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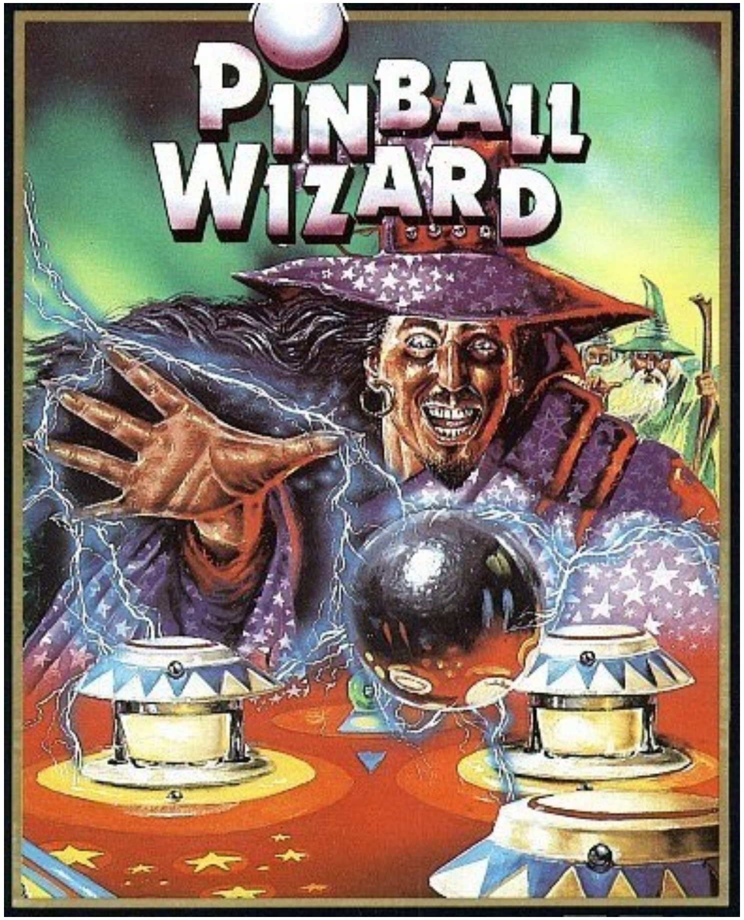

Solving for Gravity

- full set of equations

  - collisionless matter (e.g. dark matter)

$$\frac{d\vec{x}_{DM}}{dt} = \vec{v}_{DM}$$

$$\frac{d\vec{v}_{DM}}{dt} = -\nabla\phi$$

  - Poisson's equation

$$\Delta\phi = 4\pi G\rho_{tot}$$

  - collisional matter (e.g. gas)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\vec{v}\right) = 0$$

$$\frac{\partial\left(\rho\vec{v}\right)}{\partial t} + \nabla\cdot\left(\rho\vec{v}\otimes\vec{v} + \left(p + \frac{1}{2\mu}B^2\right)\overleftrightarrow{1} - \frac{1}{\mu}\vec{B}\otimes\vec{B}\right) = \rho\ \left(-\nabla\phi\right)$$

$$\frac{\partial\left(\rho E\right)}{\partial t} + \nabla\cdot\left(\left[\rho E + p + \frac{1}{2\mu}B^2\right]\vec{v} - \frac{1}{\mu}\left[\vec{v}\cdot\vec{B}\right]\vec{B}\right) = \rho\vec{v}\cdot\left(-\nabla\phi\right) + \left(\Gamma - L\right)$$

  - ideal gas equations

$$p = \left(\gamma - 1\right)\rho\varepsilon$$

$$\rho\varepsilon = \rho E - \frac{1}{2}\rho v^2$$

  - Maxwell's equation

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\left(\vec{v}\times\vec{B}\right)$$

- full set of equations

  - collisionless matter (e.g. dark matter)

$$\frac{d\vec{x}_{DM}}{dt} = \vec{v}_{DM}$$
$$\frac{d\vec{v}_{DM}}{dt} = -\nabla\phi$$

leap-frog integration

  - Poisson's equation

$$\Delta\phi = 4\pi G\rho_{tot}$$

  - collisional matter (e.g. gas)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

$$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot\left(\rho\vec{v}\otimes\vec{v} + \left(p + \frac{1}{2\mu}B^2\right)\overleftrightarrow{1} - \frac{1}{\mu}\vec{B}\otimes\vec{B}\right) = \rho\ (-\nabla\phi)$$

$$\frac{\partial(\rho E)}{\partial t} + \nabla\cdot\left(\left[\rho E + p + \frac{1}{2\mu}B^2\right]\vec{v} - \frac{1}{\mu}\left[\vec{v}\cdot\vec{B}\right]\vec{B}\right) = \rho\vec{v}\cdot(-\nabla\phi) + (\Gamma - L)$$

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Solving for Gravity

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later...

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

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$$\frac{\partial(\rho E)}{\partial t} + \nabla\cdot\left(\left[\rho E + p + \frac{1}{2\mu}B^2\right]\vec{v} - \frac{1}{\mu}\left[\vec{v}\cdot\vec{B}\right]\vec{B}\right) = \rho\vec{v}\cdot(-\nabla\phi) + (\Gamma - L)$$

hyperbolic partial differential equations:
solutions are wave-like, i.e. perturbations need time to travel...

• ideal gas equations

$$p = (\gamma - 1)\rho\varepsilon$$

$$\rho\varepsilon = \rho E - \frac{1}{2}\rho v^2$$

• Maxwell's equation

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\left(\vec{v}\times\vec{B}\right)$$

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$$p = \left(\gamma - 1\right)\rho\varepsilon$$
$$\rho\varepsilon = \rho E - \frac{1}{2}\rho v^2$$

electrodynamics lecture...

  - Maxwell's equation

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\left(\vec{v}\times\vec{B}\right)$$

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  - Poisson's equation

$$\Delta\phi = 4\pi G\rho_{tot}$$

physics lecture...

  - ideal gas equations

$$p = (\gamma - 1)\rho\varepsilon$$

$$\rho\varepsilon = \rho E - \frac{1}{2}\rho v^2$$

  - Maxwell's equation

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\left(\vec{v}\times\vec{B}\right)$$

Solving for Gravity

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$$\frac{d\vec{x}_{DM}}{dt} = \vec{v}_{DM}$$

$$\frac{d\vec{v}_{DM}}{dt} = -\nabla\phi$$

now!

- Poisson's equation

$$\Delta\phi = 4\pi G\rho_{tot}$$

elliptical partial differential equation:
solution obtainable via FFT (for constant coefficients)

  - collisional matter (e.g. gas)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

$$\frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot\left(\rho\vec{v}\otimes\vec{v} + \left(p + \frac{1}{2\mu}B^2\right)\overleftrightarrow{1} - \frac{1}{\mu}\vec{B}\otimes\vec{B}\right) = \rho\,(-\nabla\phi)$$

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- Maxwell's equation

$$\frac{\partial\vec{B}}{\partial t} = -\nabla\times\left(\vec{v}\times\vec{B}\right)$$

- Poisson's equation
- direct particle-particle summation
- the tree
- force softening
- periodic boundaries

- **Poisson's equation**
- direct particle-particle summation
- the tree
- force softening
- periodic boundaries

- Poisson's equation

$$\Delta\Phi(\vec{x}) = 4\pi G\rho(\vec{x})$$

- Poisson's equation

$$\Delta\Phi(\vec{x}) = 4\pi G\rho(\vec{x})$$

- general 2nd order partial differential equation:

$$A\,\Phi_{xx} + 2B\,\Phi_{xy} + C\,\Phi_{yy} \quad + D\,\Phi_{x} + E\,\Phi_{y} + F = 0$$

  - $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ is positive definite => elliptical equation
  - if the coefficients A,B,C are constant => solutions via Fourier transforms

Solving for Gravity

- Poisson's equation

$$\Delta\Phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

$$\vec{F}(\vec{r}) = -m\nabla\Phi(\vec{r})$$

Solving for Gravity

- Poisson's equation

$$\Delta\Phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

$$\vec{F}(\vec{r}) = -m\nabla\Phi(\vec{r})$$

→

<u>grid approach</u> ($\vec{r}_{i,j,k}$ =position of centre of grid cell (i,j,k))

$$\Delta\Phi(\vec{r}_{i,j,k}) = 4\pi G\rho(\vec{r}_{i,j,k})$$

$$\vec{F}(\vec{r}_{i,j,k}) = -m\nabla\Phi(\vec{r}_{i,j,k})$$

Solving for Gravity

- Poisson's equation

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weapon of choice: AMR codes

Solving for Gravity

- Poisson's equation

$$\Delta\Phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

$$\vec{F}(\vec{r}) = -m\nabla\Phi(\vec{r})$$

particle approach

$$\vec{F}(\vec{r}_i) = -\sum_{i\neq j}\frac{Gm_i m_j}{(r_i - r_j)^3}(\vec{r}_i - \vec{r}_j)$$

grid approach ($\vec{r}_{i,j,k}$ = position of centre of grid cell (i,j,k))

$$\Delta\Phi(\vec{r}_{i,j,k}) = 4\pi G\rho(\vec{r}_{i,j,k})$$

$$\vec{F}(\vec{r}_{i,j,k}) = -m\nabla\Phi(\vec{r}_{i,j,k})$$

Solving for Gravity

- Poisson's equation

$$\Delta\Phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

$$\vec{F}(\vec{r}) = -m\nabla\Phi(\vec{r})$$

weapon of choice: tree codes

particle approach

$$\vec{F}(\vec{r}_i) = -\sum_{i\neq j}\frac{Gm_i m_j}{(r_i - r_j)^3}(\vec{r}_i - \vec{r}_j)$$

grid approach ($\vec{r}_{i,j,k}$ =position of centre of grid cell (i,j,k))

$$\Delta\Phi(\vec{r}_{i,j,k}) = 4\pi G\rho(\vec{r}_{i,j,k})$$

$$\vec{F}(\vec{r}_{i,j,k}) = -m\nabla\Phi(\vec{r}_{i,j,k})$$

Solving for Gravity

- Poisson's equation

...but where is this formula actually coming from?

weapon of choice: tree codes

$$\Delta\Phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

$$\vec{F}(\vec{r}) = -m\nabla\Phi(\vec{r})$$

particle approach

$$\vec{F}(\vec{r}_i) = -\sum_{i\neq j}\frac{Gm_i m_j}{(r_i - r_j)^3}(\vec{r}_i - \vec{r}_j)$$

grid approach ($\vec{r}_{i,j,k}$ =position of centre of grid cell (i,j,k))

$$\Delta\Phi(\vec{r}_{i,j,k}) = 4\pi G\rho(\vec{r}_{i,j,k})$$

$$\vec{F}(\vec{r}_{i,j,k}) = -m\nabla\Phi(\vec{r}_{i,j,k})$$

Solving for Gravity

- Poisson's equation

$$\Delta\Phi(\vec{r}) = 4\pi G\rho(\vec{r})$$

Green's function method

$$\vec{F}(\vec{r}) = -m\nabla\Phi(\vec{r})$$

- Poisson's equation – Green's function method

$$\Delta\Phi = S \qquad \rightarrow \text{ equation we wish to solve}$$

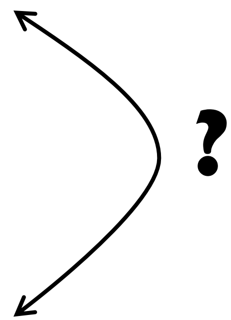
- Poisson's equation – Green's function method

$$\Delta\Phi = S$$ → equation we wish to solve

$$\Delta\mathcal{G} = \delta$$ → equation way easier to solve…

- Poisson's equation – Green's function method

$$\Delta\Phi = S$$



$$\Delta\mathcal{G} = \delta$$

Solving for Gravity

- Poisson's equation – Green's function method

$$\Delta \Phi = S$$

$$\Phi(\vec{x}) = \iiint \mathcal{G}(\vec{x} - \vec{x}')\, S(\vec{x}')\, d^3 x'$$

$$\Delta \mathcal{G} = \delta$$

Solving for Gravity

- Poisson's equation – Green's function method

$$\Delta\Phi = S$$

$$\Delta\mathcal{G} = \delta$$

$$\Phi(\vec{x}) = \iiint \mathcal{G}(\vec{x}-\vec{x}')\, S(\vec{x}')d^3x'$$

$$\begin{aligned}\Delta\Phi(\vec{x}) &= \Delta\iiint \mathcal{G}(\vec{x}-\vec{x}')S(\vec{x}')d^3x' \\ &= \iiint \Delta\mathcal{G}(\vec{x}-\vec{x}')S(\vec{x}')d^3x' \\ &= \iiint \delta(\vec{x}-\vec{x}')S(\vec{x}')d^3x' \\ &= S(\vec{x})\end{aligned}$$

