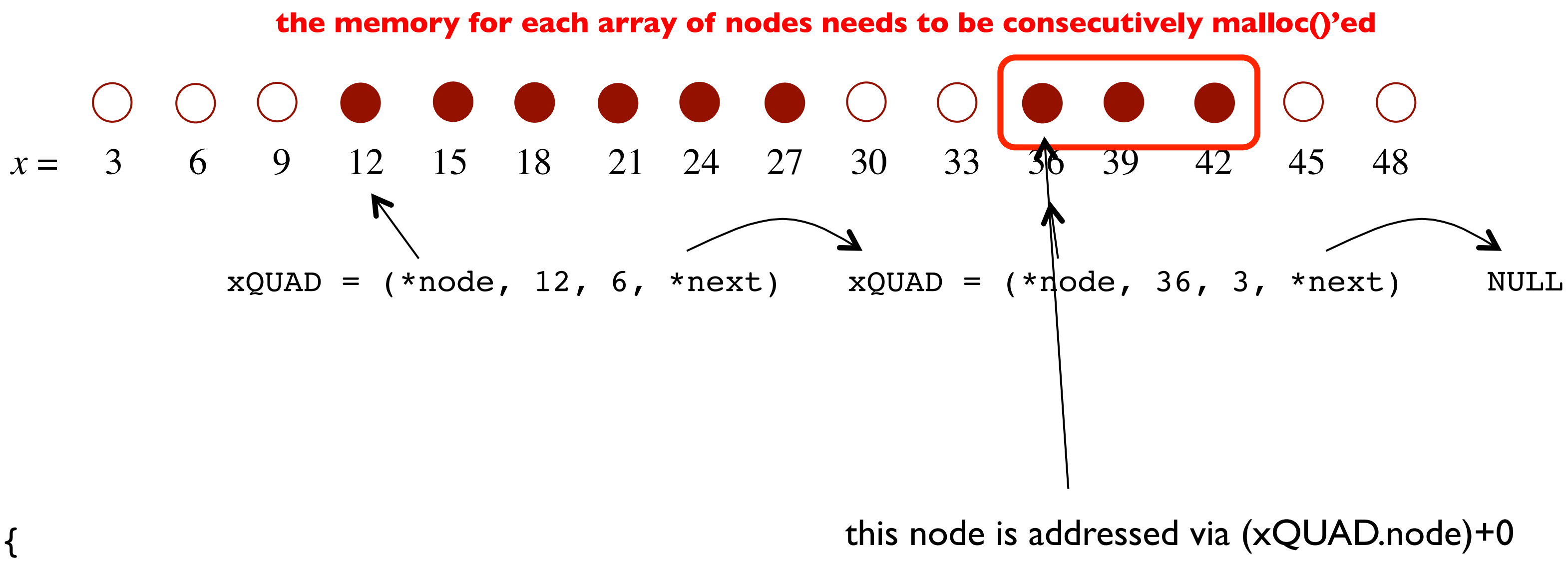



```
struct {

 float rho;
 …
} node;
```

Solving for Gravity

- handling irregular grids (1D) ***quad's***



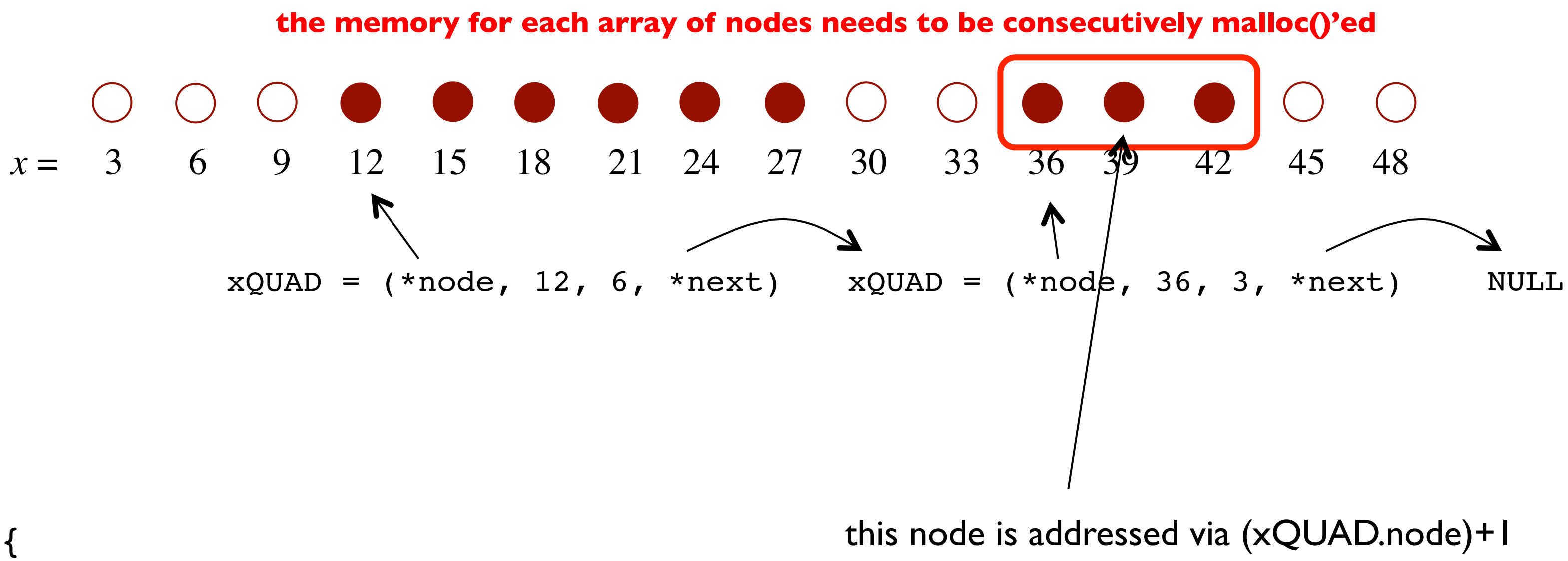

```
struct {

 float rho;
 …
} node;
```

Solving for Gravity

- handling irregular grids (1D)

quad's



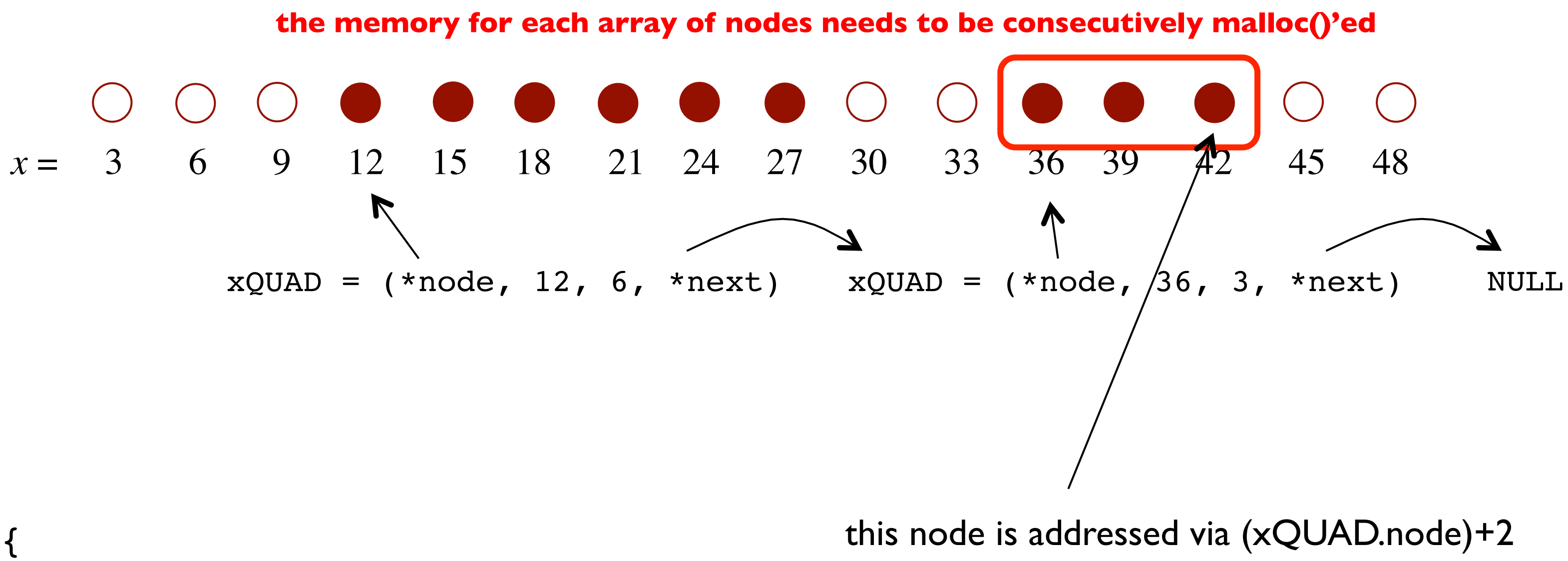

```
struct {

 float rho;
 …
} node;
```

Solving for Gravity

- handling irregular grids (1D)

quad's



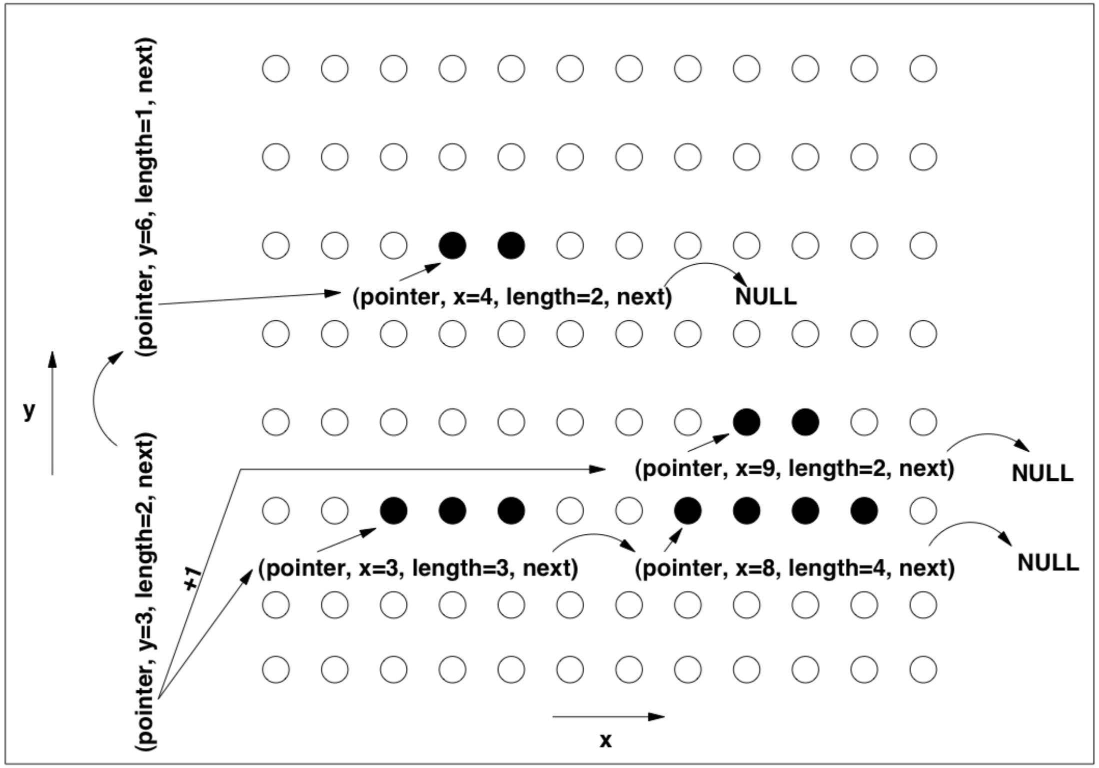

```
struct {

 float rho;
 …
} node;
```

Solving for Gravity

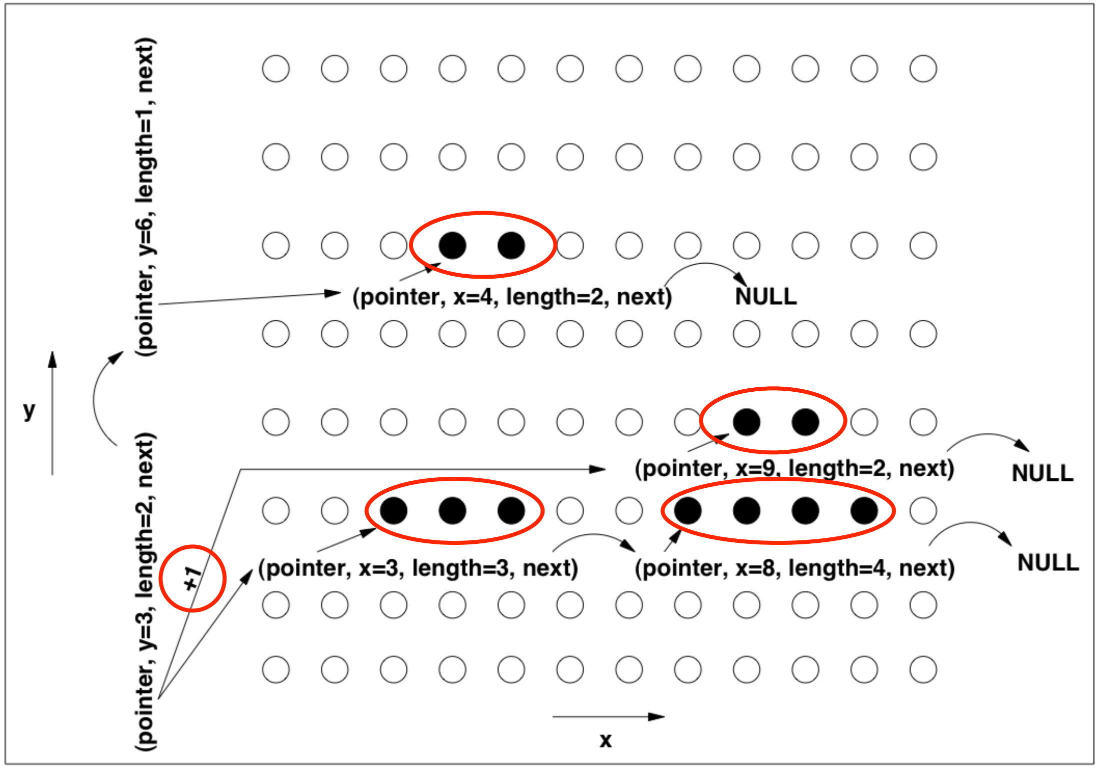
- handling irregular grids (2D)

quad's



Solving for Gravity

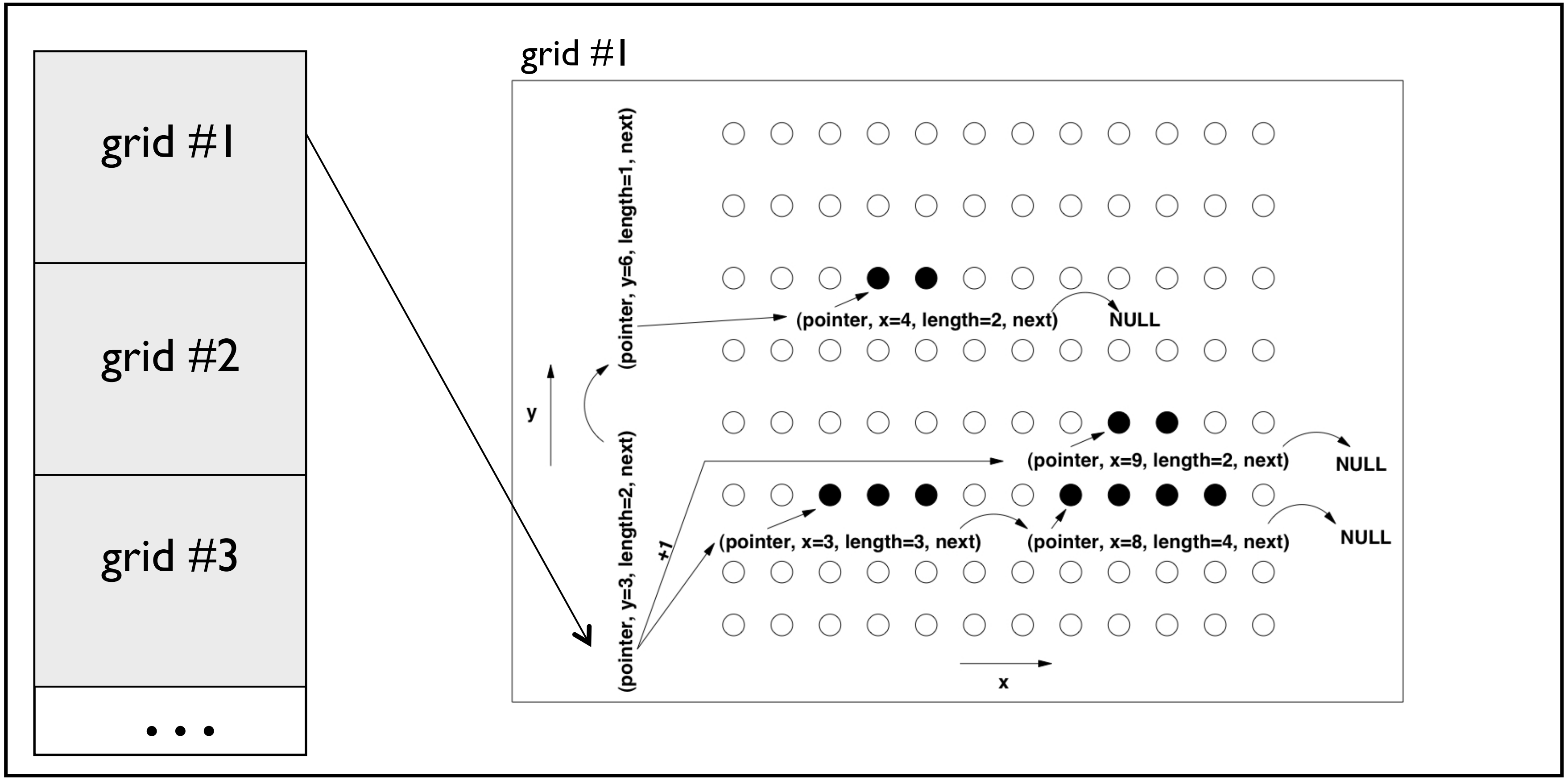
- handling irregular grids (2D) ***quad's***



the memory for each quad array needs to be malloc()'ed consecutively!

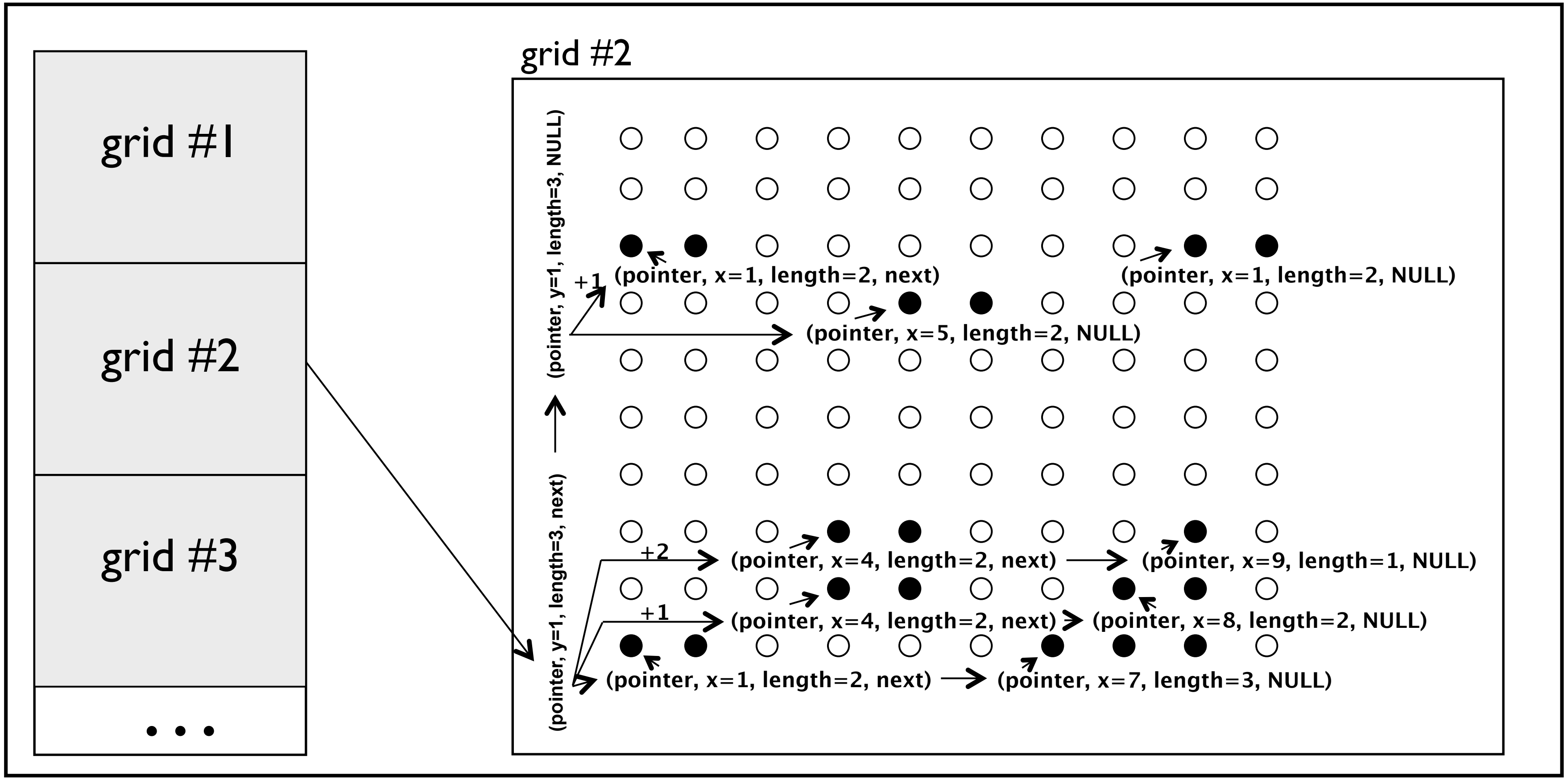
Solving for Gravity

- handling irregular grids (2D) ***quad's***
  - store "grid structures" as a consecutive memory block
  - each "grid" points to the first yQUAD which in turns gives access to all nodes



Solving for Gravity

- handling irregular grids (2D) ***quad's***
  - store "grid structures" as a consecutive memory block
  - each "grid" points to the first yQUAD which in turns gives access to all nodes



Solving for Gravity

- handling irregular grids (3D) ***quad's***

too complicated to sketch…

Solving for Gravity

- handling irregular grids (3D) ***quad's***
  - C-code example of how to loop over all nodes attached to a "grid"

```
for (zquad=grid.first_zquad; zquad != NULL; zquad=zquad->next) {
  for (yquad=zquad->first_yquad; yquad < yquad->pointer+yquad->length; yquad++)

    for (iyquad=yquad; iyquad != NULL; iyquad=iyquad->next) {
      for (xquad=yquad->first_xquad; xquad < xquad->pointer+xquad->length; xquad++)

        for (ixquad=xquad; ixquad != NULL; ixquad=ixquad->next) {
          for (node=ixquad->pointer; node < ixquad->x+ixquad->length; node++) {

            /* the node is at your disposal */
            density       = node->density;
            potential     = node->potential;
            forceX        = node->force[X];

            for(part=node->first_particle; part != NULL; part=part->next)
            /* use particle structure to access particle position, velocity, etc. */ }}}
```