- Poisson's equation – Green's function method

$$\Delta \mathcal{G} = \delta$$

Ansatz : $\mathcal{G} = \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} d^3 k$ (spectral decomposition of $\mathcal{G}$)

Solving for Gravity

- Poisson's equation – Green's function method

$$\Delta \mathcal{G} = \delta$$

Ansatz : $\mathcal{G} = \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} d^3k$

$$\begin{aligned}
\delta &= \Delta \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} d^3k \\
&= \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, \Delta e^{i\vec{k}\cdot\vec{x}} d^3k \\
&= \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, (-k^2) e^{i\vec{k}\cdot\vec{x}} d^3k
\end{aligned}$$

- Poisson's equation – Green's function method

$$\Delta \mathcal{G} = \delta$$

Ansatz : $\mathcal{G} = \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} d^3k$

$$\delta = \Delta \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} d^3k$$

$$= \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, \Delta e^{i\vec{k}\cdot\vec{x}} d^3k$$

$$= \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, (-k^2) e^{i\vec{k}\cdot\vec{x}} d^3k \quad \xrightarrow{e^{-i(\vec{k}'\cdot\vec{x})}} \quad \delta\, e^{-i(\vec{k}'\cdot\vec{x})} = e^{-i(\vec{k}'\cdot\vec{x})} \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, (-k^2) e^{i\vec{k}\cdot\vec{x}} d^3k$$

$$= \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, (-k^2) e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} d^3k$$

- Poisson's equation – Green's function method

$$\Delta \mathcal{G} = \delta$$

Ansatz : $\mathcal{G} = \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} d^3k$

$$\delta = \Delta \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} d^3k$$

$$= \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, \Delta e^{i\vec{k}\cdot\vec{x}} d^3k$$

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$$= \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, (-k^2) e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} d^3k$$

$$\xrightarrow{\iiint d^3x}$$

Solving for Gravity

- Poisson's equation – Green's function method

$$\Delta \mathcal{G} = \delta$$

$$\iiint \delta\, e^{-i(\vec{k}'\cdot\vec{x})} d^3x = \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\, (-k^2) \iiint e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} d^3x d^3k$$

$$1 = \iiint \hat{\mathcal{G}}(\vec{k})\, (-k^2) \frac{1}{(2\pi)^3} \iiint e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} d^3x d^3k$$

$$= \iiint \hat{\mathcal{G}}(\vec{k})\, (-k^2) \delta(\vec{k}-\vec{k}') d^3k$$

$$= -k^2 \hat{\mathcal{G}}(\vec{k})$$

Solving for Gravity

- Poisson's equation – Green's function method

$$\Delta \mathcal{G} = \delta$$

$$\iiint \delta\, e^{-i(\vec{k}'\cdot\vec{x})} d^3x = \frac{1}{(2\pi)^3} \iiint \hat{\mathcal{G}}(\vec{k})\,(-k^2) \iiint e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} d^3x d^3k$$

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$$= \iiint \hat{\mathcal{G}}(\vec{k})\,(-k^2)\delta(\vec{k}-\vec{k}')d^3k$$

$$= -k^2 \hat{\mathcal{G}}(\vec{k})$$

$$\hat{\mathcal{G}}(\vec{k}) = -\frac{1}{k^2}$$

Solving for Gravity

- Poisson's equation – Green's function method

$$\Delta G = \delta$$

$$\iiint \delta \, e^{-i(\vec{k}'\cdot\vec{x})} d^3x = \frac{1}{(2\pi)^3} \iiint \hat{G}(\vec{k}) \, (-k^2) \iiint e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} d^3x d^3k$$

$$1 = \iiint \hat{G}(\vec{k}) \, (-k^2) \frac{1}{(2\pi)^3} \iiint e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} d^3x d^3k$$

$$= \iiint \hat{G}(\vec{k}) \, (-k^2) \delta(\vec{k}-\vec{k}') d^3k$$

$$= -k^2 \hat{G}(\vec{k})$$

$$\hat{G}(\vec{k}) = -\frac{1}{k^2}$$

$$\xrightarrow{FFT^{-1}(\hat{G}(\vec{k}))}$$

Solving for Gravity

- Poisson's equation – Green's function method

$$\Delta \mathcal{G} = \delta$$

$$\hat{\mathcal{G}}(\vec{k}) = -\frac{1}{k^2} \qquad \mathcal{G}(\vec{x}) = \frac{1}{4\pi x}$$

Solving for Gravity

- Poisson's equation – Green's function method

$$\Delta \mathcal{G} = \delta$$

$$\hat{\mathcal{G}}(\vec{k}) = -\frac{1}{k^2} \qquad \mathcal{G}(\vec{x}) = \frac{1}{4\pi x}$$

$$\Delta \Phi = S$$

=>

$$\Phi(\vec{x}) = \iiint \mathcal{G}(\vec{x} - \vec{x}')\, S(\vec{x}') d^3 x'$$

- Poisson's equation – Green's function method

$$\Delta\Phi = S$$

$$\Phi(\vec{x}) = \iiint \mathcal{G}(\vec{x} - \vec{x}')\, S(\vec{x}')\, d^3x'$$

$$\mathcal{G}(\vec{x}) = \frac{1}{4\pi\, x}$$

Solving for Gravity

- Poisson's equation – Green's function method

Poisson's equation

$$\Delta\Phi = 4\pi G\rho$$

$$\Phi(\vec{x}) = 4\pi G \iiint \mathcal{G}(\vec{x}-\vec{x}')\, \rho(\vec{x}')d^3x'$$

$$\mathcal{G}(\vec{x}) = \frac{1}{4\pi x}$$

- Poisson's equation – Green's function method

Poisson's equation

$$\Delta\Phi = 4\pi G\rho$$

$$\vec{F} = -\nabla\Phi \quad \longleftrightarrow$$

Poisson's integral

$$\vec{F}(\vec{x}) = -G\iiint \frac{\rho(\vec{x}')}{\left|\vec{x}-\vec{x}'\right|^3}(\vec{x}-\vec{x}')d^3x'$$

$$\Phi(\vec{x}) = 4\pi G\iiint \mathcal{G}(\vec{x}-\vec{x}')\,\rho(\vec{x}')d^3x'$$

$$\mathcal{G}(\vec{x}) = \frac{1}{4\pi x}$$

$$\begin{aligned}
\nabla_x\Phi &= \nabla_x\left(4\pi G\iiint \mathcal{G}(\vec{x}-\vec{x}')\,\rho(\vec{x}')d^3x'\right) \\
&= 4\pi G\iiint \rho(\vec{x}')\nabla_x\mathcal{G}(\vec{x}-\vec{x}')\,d^3x' \\
&= 4\pi G\iiint \rho(\vec{x}')\nabla_x\left(\frac{1}{4\pi\left(x-x'\right)}\right)d^3x' \\
&= G\iiint \rho(\vec{x}')\frac{(\vec{x}-\vec{x}')}{\left|\vec{x}-\vec{x}'\right|^3}\,d^3x'
\end{aligned}$$

Solving for Gravity

- Poisson's equation – Green's function method

Poisson's equation

$$\Delta\Phi = 4\pi G\rho$$

$$\Phi(\vec{x}) = 4\pi G \iiint \mathcal{G}(\vec{x}-\vec{x}')\,\rho(\vec{x}')d^3x'$$

$$\mathcal{G}(\vec{x}) = \frac{1}{4\pi x}$$

Poisson's integral

$$\vec{F}(\vec{x}) = -G\iiint \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|^3}(\vec{x}-\vec{x}')d^3x'$$

↑

particle approach

$$\rho(\vec{r}) = \sum_{i=1}^{N} m_i \delta_{\mathrm{Dirac}}(\vec{r}-\vec{r}_i)$$

Note:

- we are already using particles to sample phase-space!
- we explicitly use subscript 'Dirac' to avoid confusion with the density contrast...

Solving for Gravity

- Poisson's equation – Green's function method

Poisson's equation

$$\Delta\Phi = 4\pi G\rho$$

$$\Phi(\vec{x}) = 4\pi G \iiint \mathcal{G}(\vec{x}-\vec{x}')\,\rho(\vec{x}')d^3x'$$

$$\mathcal{G}(\vec{x}) = \frac{1}{4\pi x}$$

$$\begin{aligned}
\Phi(\vec{r}) &= 4\pi G \iiint \mathcal{G}(\vec{r}-\vec{r}')\,\rho(\vec{r}')d^3r' \\
&= 4\pi G \iiint \mathcal{G}(\vec{r}-\vec{r}')\sum_{i=1}^{N} m_i \delta_{\mathtt{Dirac}}(\vec{r}'-\vec{r}_i)d^3r' \\
&= 4\pi G \sum_{i=1}^{N} m_i \iiint \mathcal{G}(\vec{r}-\vec{r}')\delta_{\mathtt{Dirac}}(\vec{r}'-\vec{r}_i)\,d^3r' \\
&= \sum_{i=1}^{N} m_i \iiint \frac{G}{|\vec{r}-\vec{r}'|}\delta_{\mathtt{Dirac}}(\vec{r}'-\vec{r}_i)\,d^3r' \\
&= \sum_{i=1}^{N} m_i \iiint \frac{G}{|(\vec{r}-\vec{r}_i)-(\vec{r}'-\vec{r}_i)|}\delta_{\mathtt{Dirac}}(\vec{r}'-\vec{r}_i)\,d^3r' \\
&= -\sum_{i=1}^{N} m_i \iiint \frac{G}{|(\vec{r}'-\vec{r}_i)-(\vec{r}-\vec{r}_i)|}\delta_{\mathtt{Dirac}}(\vec{r}'-\vec{r}_i)\,d^3r' \\
&= -\sum_{i=1}^{N} m_i \iiint \frac{G}{|\vec{y}-(\vec{r}-\vec{r}_i)|}\delta_{\mathtt{Dirac}}(\vec{y})\,d^3y = -\sum_{i=1}^{N}\frac{Gm_i}{|\vec{r}-\vec{r}_i|}
\end{aligned}$$

=>

$$\begin{aligned}
\vec{F}_i(\vec{r}_i) &= -m_i\vec{\nabla}\Phi(\vec{r}_i) \\
&= -m_i\vec{\nabla}\left(-\sum_{j=1}^{N}\frac{Gm_j}{|\vec{r}_i-\vec{r}_j|}\right) \\
&= \sum_{j=1}^{N} Gm_i m_j \vec{\nabla}\frac{1}{|\vec{r}_i-\vec{r}_j|} \\
&= -\sum_{j=1}^{N} Gm_i m_j \frac{(\vec{r}_i-\vec{r}_j)}{|\vec{r}_i-\vec{r}_j|^3} \\
&= -\sum_{j=1}^{N} \frac{Gm_i m_j}{|\vec{r}_i-\vec{r}_j|^3}(\vec{r}_i-\vec{r}_j)
\end{aligned}$$

Poisson's integral

$$\vec{F}_i(\vec{r}_i) = -\sum_{i\neq j}\frac{Gm_i m_j}{(r_i-r_j)^3}(\vec{r}_i-\vec{r}_j) \quad \forall i \in N$$

↑

particle approach

$$\rho(\vec{r}) = \sum_{i=1}^{N} m_i \delta_{\mathrm{Dirac}}(\vec{r}-\vec{r}_i)$$

- Poisson's equation
- **direct particle-particle summation**
- the tree
- force softening
- periodic boundaries

Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{(r_i - r_j)^3} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

Solving for Gravity

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✓ advantage:	easy to code
↯ drawback:	extremely time consuming (N^2 operations)

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$$N \times N = N^2$$

✓ advantage:	easy to code
⚡ drawback:	extremely time consuming (N^2 operations)

Solving for Gravity

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$$N \times N = N^2$$

overcoming the "N^2" issue?!