`loc` = **loc**ation of first quad

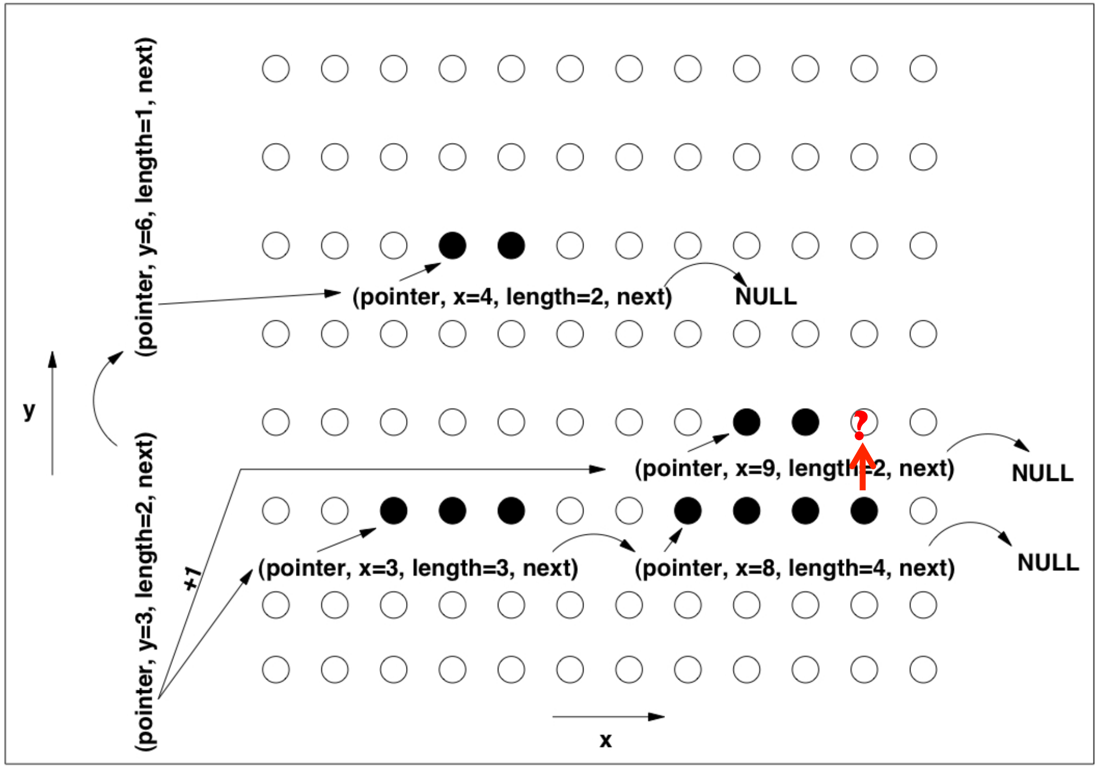
Solving for Gravity

- handling irregular grids

***quad*'s**

  - drawback:

    no direct access to neighbouring nodes…

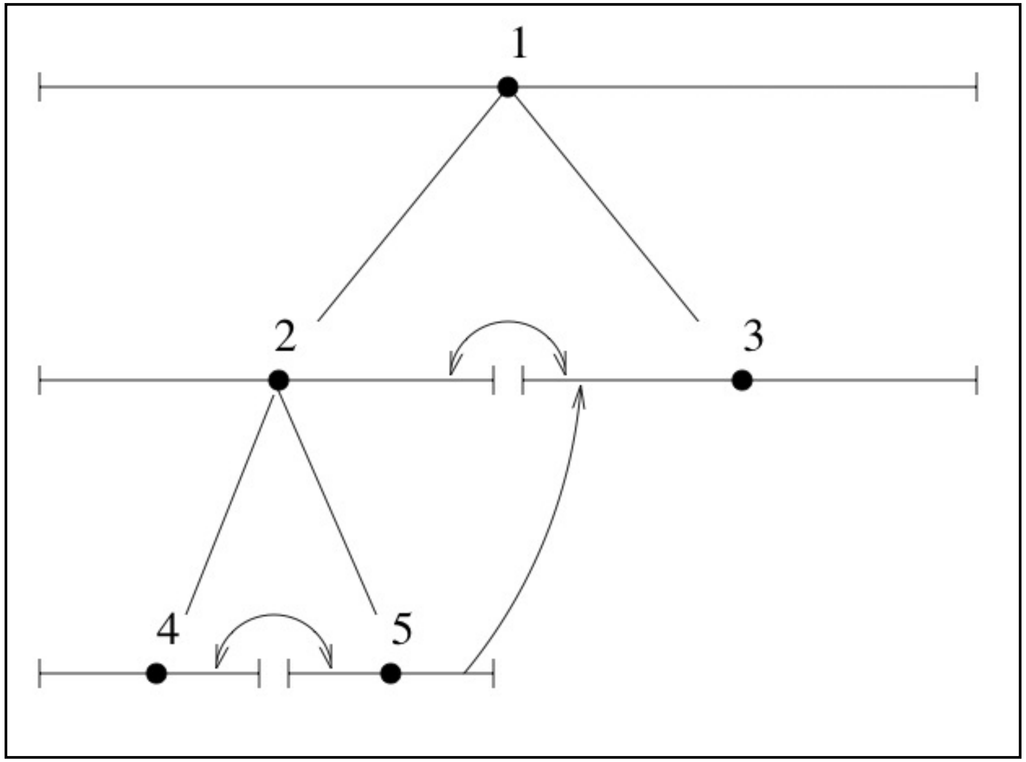


Solving for Gravity

- handling irregular grids **FTT**
  - other schemes possible:

```
struct {

 NODE *daughter;
 float rho;
 …
} node;
```



Fully-Threaded-Tree (FTT) by Khokhlov, 1998, J. Comp. Phy. 143, 519

(used with Andrey Kravtsov's ART code…)

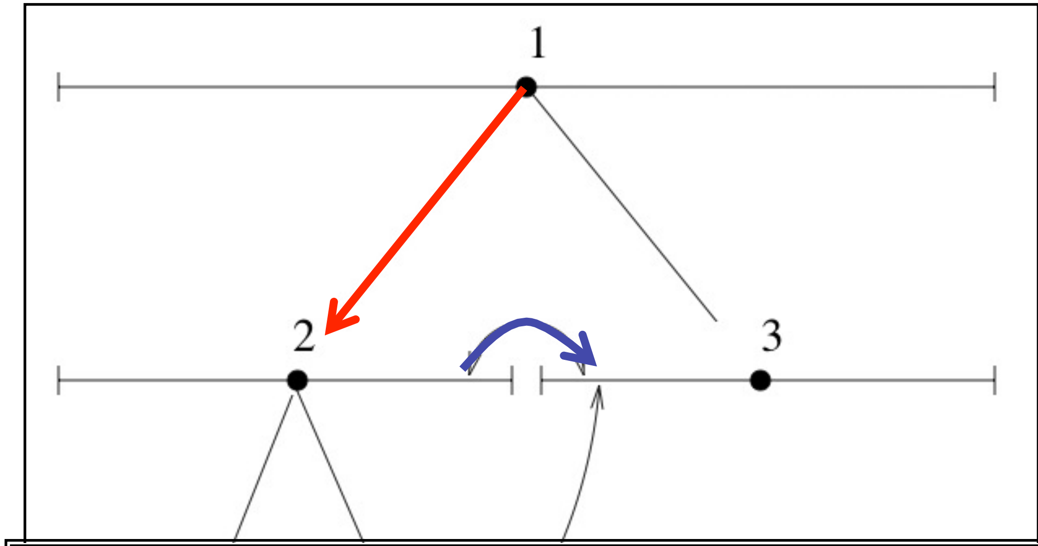
Solving for Gravity

- handling irregular grids **FTT**

  - other schemes possible:

```
struct {

 NODE *daughter;
 float rho;
 …
} node;
```



- **each cell stores pointers to 1st daughter**
- **daughters are malloc()'ed consecutively**

Fully-Threaded-Tree (FTT) by Khokhlov, 1998, J. Comp. Phy. 143, 519

(used with Andrey Kravtsov's ART code…)

- mesh refinements
- adaptive mesh refinement
- adaptive mesh refinement for N-body codes
- handling irregular grids
- **adaptive leap-frog integration**

Solving for Gravity

- full set of equations

• collisionless matter (e.g. dark matter)

$$\frac{d\vec{x}_{DM}}{dt} = \vec{v}_{DM}$$
$$\frac{d\vec{v}_{DM}}{dt} = -\nabla\phi$$

leap-frog integration

AMR solver

• Poisson's equation

$$\Delta\phi = 4\pi G\rho_{tot}$$

• collisional matter (e.g. gas)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

$$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot\left(\rho\vec{v}\otimes\vec{v} + \left(p + \frac{1}{2\mu}B^2\right)\vec{1} - \frac{1}{\mu}\vec{B}\otimes\vec{B}\right) = \rho\ (-\nabla\phi)$$

$$\frac{\partial(\rho E)}{\partial t} + \nabla\cdot\left(\left[\rho E + p + \frac{1}{2\mu}B^2\right]\vec{v} - \frac{1}{\mu}\left[\vec{v}\cdot\vec{B}\right]\vec{B}\right) = \rho\vec{v}\cdot(-\nabla\phi) + (\Gamma - L)$$

• ideal gas equations

$$p = (\gamma - 1)\rho\varepsilon$$

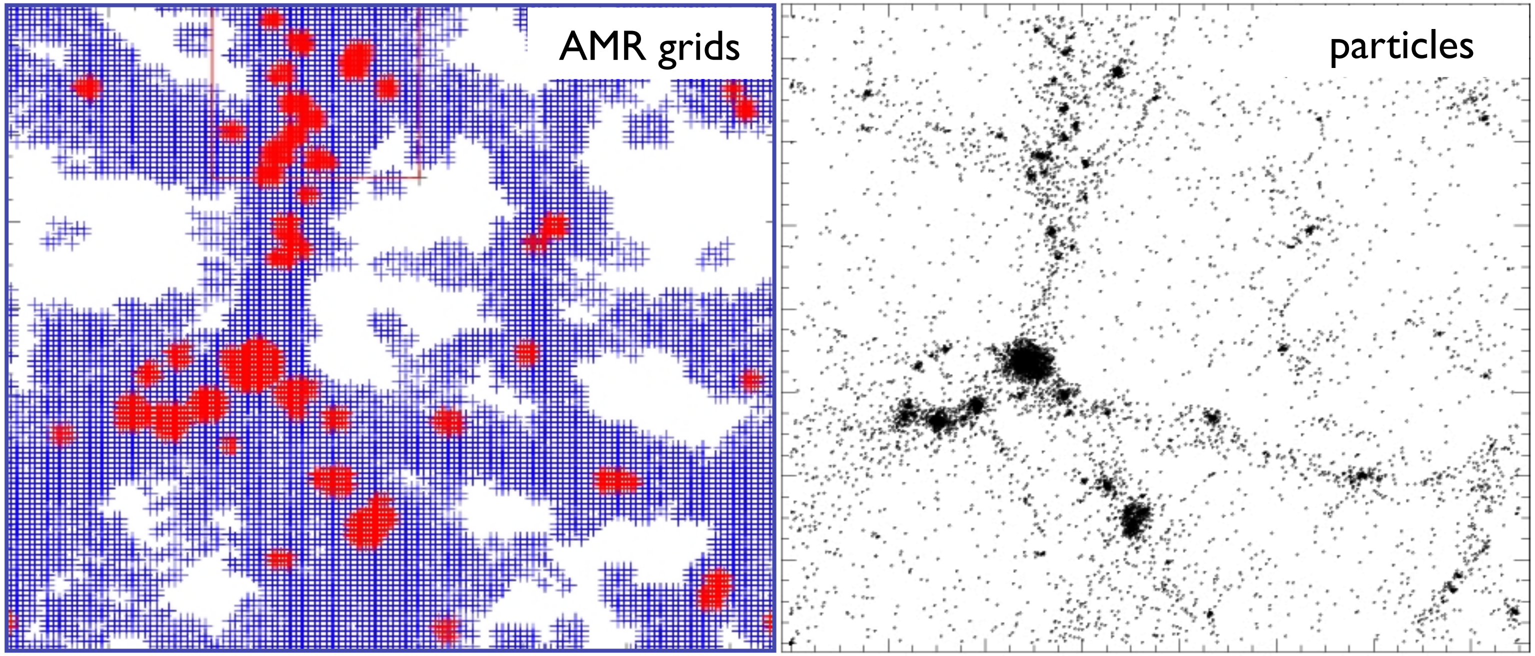
$$\rho\varepsilon = \rho E - \frac{1}{2}\rho v^2$$

• Maxwell's equation

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times(\vec{v}\times\vec{B})$$

Solving for Gravity

- moving particles on the AMR hierarchy



$$\Delta\phi = 4\pi G\rho_{tot}$$

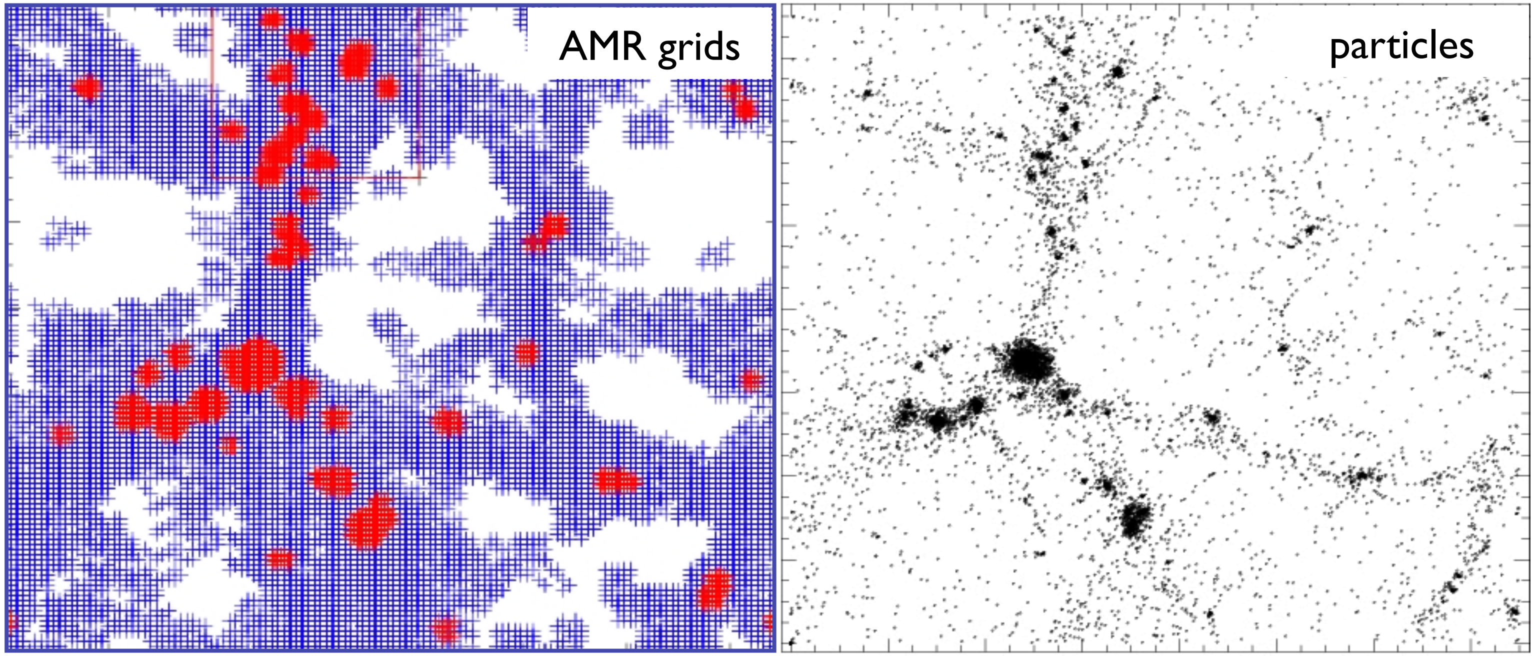
$$\frac{d\vec{x}_{DM}}{dt} = \vec{v}_{DM}$$

$$\frac{d\vec{v}_{DM}}{dt} = -\nabla\phi$$

Solving for Gravity

- moving particles on the AMR hierarchy

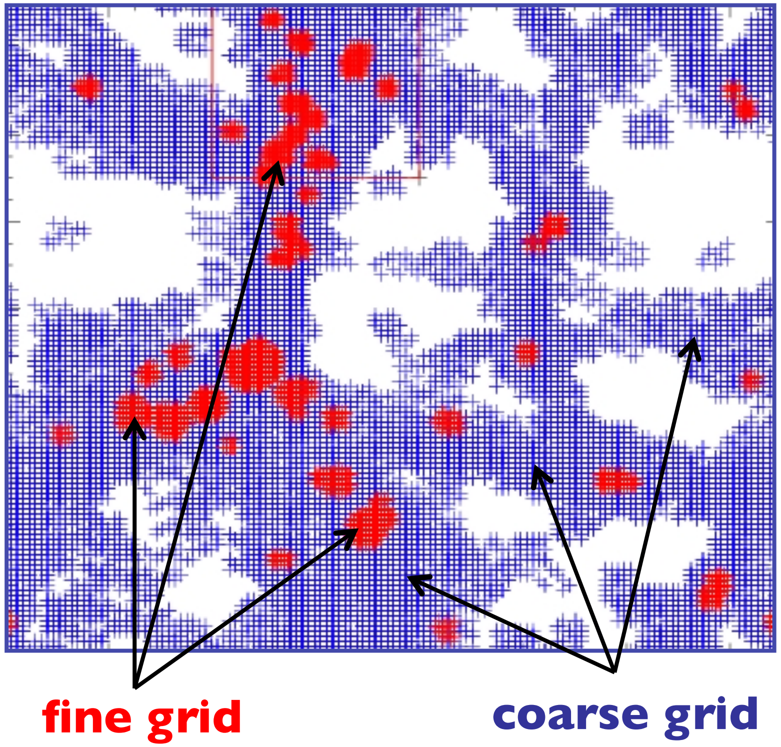
move particles on fine grids with smaller time step
to better resolve the dynamics, too!

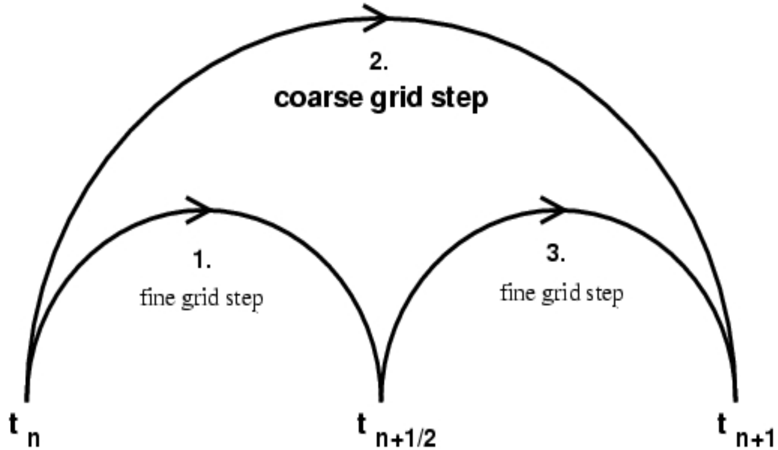


$$\Delta\phi = 4\pi G\rho_{tot}$$

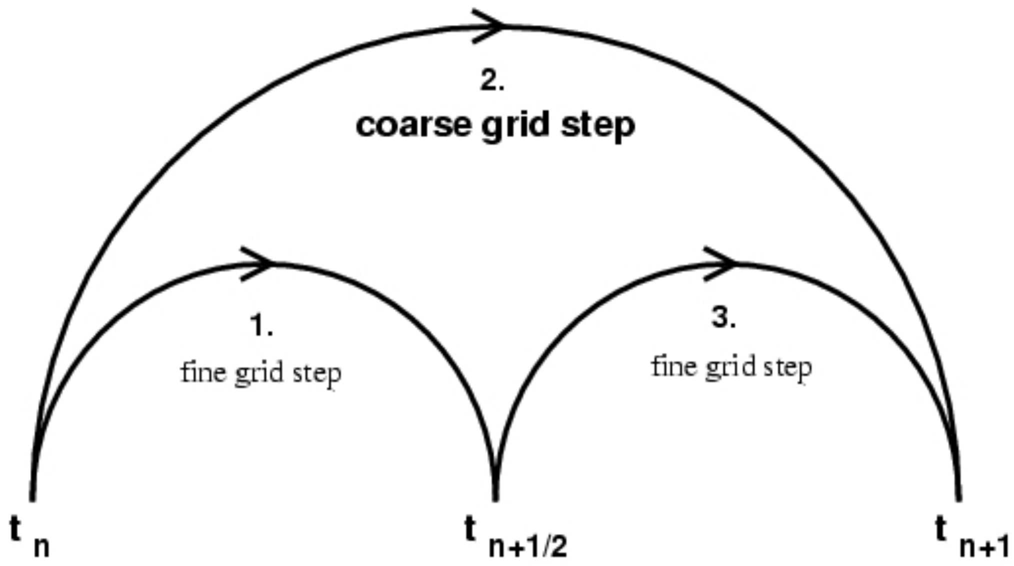
$$\frac{d\vec{x}_{DM}}{dt} = \vec{v}_{DM}$$
$$\frac{d\vec{v}_{DM}}{dt} = -\nabla\phi$$

- moving particles on the AMR hierarchy
    - fully recursive approach:





- moving particles on the AMR hierarchy
  - fully recursive approach:

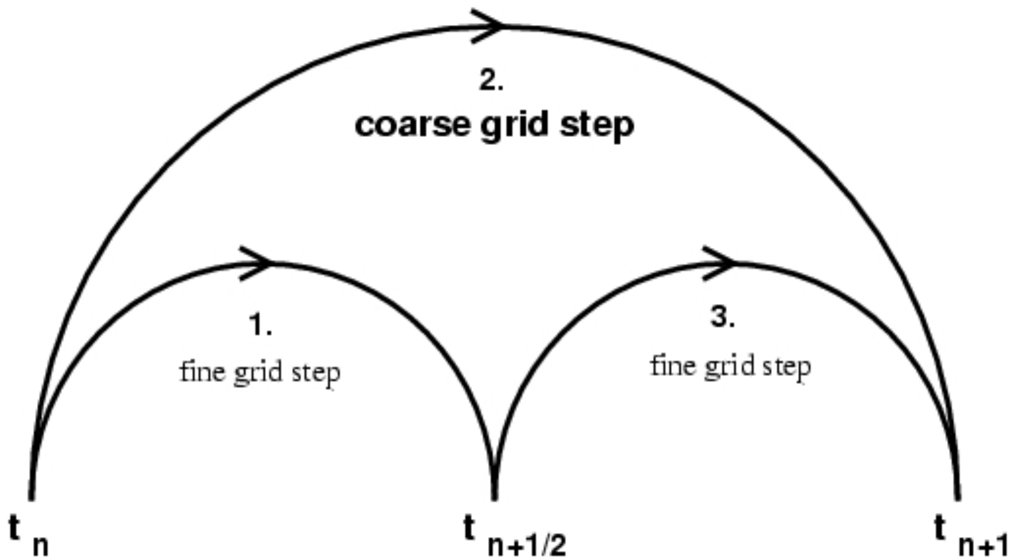


Drift-Kick-Drift variant of the leap-frog integrator:

time synchronisation between different grid levels rather than “leap-frogging”!