✓ advantage: easy to code

drawback: extremely time consuming (N^2 operations)

Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{(r_i - r_j)^3} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

$$N \times N = N^2$$

overcoming the "N^2" issue?!

organizing particles into a "tree structure" will give $N \log(N)$ operations

✓ advantage: easy to code

drawback: extremely time consuming (N^2 operations)

Solving for Gravity

- direct particle-particle summation (PP) – the idea behind tree codes:

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{(r_i - r_j)^3} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

$$\vec{F}_i(\vec{r}_i) = \cdots + \frac{-G m_i m_n}{(r_i - r_n)^3} (\vec{r}_i - \vec{r}_n) + \cdots + \frac{-G m_i m_j}{\left(r_i - r_j\right)^3} \left(\vec{r}_i - \vec{r}_j\right) + \cdots + \frac{-G m_i m_k}{(r_i - r_k)^3} (\vec{r}_i - k) + \cdots$$

Solving for Gravity

- direct particle-particle summation (PP) – the idea behind tree codes:

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{(r_i - r_j)^3} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

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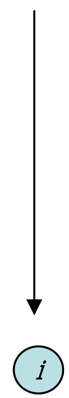

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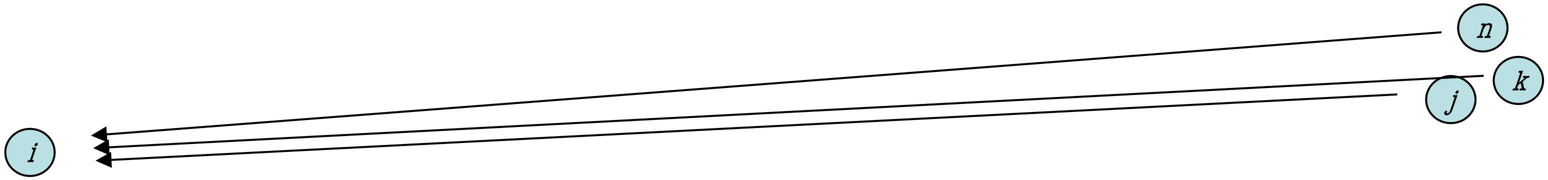

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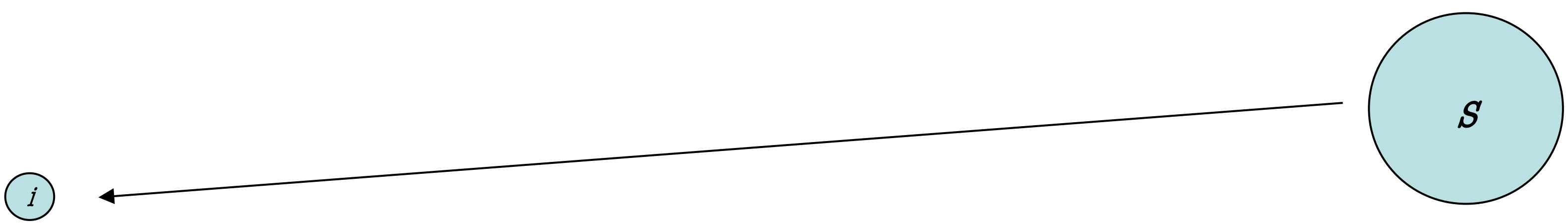
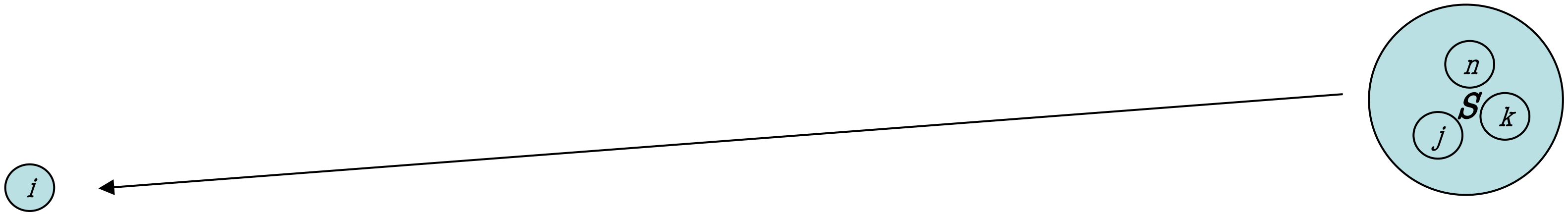


Solving for Gravity

- direct particle-particle summation (PP) – the idea behind tree codes:

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{(r_i - r_j)^3} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

$$\vec{F}_i(\vec{r}_i) = \cdots + \cdots + \cdots + \cdots + \cdots + \cdots + \underbrace{\frac{-G m_i M_s}{(r_i - r_s)^3} (\vec{r}_i - \vec{r}_s)}_{s = n \cap j \cap k} + \cdots + \cdots + \cdots + .. + .. + ..$$



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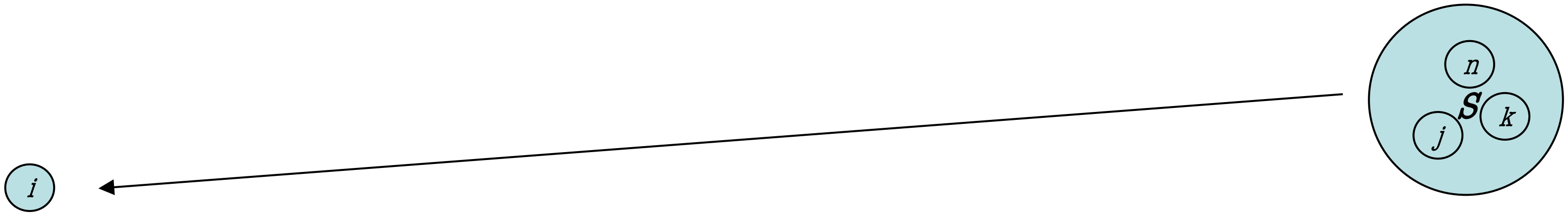


a) how to find $s = n \cap j \cap k$?

b) how get get M_s and r_s?

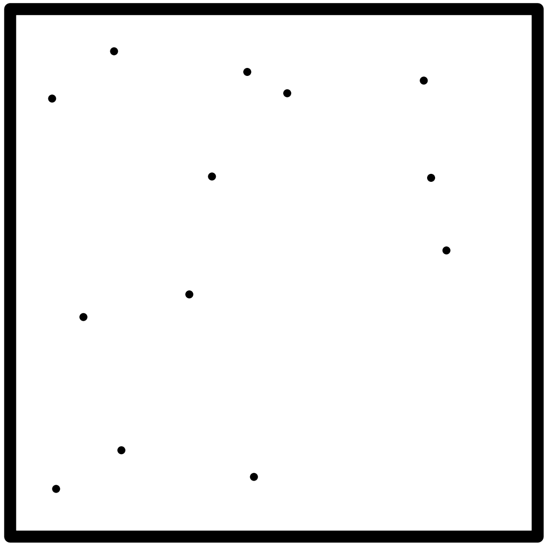
- Poisson's equation
- direct particle-particle summation
- **the tree**
- force softening
- periodic boundaries

Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{(r_i - r_j)^3} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

  - generating the tree:



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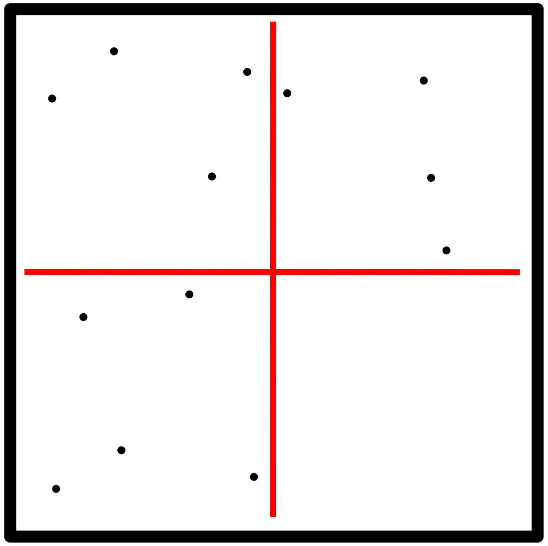

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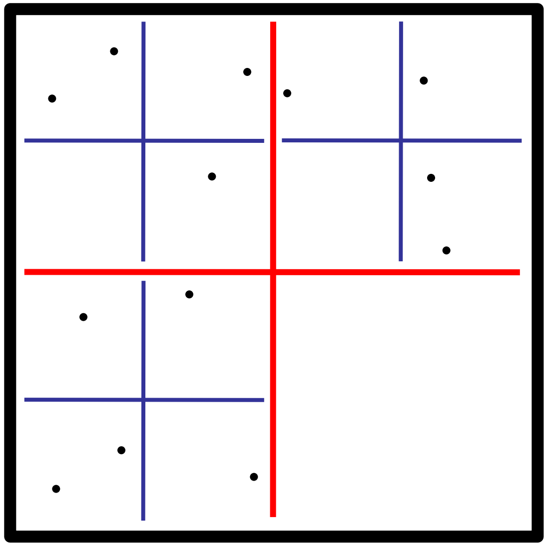

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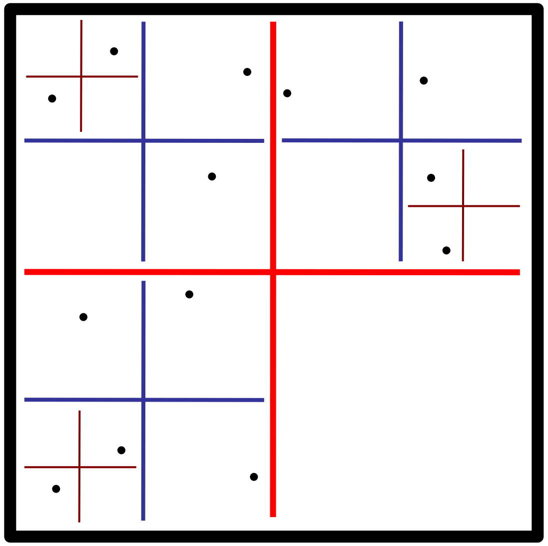