- moving particles on the AMR hierarchy

  - fully recursive approach:



1. fine grid step:

$$\text{Drift}: \quad \vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n} \int_{t_n}^{t_n+\Delta t/4} dt$$

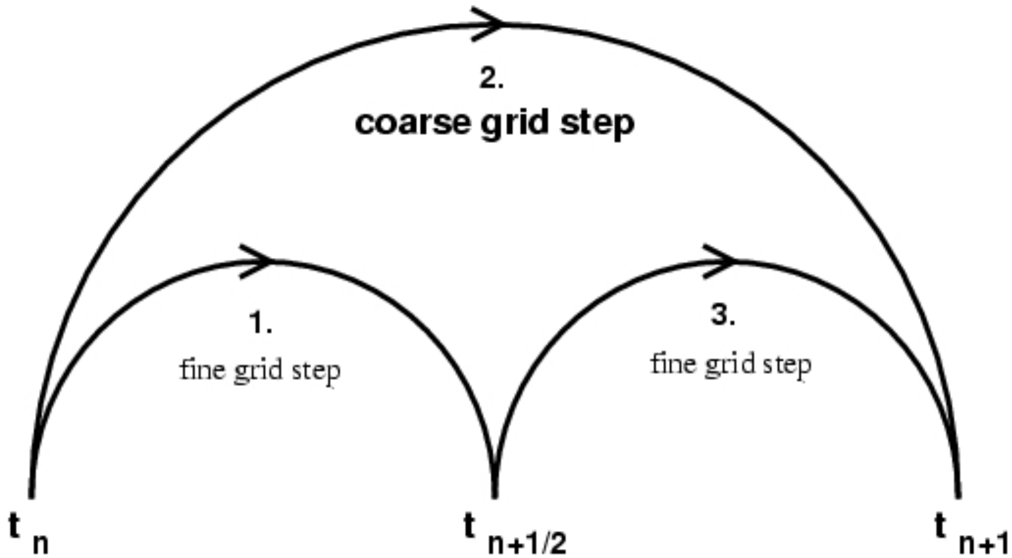
$$\leftarrow \quad \text{Kick}: \quad \vec{p}^{n+1/2} = \vec{p}^{n} - \vec{\nabla}\Phi^{n+1/4} \int_{t_n}^{t_n+\Delta t/2} dt \quad \rightarrow$$

$$\text{Drift}: \quad \vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2} \int_{t_n+\Delta t/4}^{t_n+\Delta t/2} dt$$

Solving for Gravity

- moving particles on the AMR hierarchy

    - fully recursive approach:



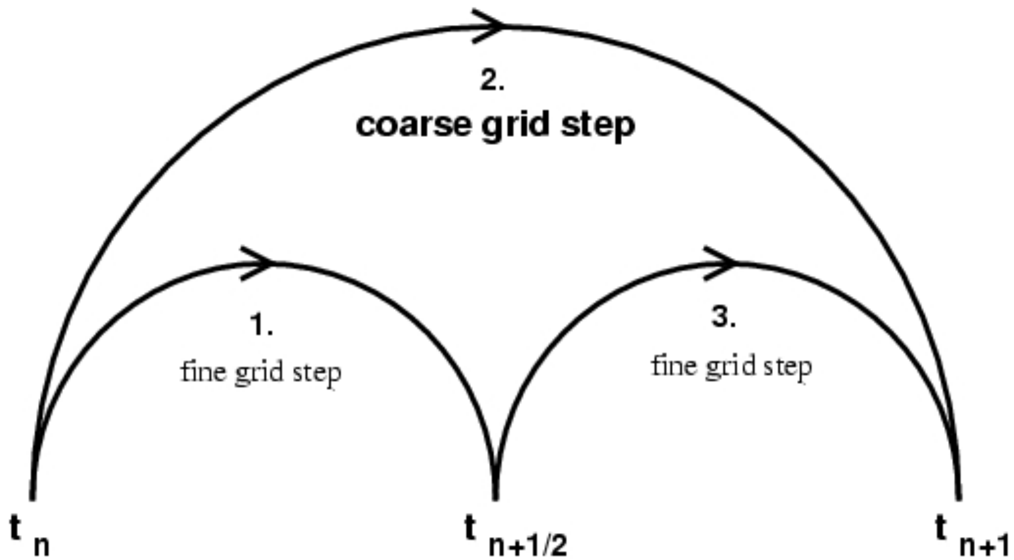
2. coarse grid step:

$$\text{Drift}: \quad \vec{x}^{n+1/2} = \vec{x}^{n} + \vec{p}^{n} \int_{t_n}^{t_n+\Delta t/2} dt$$

$$\longleftarrow \qquad \text{Kick}: \quad \vec{p}^{n+1} = \vec{p}^{n} - \vec{\nabla}\Phi^{n+1/2} \int_{t_n}^{t_n+\Delta t} dt \qquad \longrightarrow$$

$$\text{Drift}: \quad \vec{x}^{n+1} = \vec{x}^{n+1/2} + \vec{p}^{n+1} \int_{t_n+\Delta t/2}^{t_n+\Delta t} dt$$

- moving particles on the AMR hierarchy
    - fully recursive approach:



3. fine grid step:

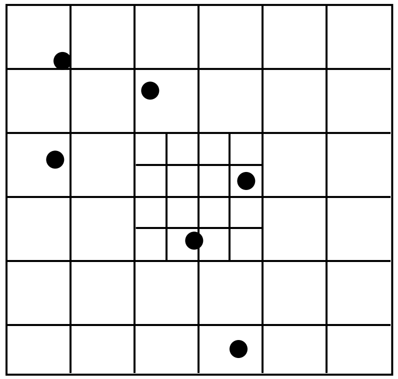
$$\text{Drift}: \quad \vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n} \int_{t_n+\Delta t/2}^{t_n+3\Delta t/4} dt$$

$$\leftarrow \quad \text{Kick}: \quad \vec{p}^{n+1} = \vec{p}^{n+1/2} - \vec{\nabla}\Phi^{n+3/4} \int_{t_n+\Delta t/2}^{t_n+\Delta t} dt \quad \rightarrow$$

$$\text{Drift}: \quad \vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1} \int_{t_n+3\Delta t/4}^{t_n+\Delta t} dt$$

Solving for Gravity

- moving particles on the AMR hierarchy



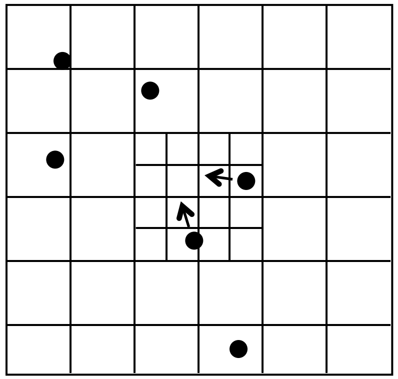
- moving particles on the AMR hierarchy

1. fine grid DKD step:

$$\text{Drift}: \quad \vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n} \frac{\Delta t}{4}$$

$$\text{Kick}: \quad \vec{p}^{n+1/2} = \vec{p}^{n} - \vec{\nabla}\Phi^{n+1/4} \frac{\Delta t}{2}$$

$$\text{Drift}: \quad \vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2} \frac{\Delta t}{4}$$



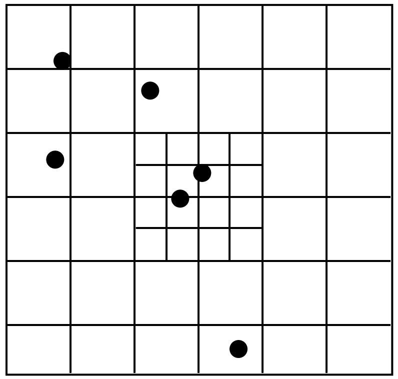
- moving particles on the AMR hierarchy

1. fine grid DKD step:

$$\text{Drift}: \quad \vec{x}^{\,n+1/4} = \vec{x}^{\,n} + \vec{p}^{\,n} \frac{\Delta t}{4}$$

$$\text{Kick}: \quad \vec{p}^{\,n+1/2} = \vec{p}^{\,n} - \vec{\nabla}\Phi^{n+1/4} \frac{\Delta t}{2}$$

$$\text{Drift}: \quad \vec{x}^{\,n+1/2} = \vec{x}^{\,n+1/4} + \vec{p}^{\,n+1/2} \frac{\Delta t}{4}$$



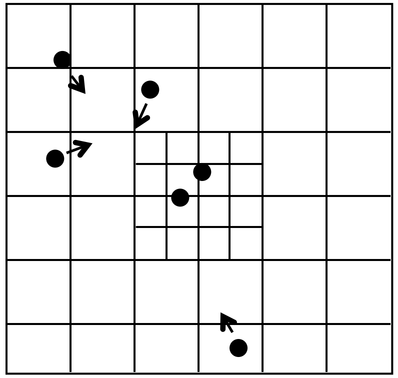
- moving particles on the AMR hierarchy

2. coarse grid DKD step:

$$\text{Drift}: \quad \vec{x}^{\,n+1/2} = \vec{x}^{\,n} + \vec{p}^{\,n}\,\frac{\Delta t}{2}$$

$$\text{Kick}: \quad \vec{p}^{\,n+1} = \vec{p}^{\,n} - \vec{\nabla}\Phi^{n+1/2}\Delta t$$

$$\text{Drift}: \quad \vec{x}^{\,n+1} = \vec{x}^{\,n+1/2} + \vec{p}^{\,n+1}\,\frac{\Delta t}{2}$$



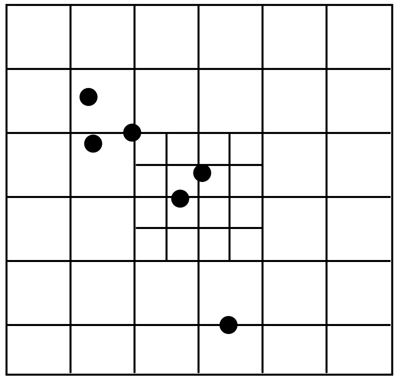
- moving particles on the AMR hierarchy

2. coarse grid DKD step:

$$\text{Drift}: \quad \vec{x}^{n+1/2} = \vec{x}^{n} + \vec{p}^{n}\,\frac{\Delta t}{2}$$

$$\text{Kick}: \quad \vec{p}^{n+1} = \vec{p}^{n} - \vec{\nabla}\Phi^{n+1/2}\Delta t$$

$$\text{Drift}: \quad \vec{x}^{n+1} = \vec{x}^{n+1/2} + \vec{p}^{n+1}\,\frac{\Delta t}{2}$$



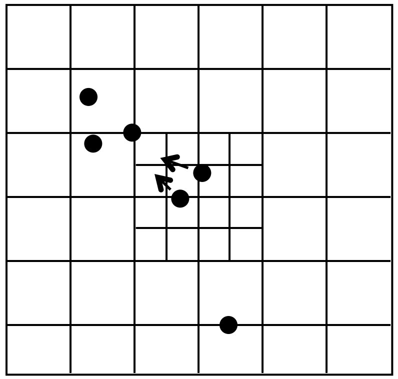
- moving particles on the AMR hierarchy

3. fine grid DKD step:

$$\text{Drift}: \quad \vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n} \frac{\Delta t}{4}$$

$$\text{Kick}: \quad \vec{p}^{n+1} = \vec{p}^{n+1/2} - \vec{\nabla}\Phi^{n+3/4} \frac{\Delta t}{2}$$

$$\text{Drift}: \quad \vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1} \frac{\Delta t}{4}$$



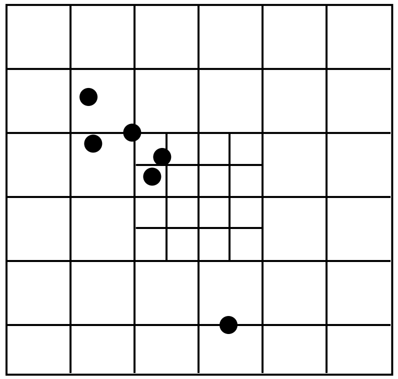
- moving particles on the AMR hierarchy

3. fine grid DKD step:

$$\text{Drift}: \quad \vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n}\frac{\Delta t}{4}$$

$$\text{Kick}: \quad \vec{p}^{n+1} = \vec{p}^{n+1/2} - \vec{\nabla}\Phi^{n+3/4}\frac{\Delta t}{2}$$

$$\text{Drift}: \quad \vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1}\frac{\Delta t}{4}$$



Solving for Gravity

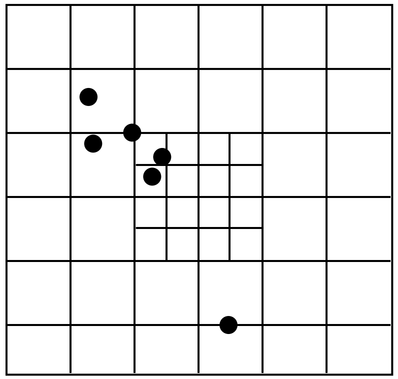
- moving particles on the AMR hierarchy

3. fine grid DKD step:

$$\text{Drift}: \quad \vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n}\frac{\Delta t}{4}$$

$$\text{Kick}: \quad \vec{p}^{n+1} = \vec{p}^{n+1/2} - \vec{\nabla}\Phi^{n+3/4}\frac{\Delta t}{2}$$

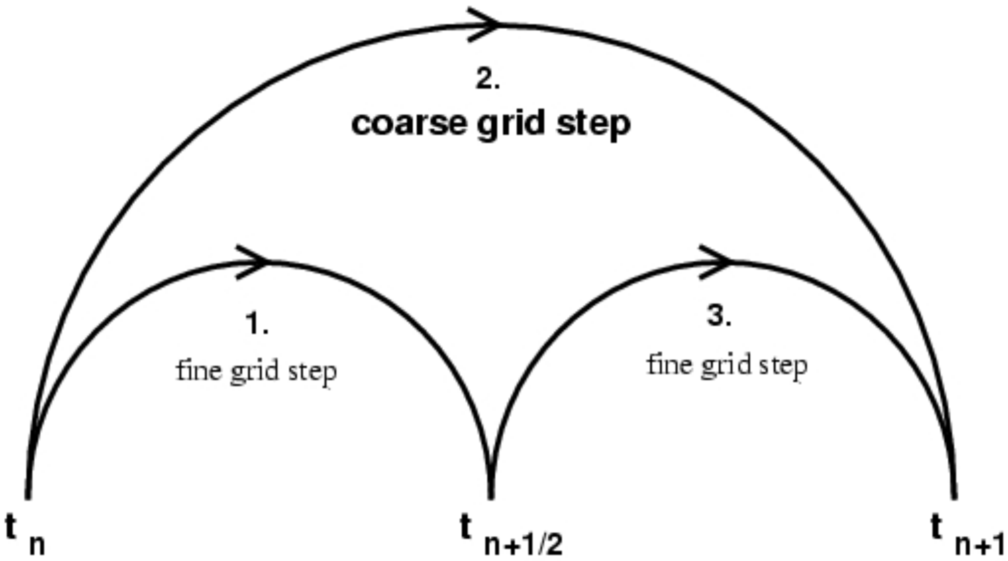
$$\text{Drift}: \quad \vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1}\frac{\Delta t}{4}$$



what about particles crossing grid boundaries?

- moving particles on the AMR hierarchy

  - particles crossing grid boundaries

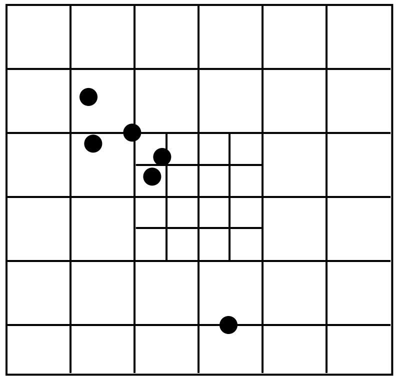


1. Drift: $\vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n}\frac{\Delta t}{4}$

2. Drift: $\vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2}\frac{\Delta t}{4}$

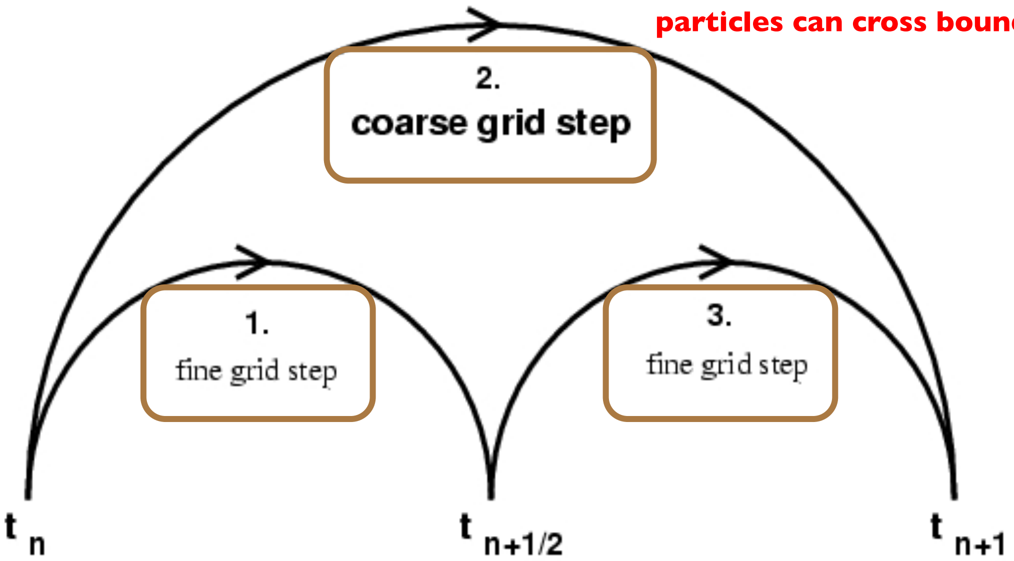
3. Drift: $\vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n}\frac{\Delta t}{4}$

4. Drift: $\vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1}\frac{\Delta t}{4}$

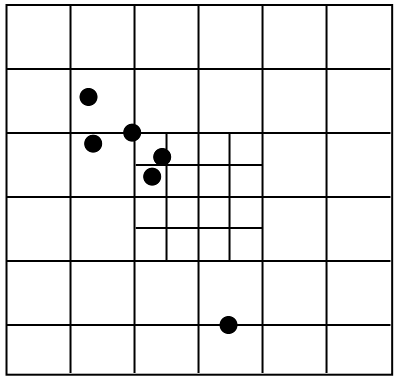


Solving for Gravity

- moving particles on the AMR hierarchy
  - particles crossing grid boundaries



particles can cross boundary during any of these steps...



1. Drift: $\vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n} \frac{\Delta t}{4}$

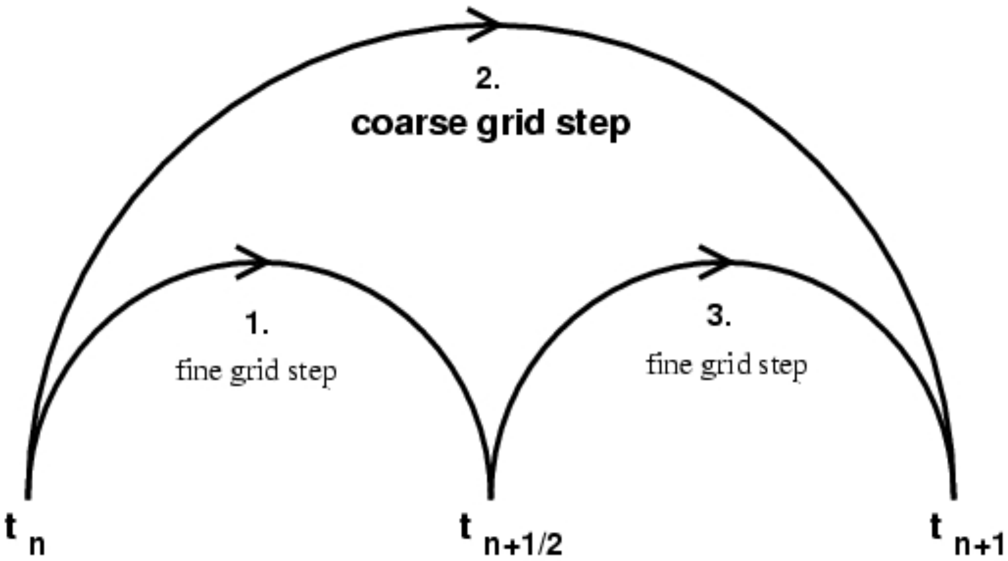
2. Drift: $\vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2} \frac{\Delta t}{4}$

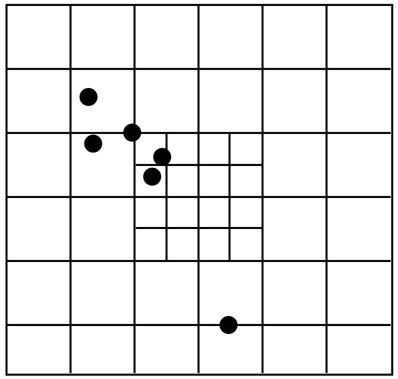
3. Drift: $\vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n} \frac{\Delta t}{4}$

4. Drift: $\vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1} \frac{\Delta t}{4}$

Solving for Gravity

- moving particles on the AMR hierarchy
  - particles crossing grid boundaries





1. Drift: $\vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n} \int_{t_n}^{t_n+\Delta t/4} dt$

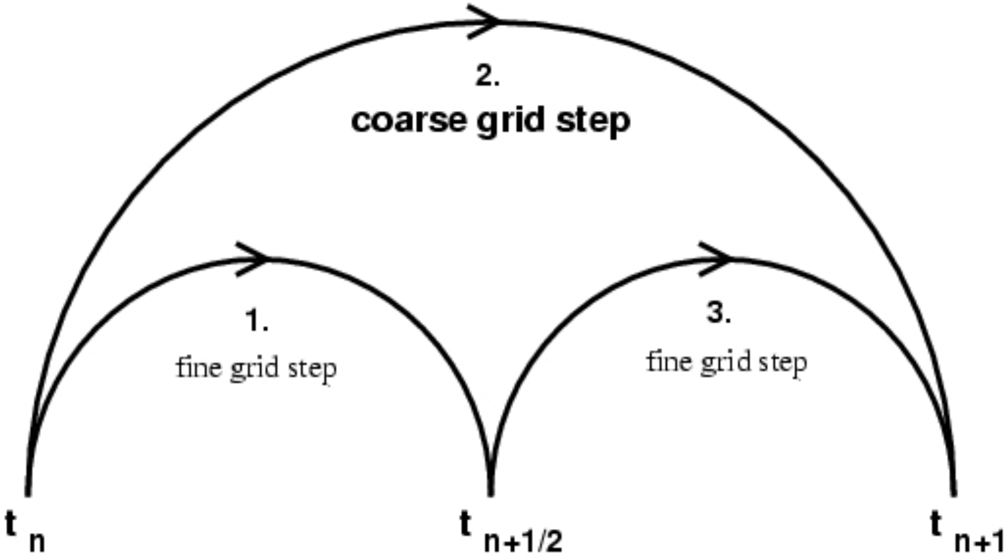
2. Drift: $\vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2} \int_{t_n+\Delta t/4}^{t_n+\Delta t/2} dt$

3. Drift: $\vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n} \int_{t_n+\Delta t/2}^{t_n+3\Delta t/4} dt$

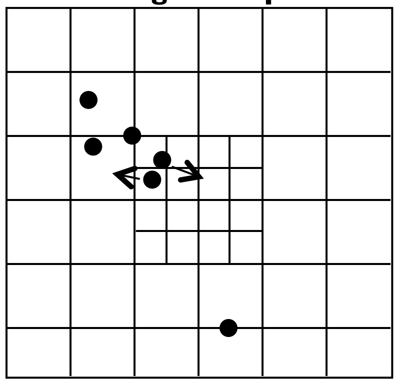
4. Drift: $\vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1} \int_{t_n+3\Delta t/4}^{t_n+\Delta t} dt$

un-drift and move with coarse grid time step to t_{n+1}...