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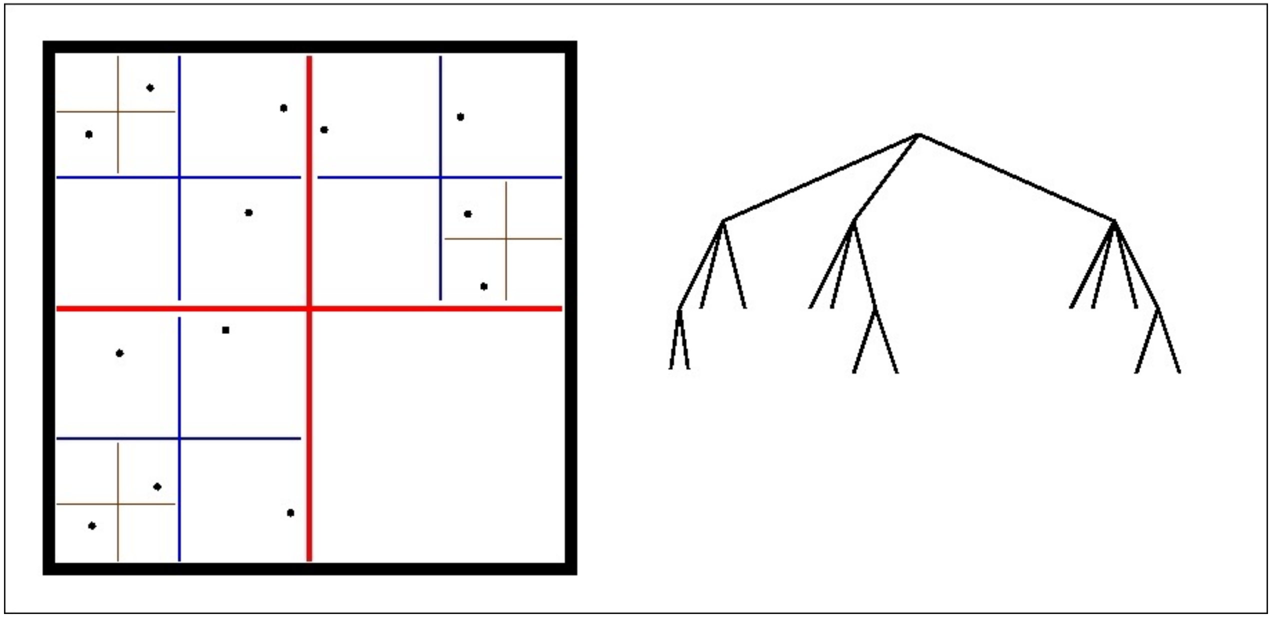

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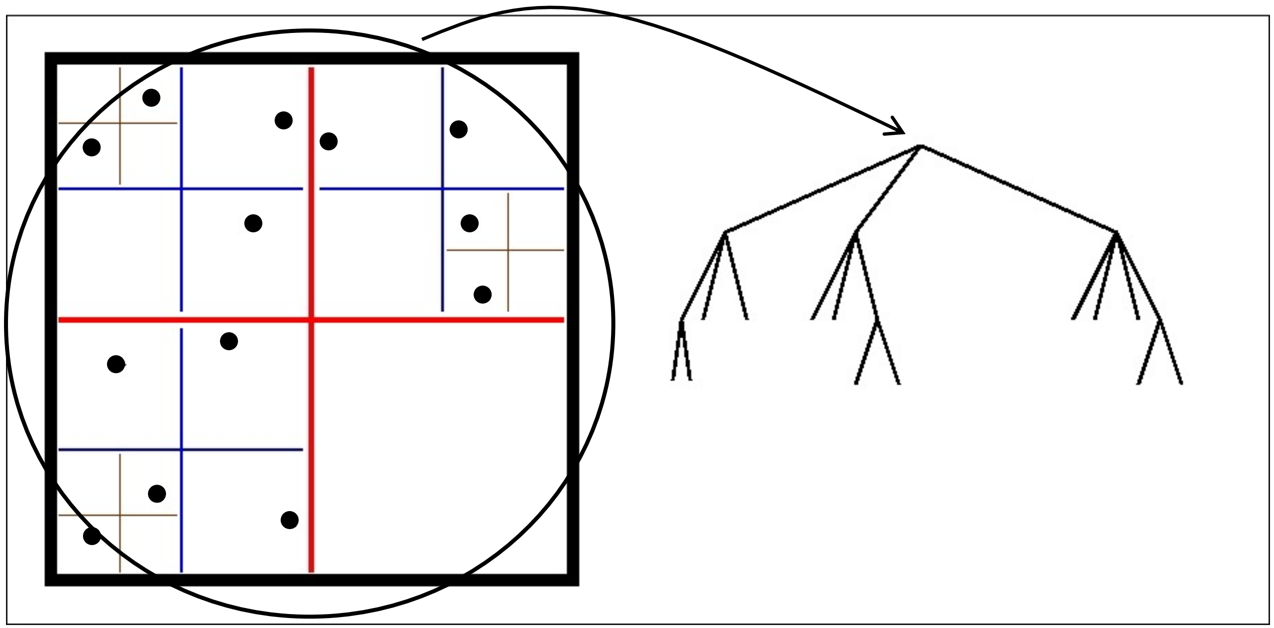
- generating the tree:

total mass, centre-of-mass, …



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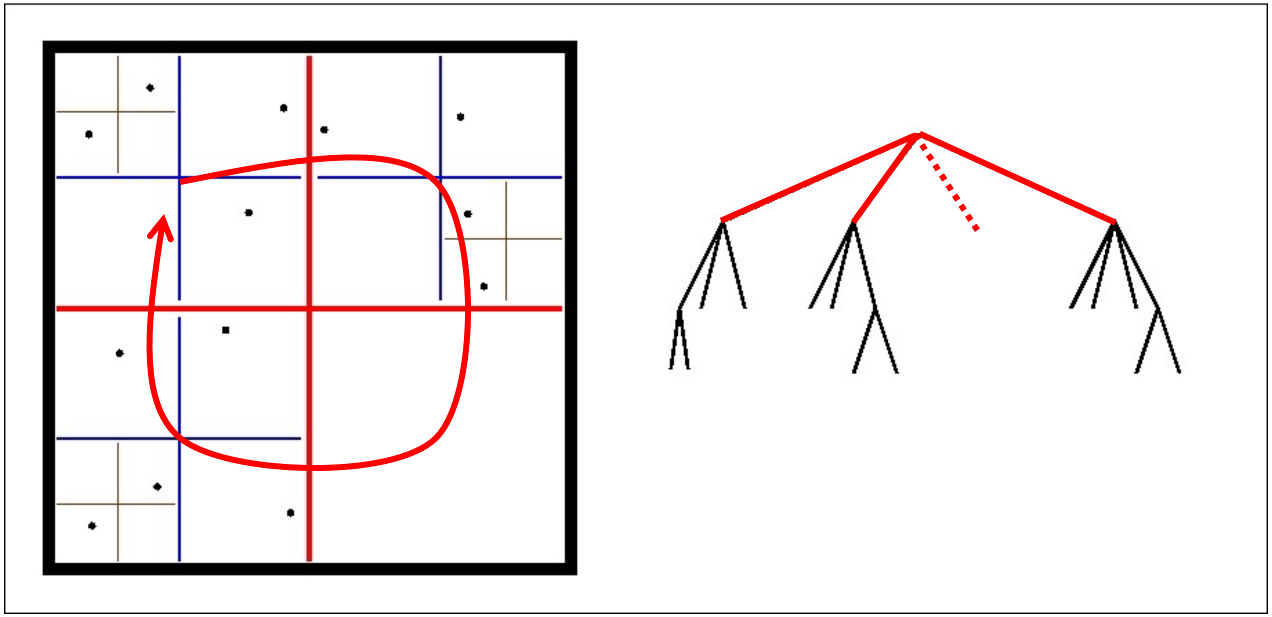

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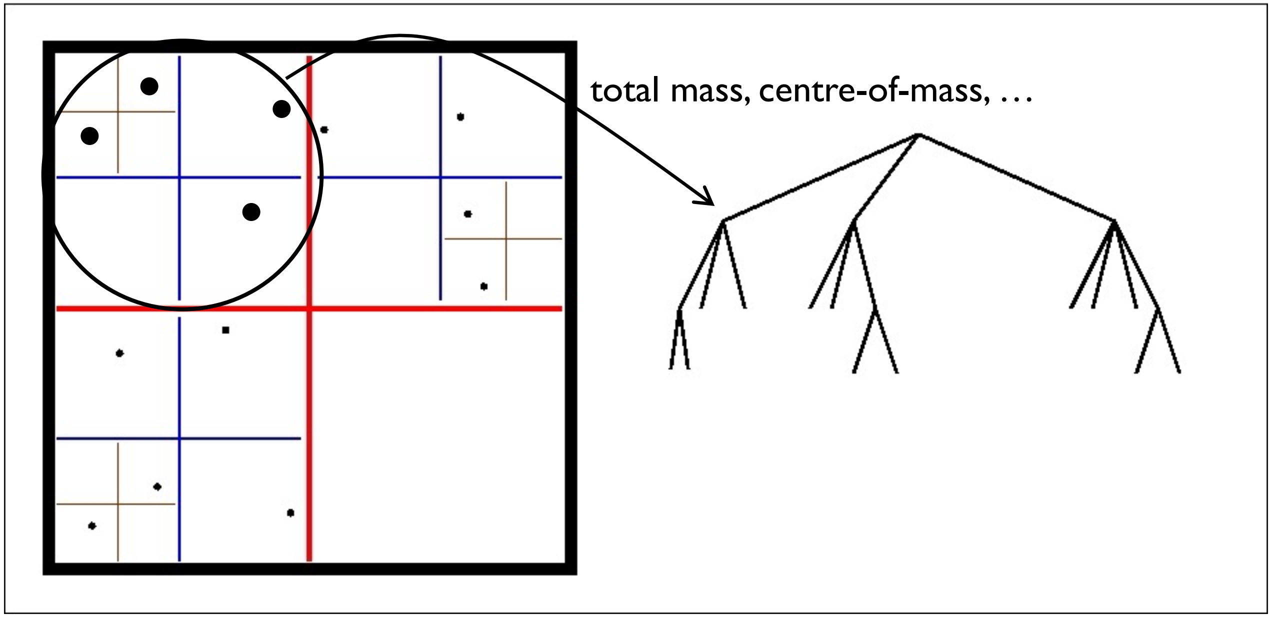

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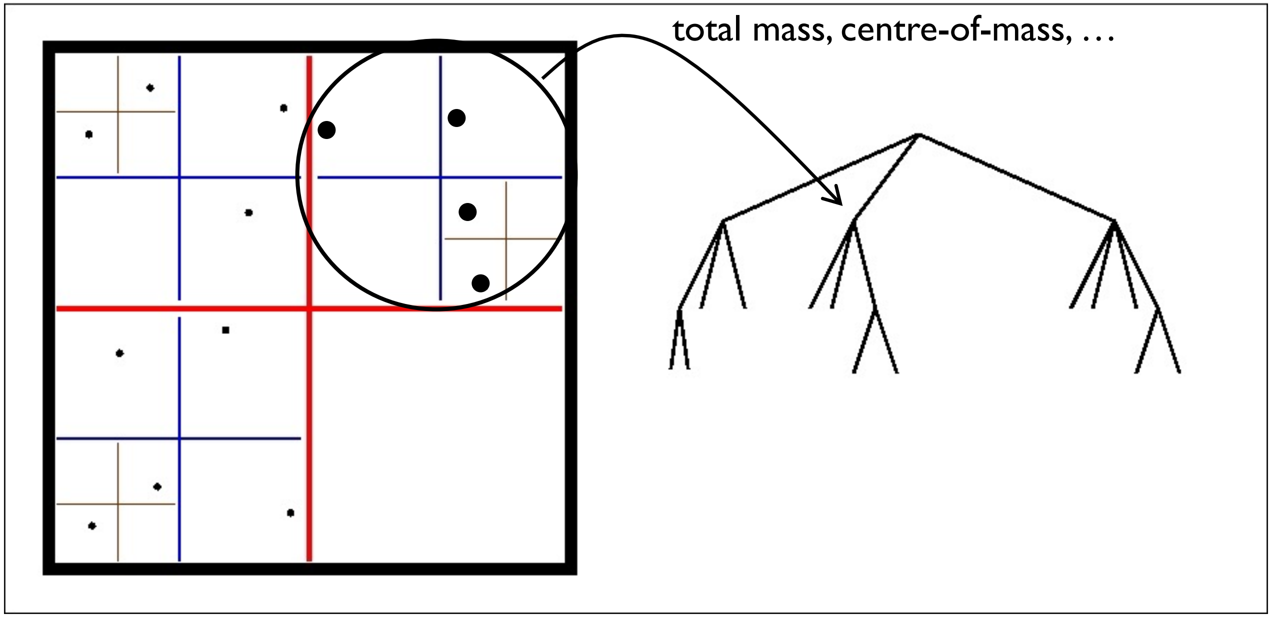

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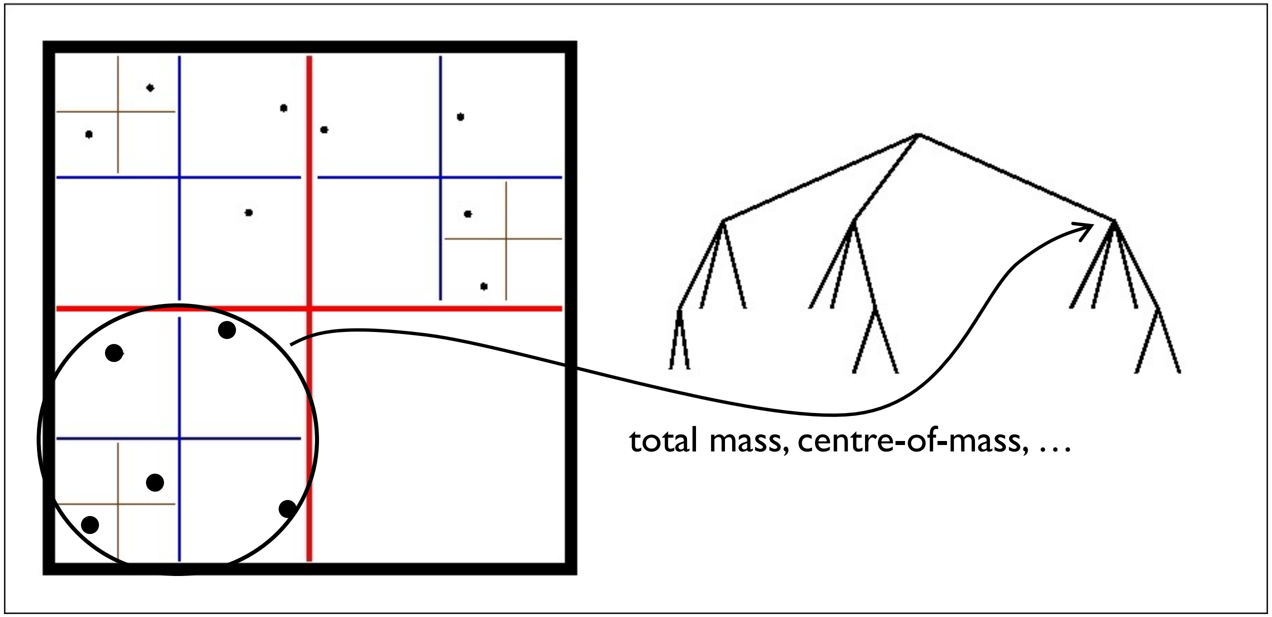

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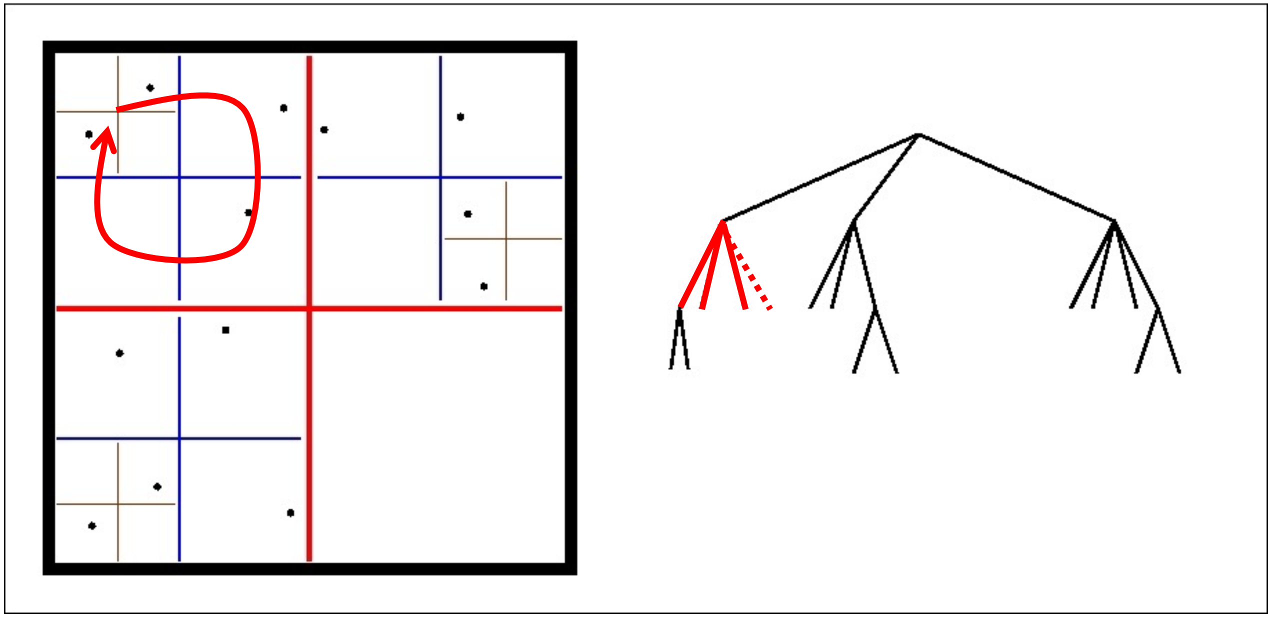

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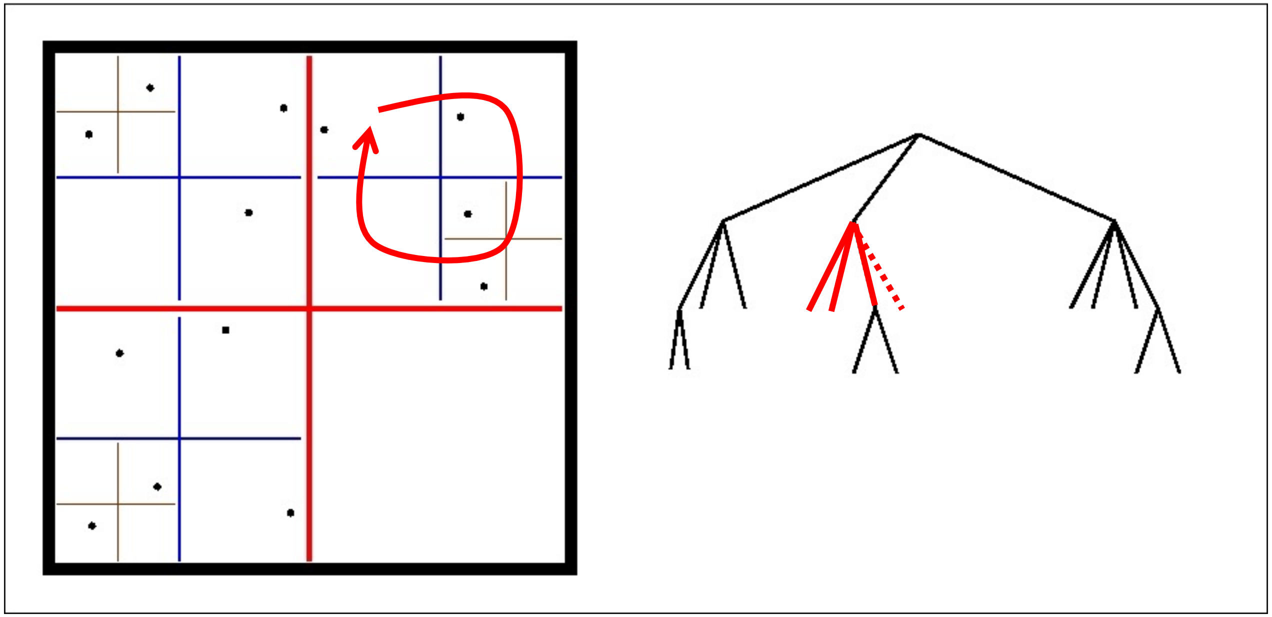

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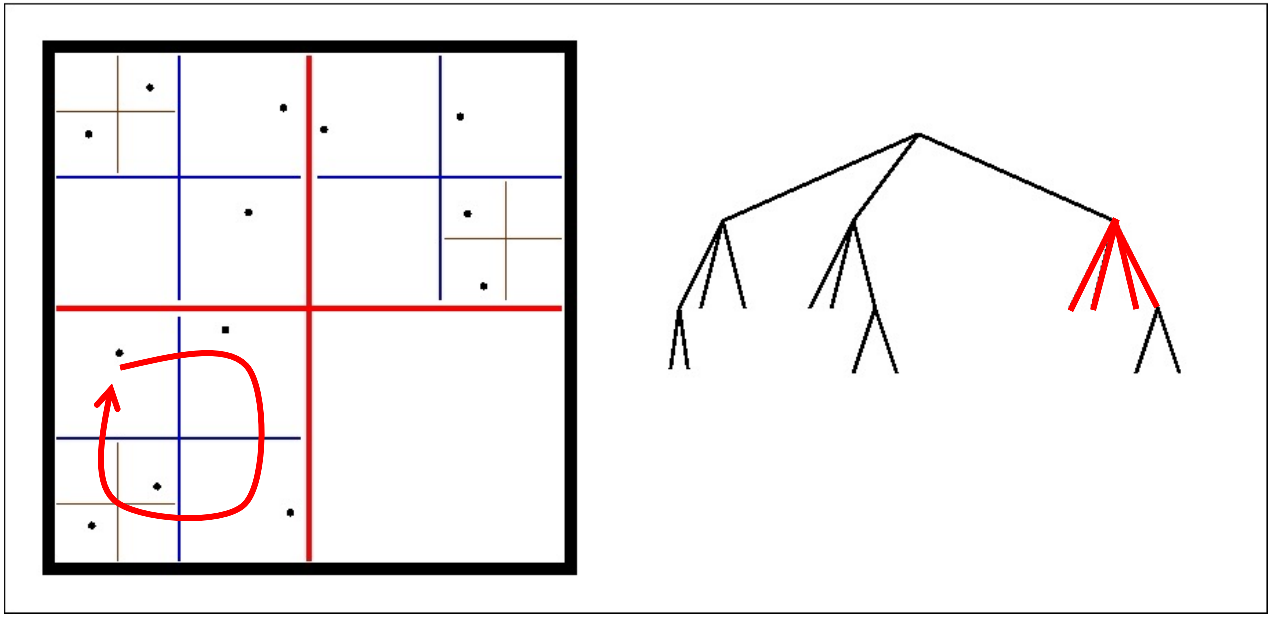