- moving particles on the AMR hierarchy
  - particles crossing grid boundaries



1. fine-grid step



1. Drift: $\vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n} \int_{t_n}^{t_n+\Delta t/4} dt$

2. Drift: $\vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2} \int_{t_n+\Delta t/4}^{t_n+\Delta t/2} dt$

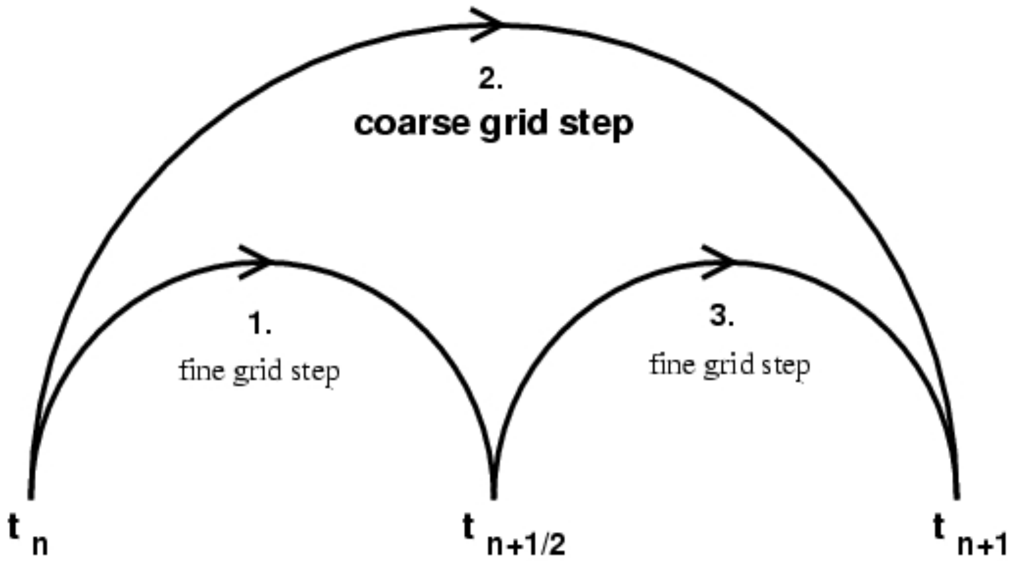
3. Drift: $\vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n} \int_{t_n+\Delta t/2}^{t_n+3\Delta t/4} dt$

4. Drift: $\vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1} \int_{t_n+3\Delta t/4}^{t_n+\Delta t} dt$

un-drift and move with coarse grid time step to t_{n+1}...

Solving for Gravity

- moving particles on the AMR hierarchy
  - particles crossing grid boundaries



1. Drift: $\vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n} \frac{\Delta t}{4}$

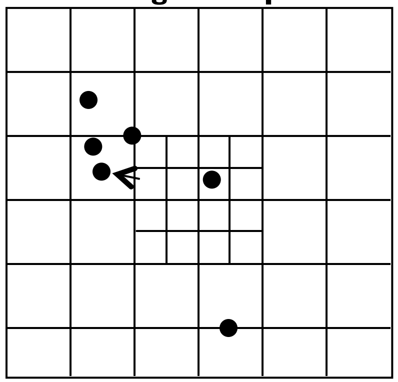
2. Drift: $\vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2} \frac{\Delta t}{4}$

3. Drift: $\vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n} \frac{\Delta t}{4}$

4. Drift: $\vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1} \frac{\Delta t}{4}$

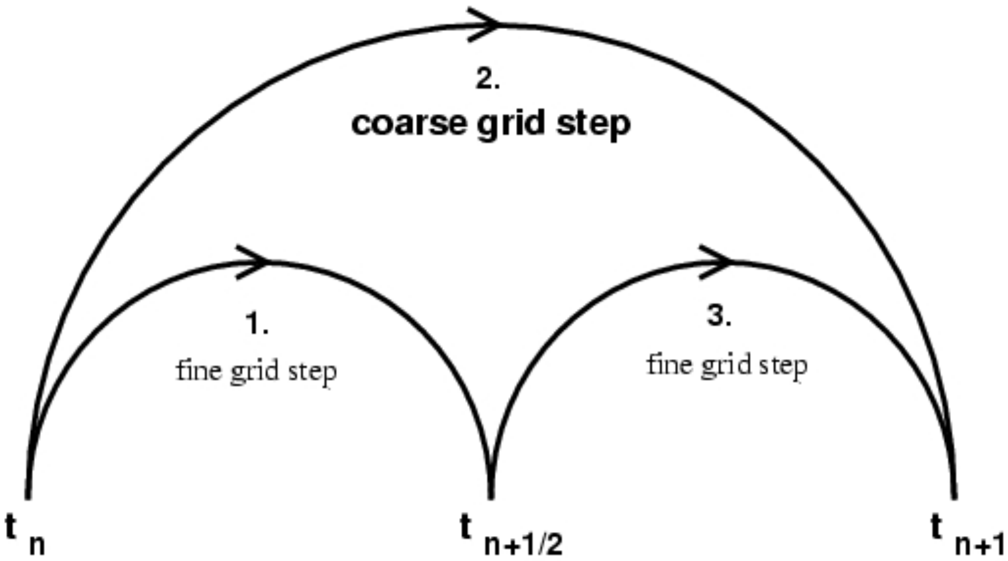
un-drift and move with coarse grid time step to t_{n+1}...

1. fine-grid step



- moving particles on the AMR hierarchy

  - particles crossing grid boundaries



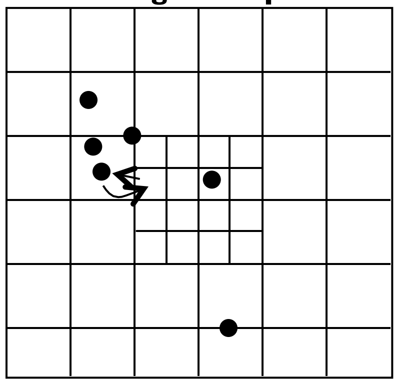
1. Drift: $\vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n}\frac{\Delta t}{4}$

2. Drift: $\vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2}\frac{\Delta t}{4}$

3. Drift: $\vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n}\frac{\Delta t}{4}$

4. Drift: $\vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1}\frac{\Delta t}{4}$

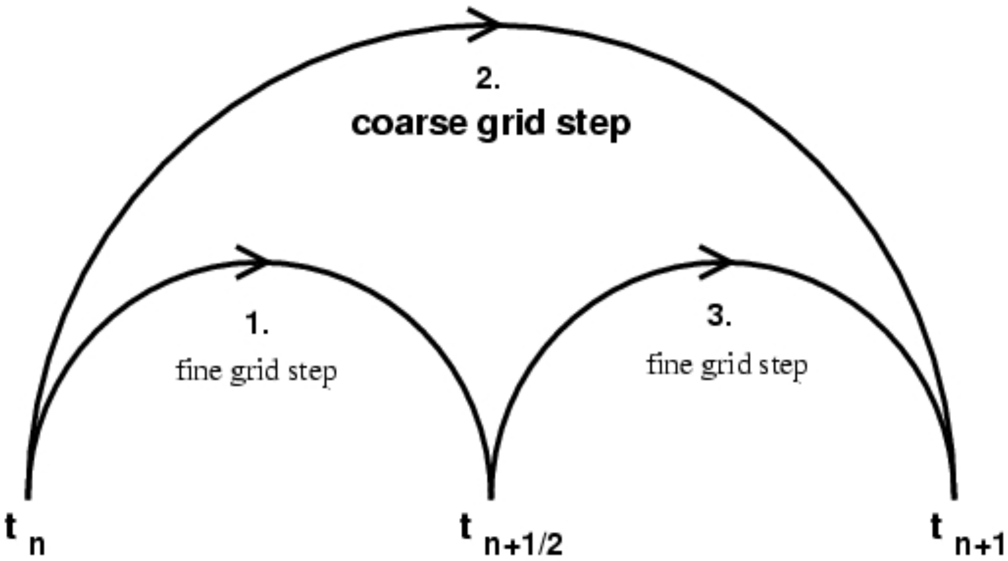
1. fine-grid step



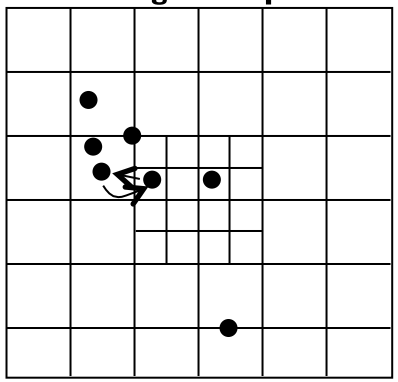
un-drift and move with coarse grid time step to t_{n+1}...

Solving for Gravity

- moving particles on the AMR hierarchy
  - particles crossing grid boundaries



1. fine-grid step



1. Drift: $\vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n} \frac{\Delta t}{4}$

2. Drift: $\vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2} \frac{\Delta t}{4}$

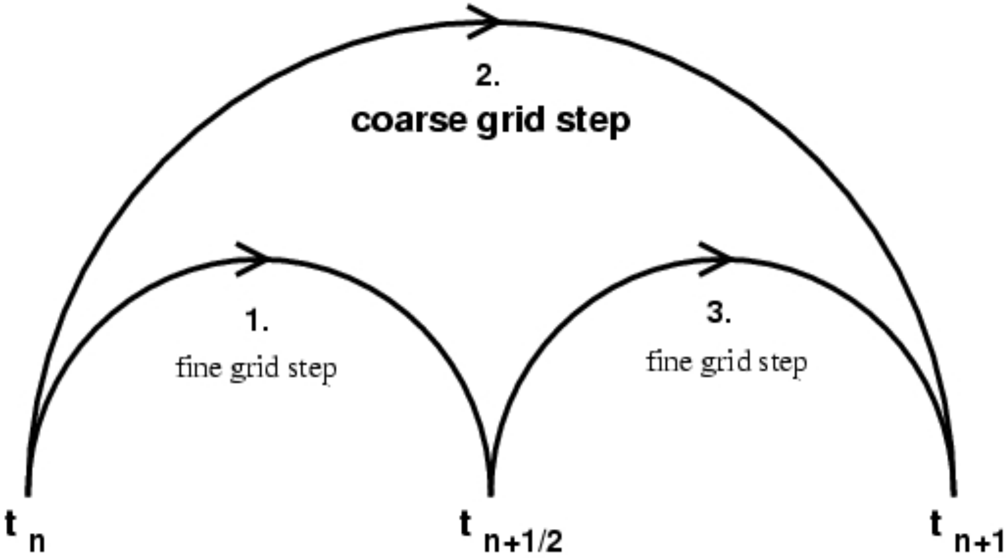
3. Drift: $\vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n} \frac{\Delta t}{4}$

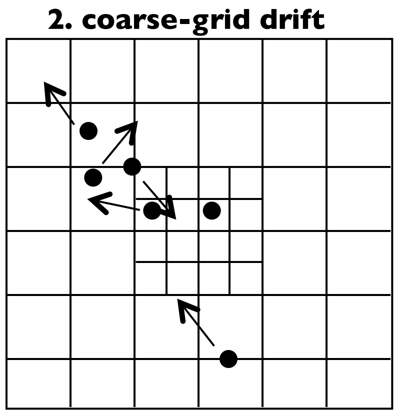
4. Drift: $\vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1} \frac{\Delta t}{4}$

un-drift and move with coarse grid time step to t_{n+1}...