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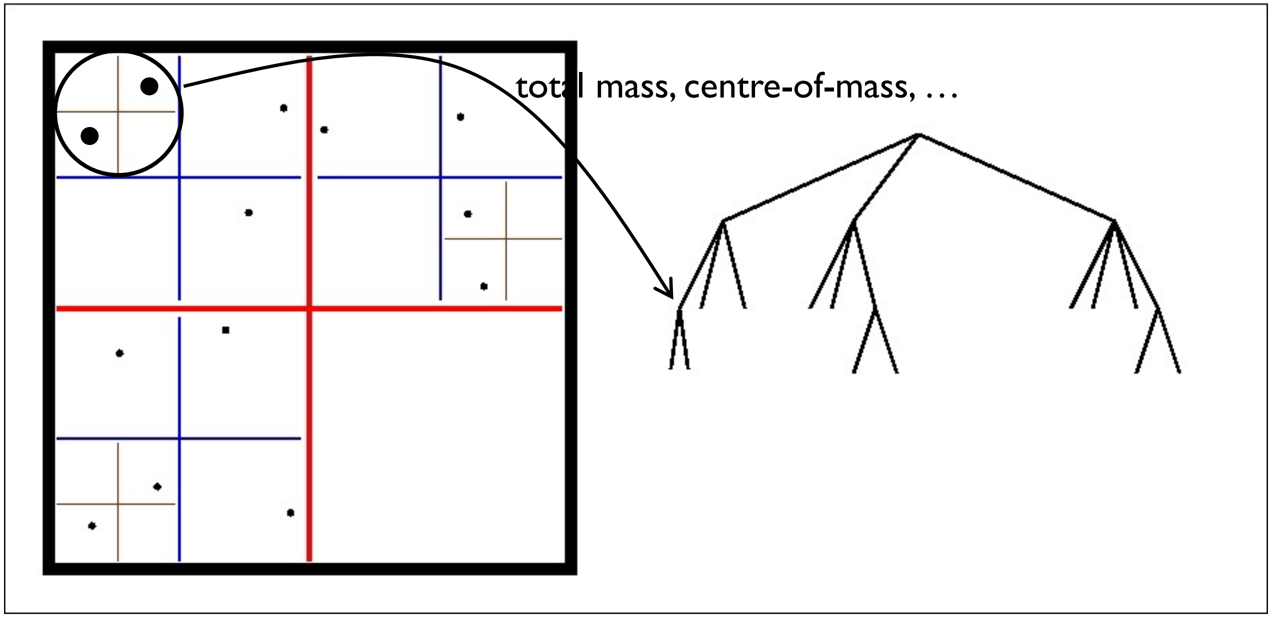

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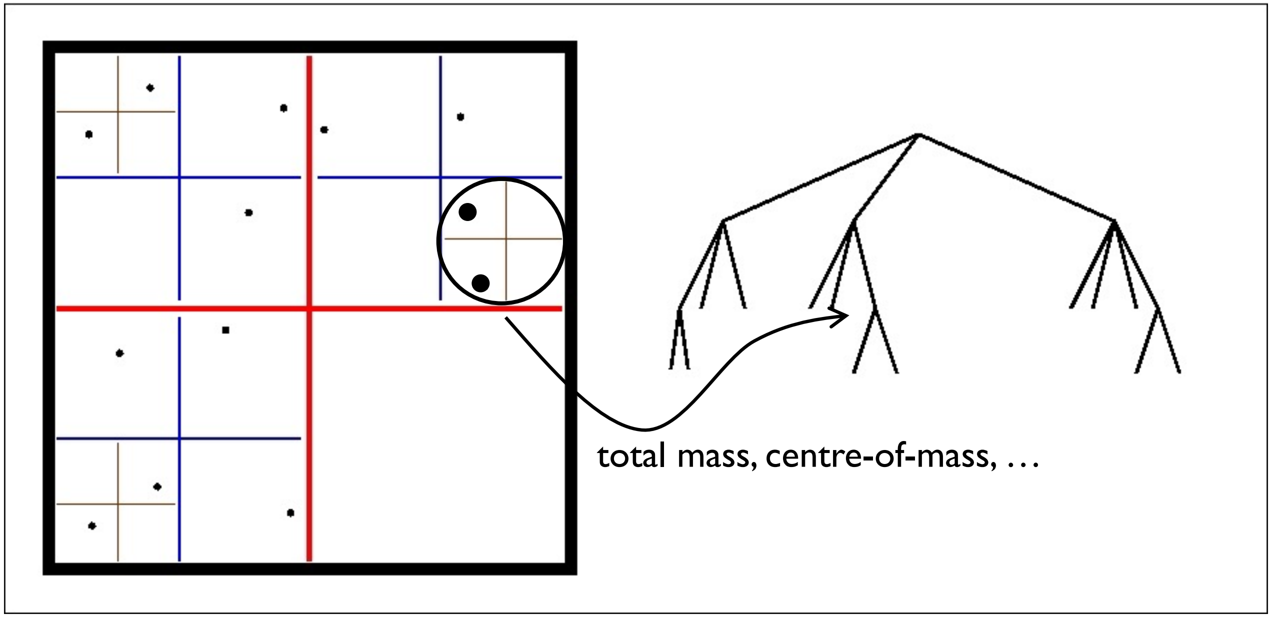

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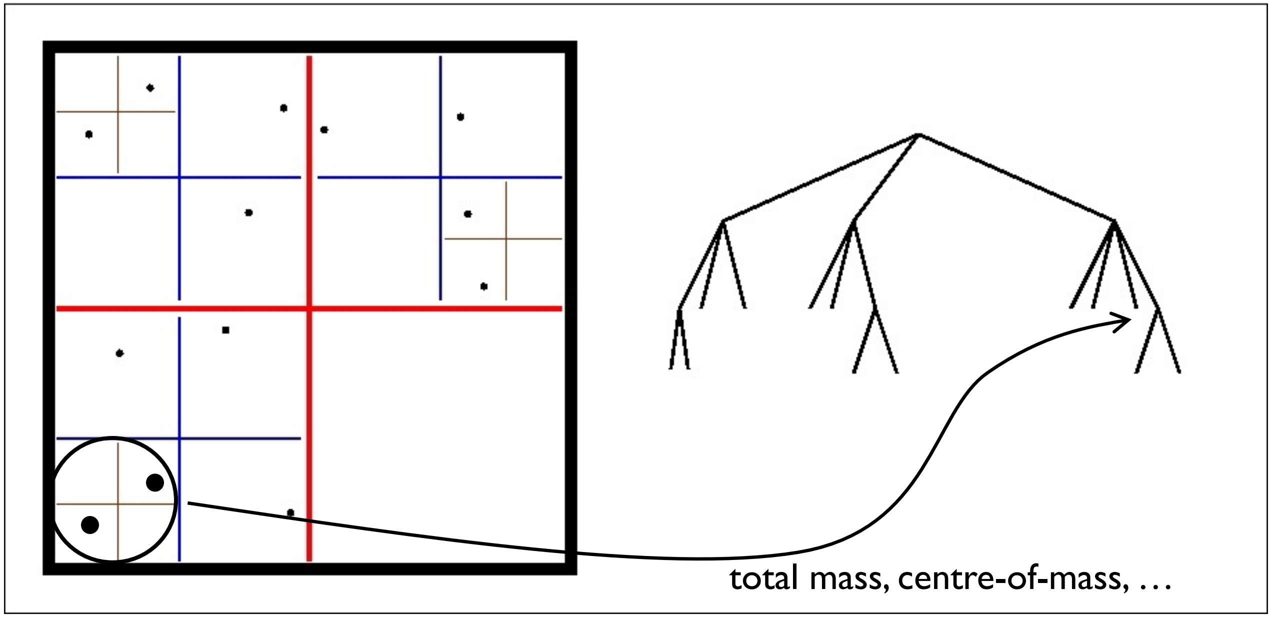

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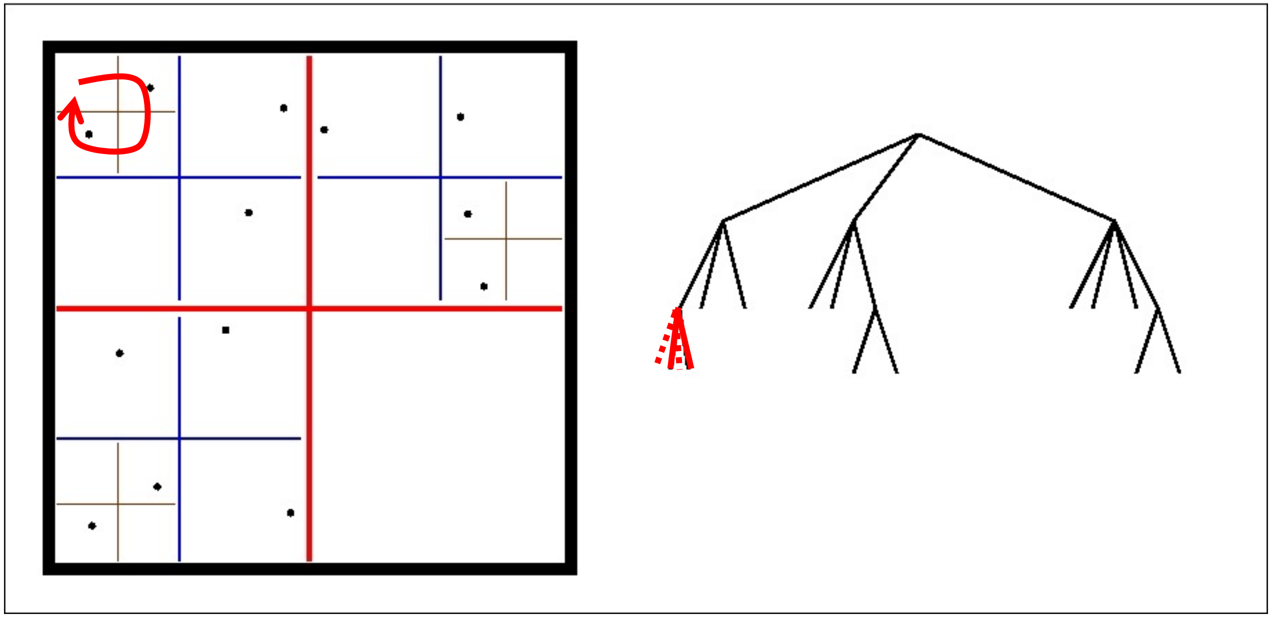

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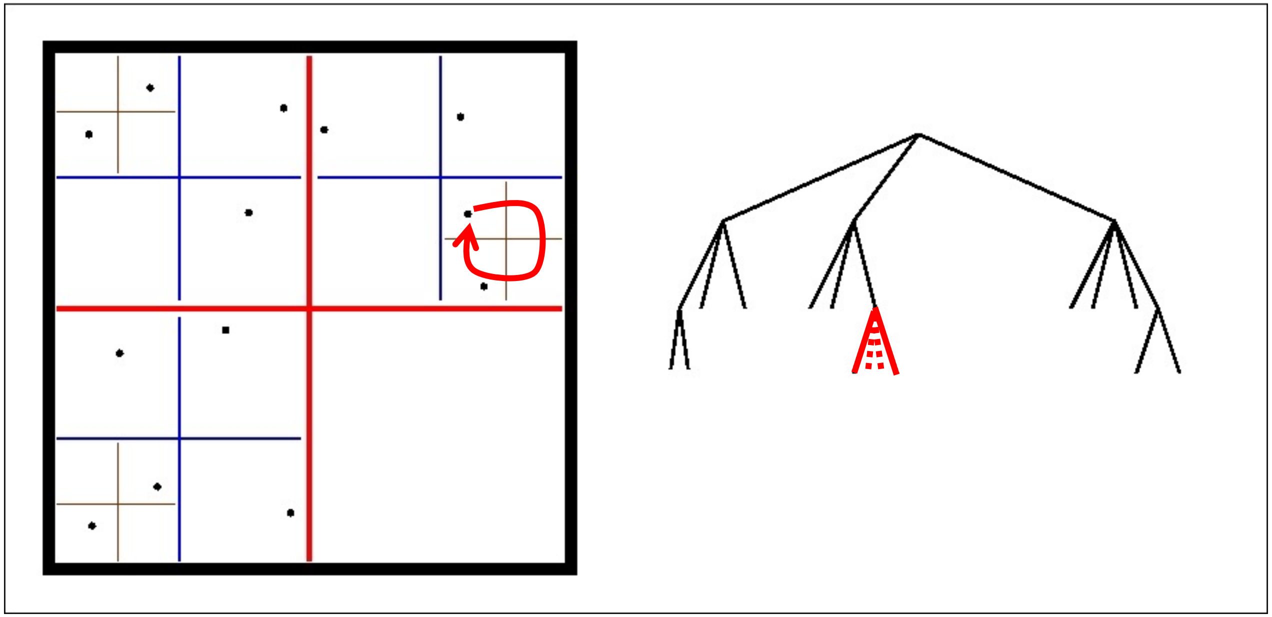

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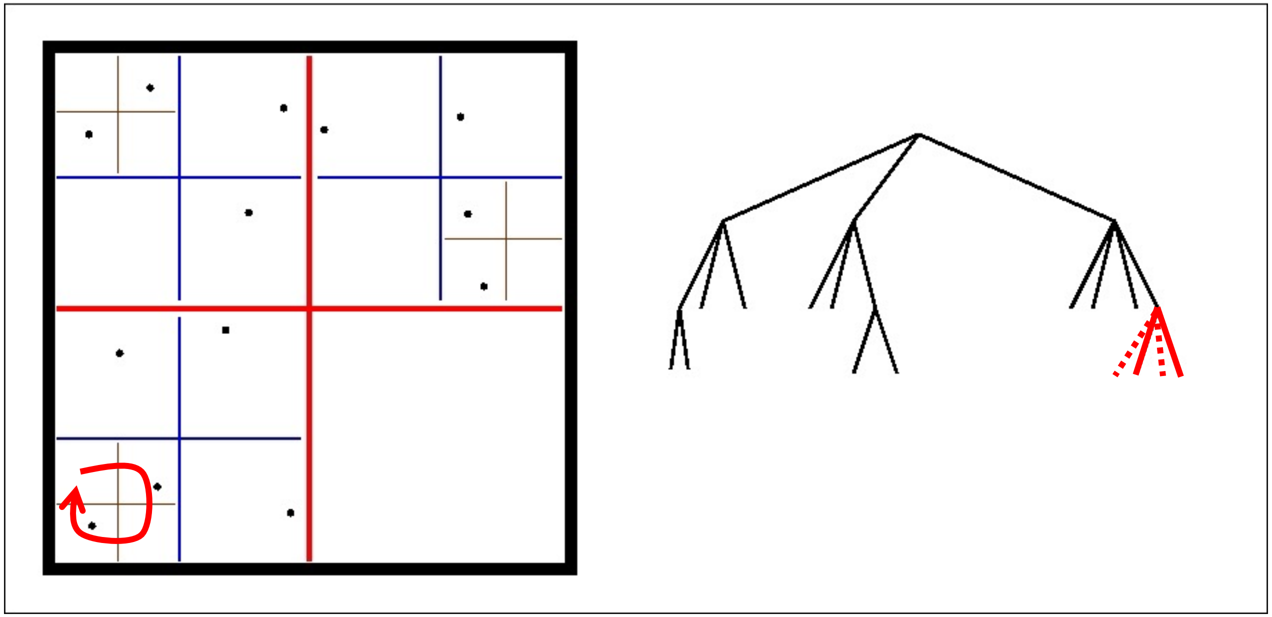

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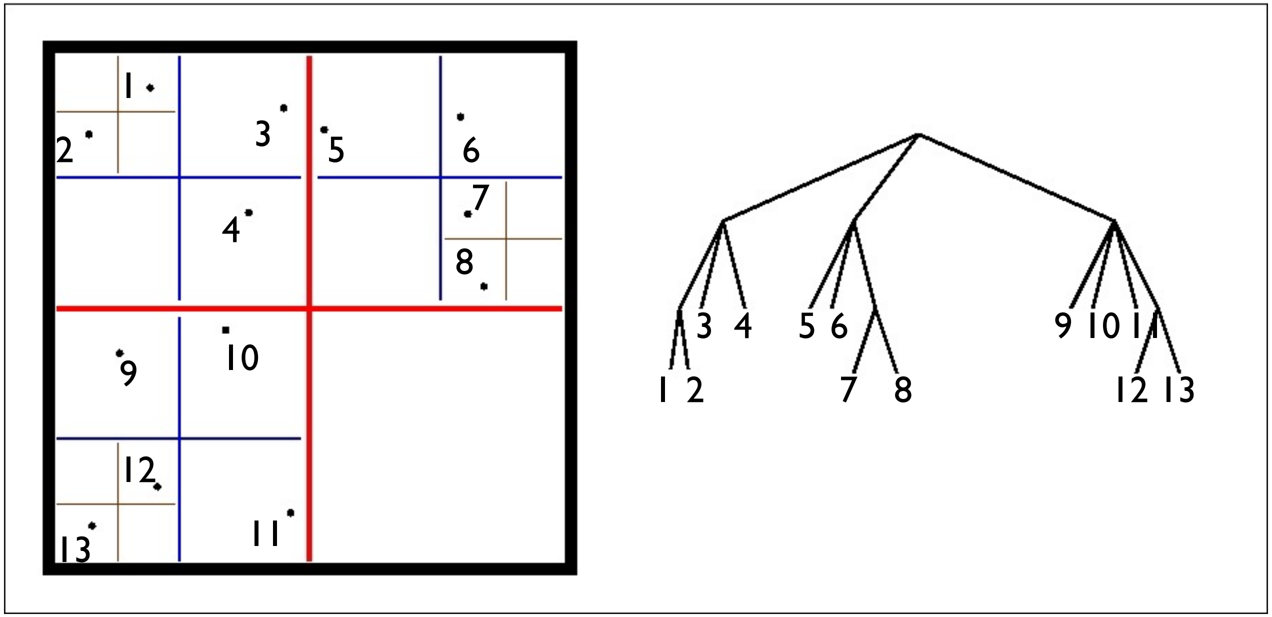

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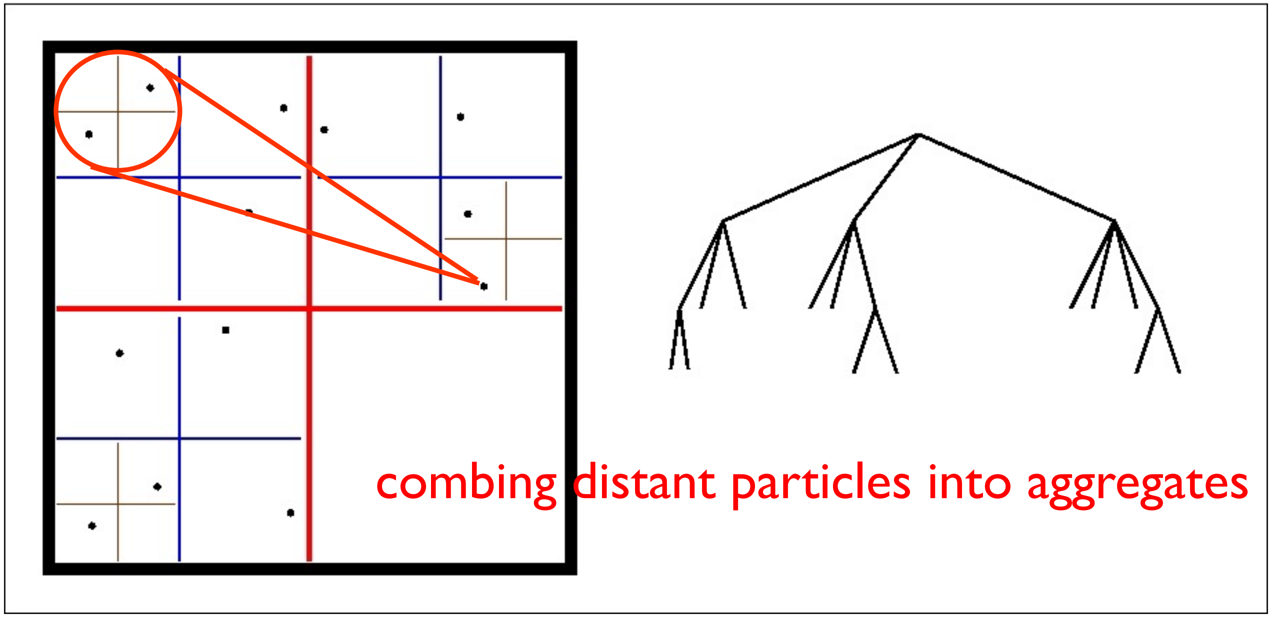
- walking the tree ($\forall i \in N$):



combing distant particles into aggregates

Solving for Gravity

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$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{(r_i - r_j)^3} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

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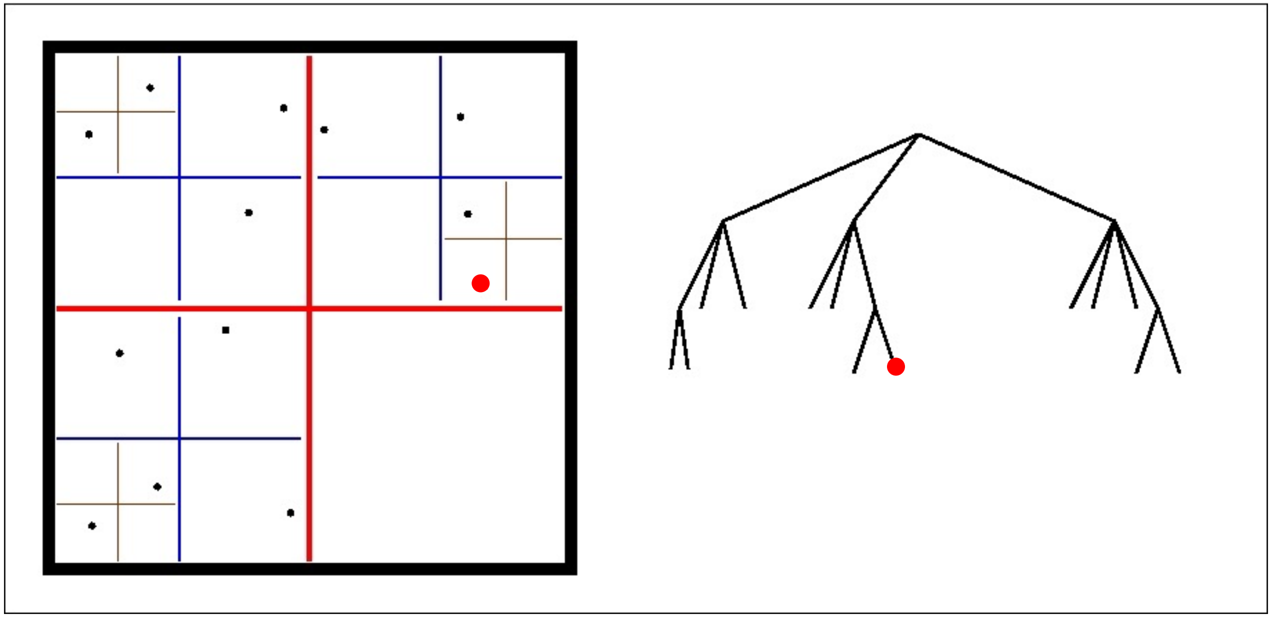