Solving for Gravity

- moving particles on the AMR hierarchy
  - particles crossing grid boundaries





1. Drift: $\vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n}\frac{\Delta t}{4}$

2. Drift: $\vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2}\frac{\Delta t}{4}$

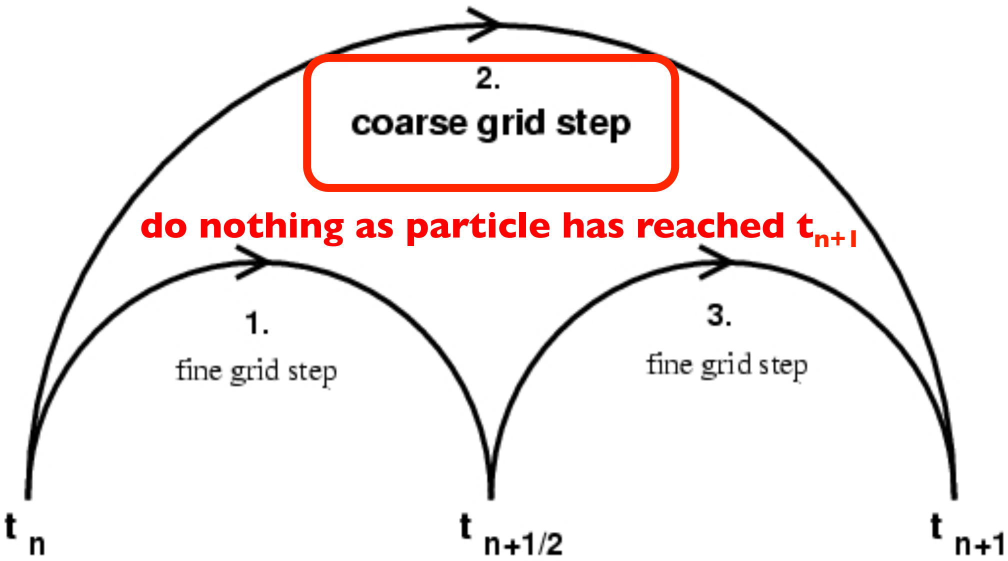
3. Drift: $\vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n}\frac{\Delta t}{4}$

4. Drift: $\vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1}\frac{\Delta t}{4}$

un-drift and move with coarse grid time step to t_{n+1}...

Solving for Gravity

- moving particles on the AMR hierarchy
  - particles crossing grid boundaries



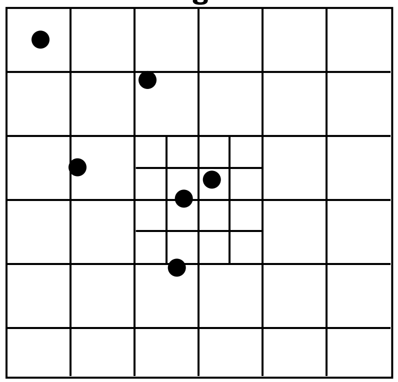
1. Drift: $\vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n}\frac{\Delta t}{4}$

2. Drift: $\vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2}\frac{\Delta t}{4}$

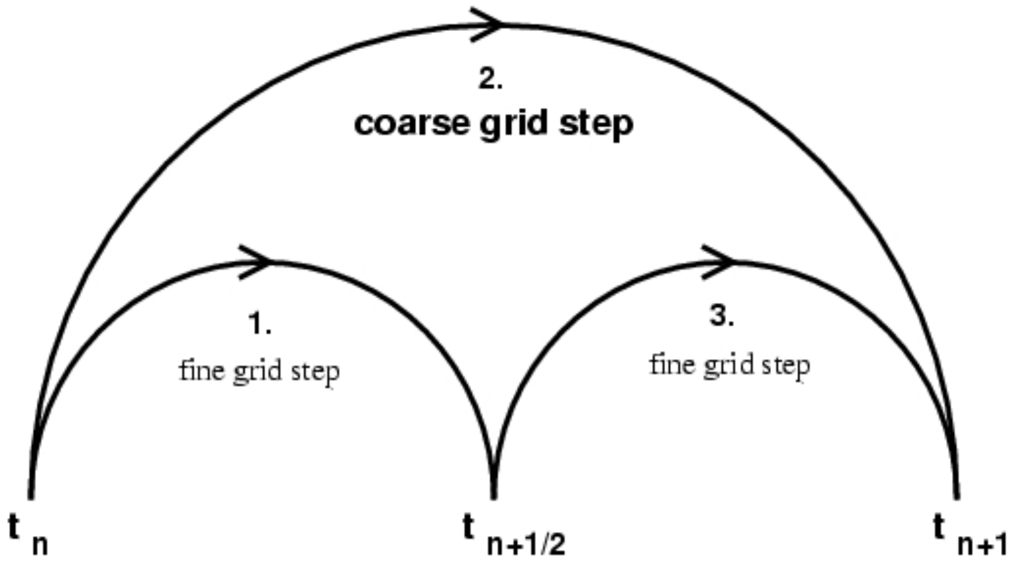
3. Drift: $\vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n}\frac{\Delta t}{4}$

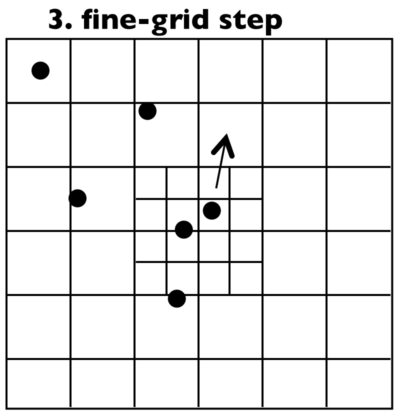
4. Drift: $\vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1}\frac{\Delta t}{4}$

2. coarse-grid drift



- moving particles on the AMR hierarchy
  - particles crossing grid boundaries





1. Drift: $\vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n}\frac{\Delta t}{4}$

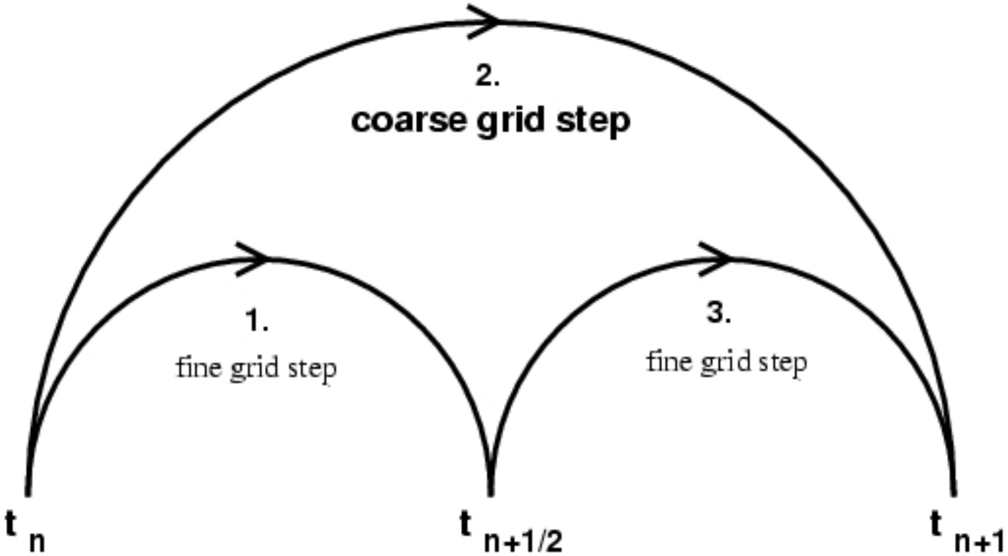
2. Drift: $\vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2}\frac{\Delta t}{4}$

3. Drift: $\vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n}\frac{\Delta t}{4}$

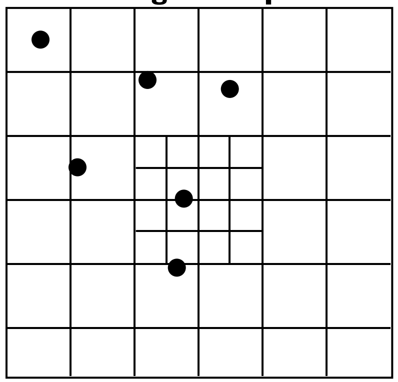
4. Drift: $\vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1}\frac{\Delta t}{4}$

keep on drift'ing as it will bring the particle to t_{n+1}

- moving particles on the AMR hierarchy
    - particles crossing grid boundaries



3. fine-grid step



1. Drift: $\vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n}\frac{\Delta t}{4}$

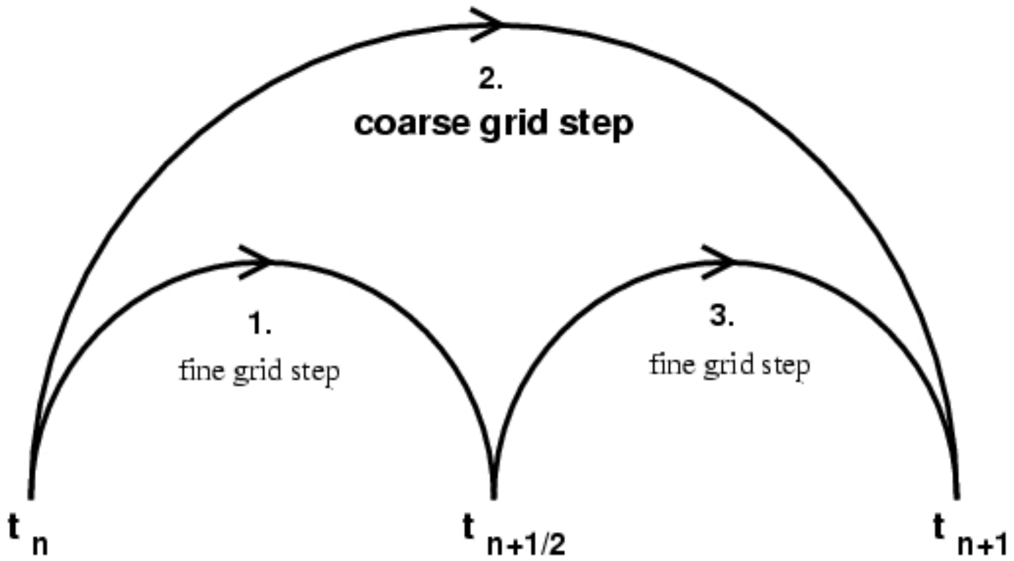
2. Drift: $\vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2}\frac{\Delta t}{4}$

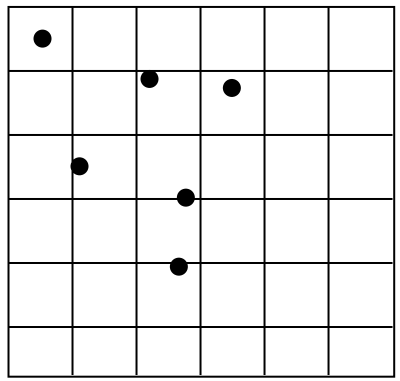
3. Drift: $\vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n}\frac{\Delta t}{4}$

4. Drift: $\vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1}\frac{\Delta t}{4}$

keep on drift'ing as it will bring the particle to t_{n+1}

- moving particles on the AMR hierarchy
  - particles crossing grid boundaries





1\. Drift: $\vec{x}^{n+1/4} = \vec{x}^{n} + \vec{p}^{n}\frac{\Delta t}{4}$

2\. Drift: $\vec{x}^{n+1/2} = \vec{x}^{n+1/4} + \vec{p}^{n+1/2}\frac{\Delta t}{4}$

3\. Drift: $\vec{x}^{n+3/4} = \vec{x}^{n+1/2} + \vec{p}^{n}\frac{\Delta t}{4}$

4\. Drift: $\vec{x}^{n+1} = \vec{x}^{n+3/4} + \vec{p}^{n+1}\frac{\Delta t}{4}$

all particles have now moved from t_n to t_{n+1} and the refinements will be re-created...

Solving for Gravity

```
Step(dt, CurrentGrid) {

   NewGrid = Refine(CurrentGrid);

   if(NewGrid) {
      Step(dt/2, NewGrid); }

   MoveParticles(dt, CurrentGrid);

   if(NewGrid) {
      Step(dt/2, NewGrid);
      DestroyGrid(NewGrid);}
}
```

Solving for Gravity