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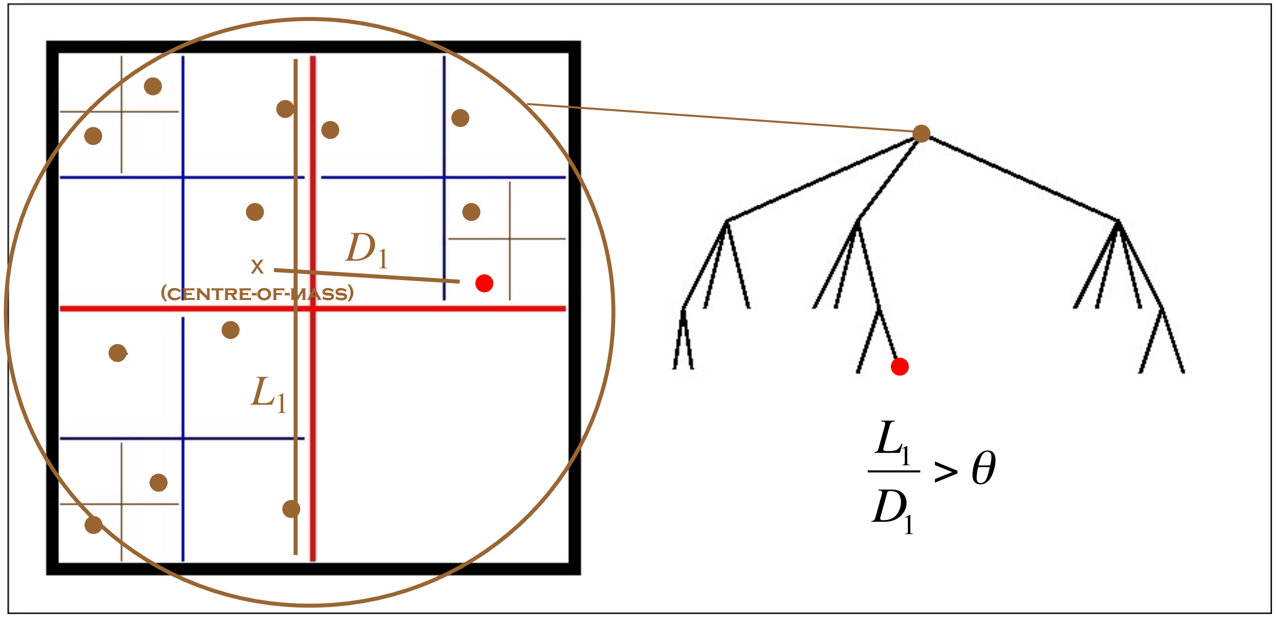

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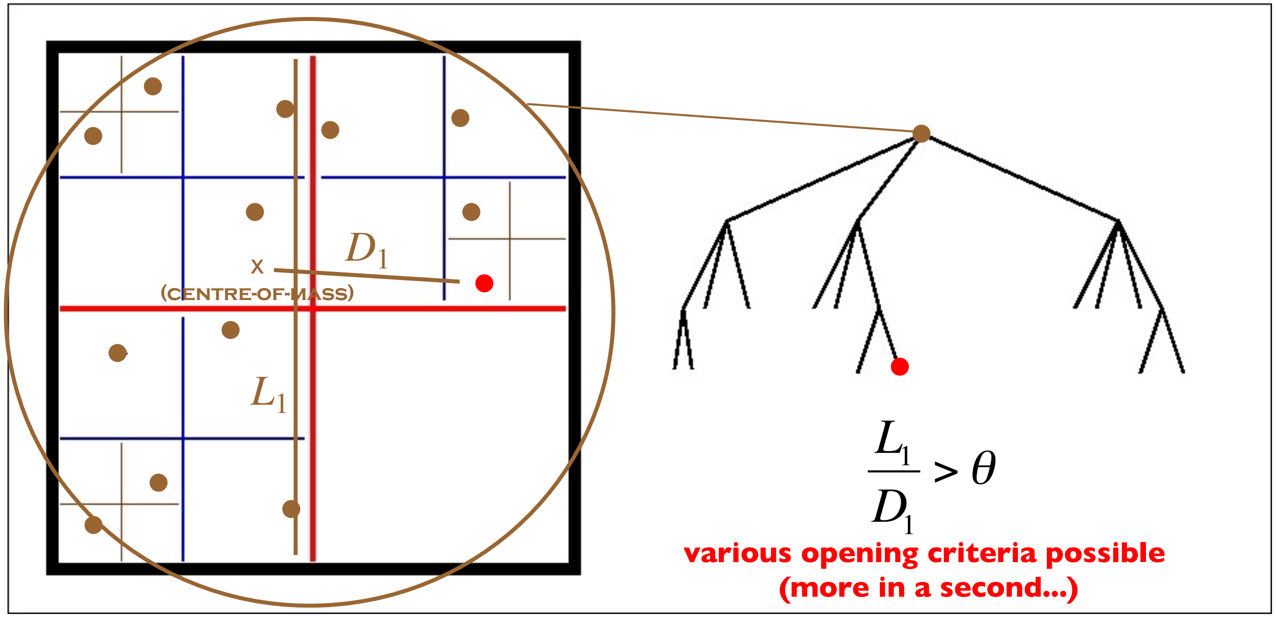

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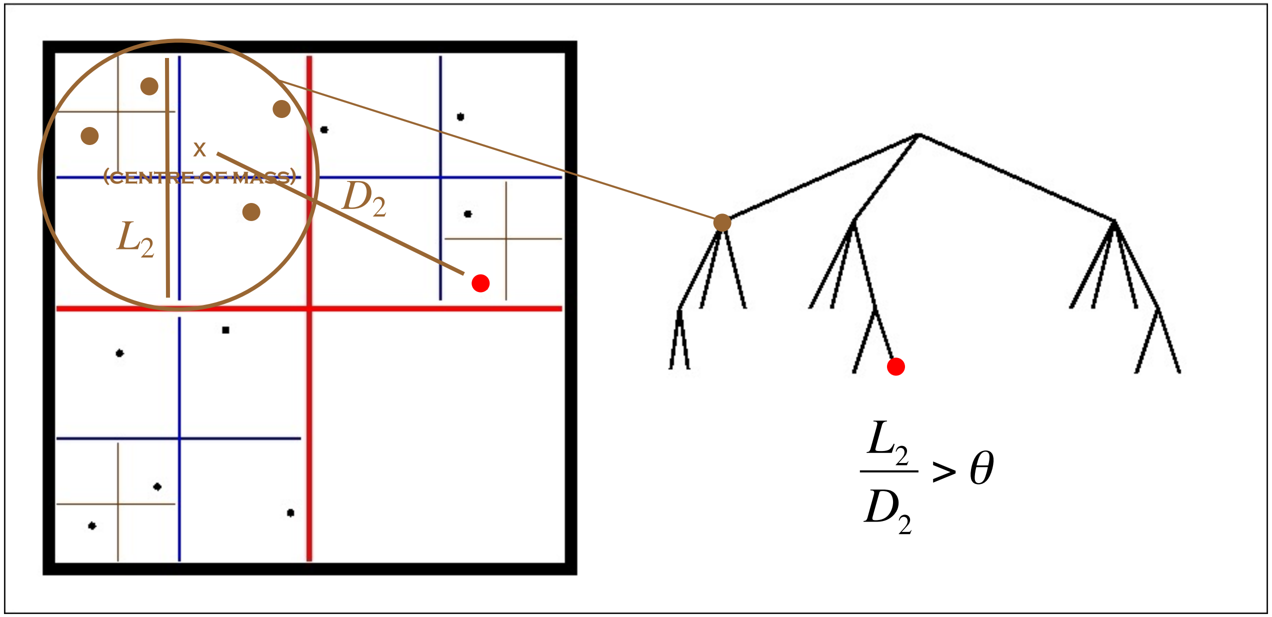

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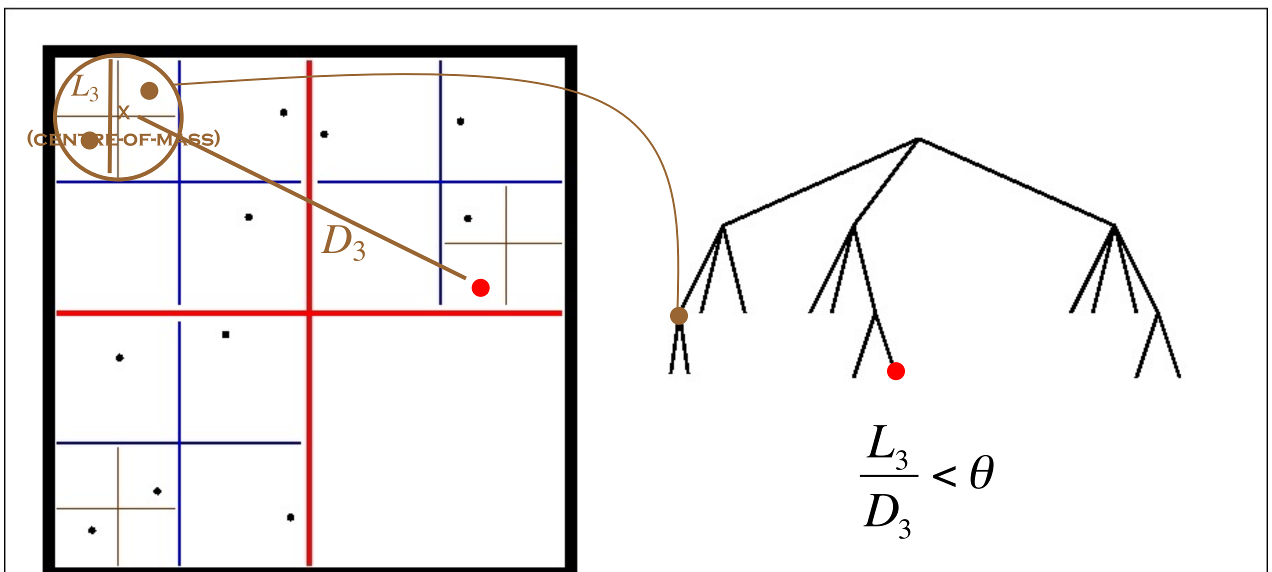
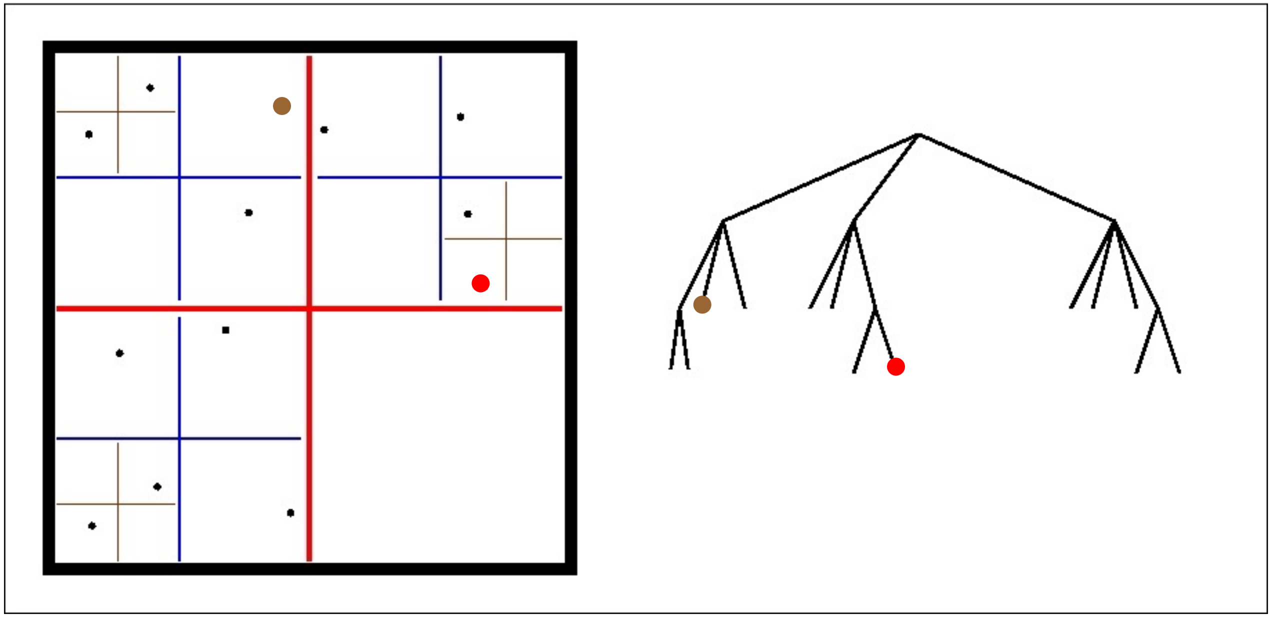


$\Rightarrow$ stop opening that branch and add force contribution from "super-particle"

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i\neq j} \frac{G m_i m_j}{(r_i - r_j)^3}(\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

- walking the tree ($\forall i \in N$):



we still need to add the remaining contributions from that branch…

Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{(r_i - r_j)^3} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

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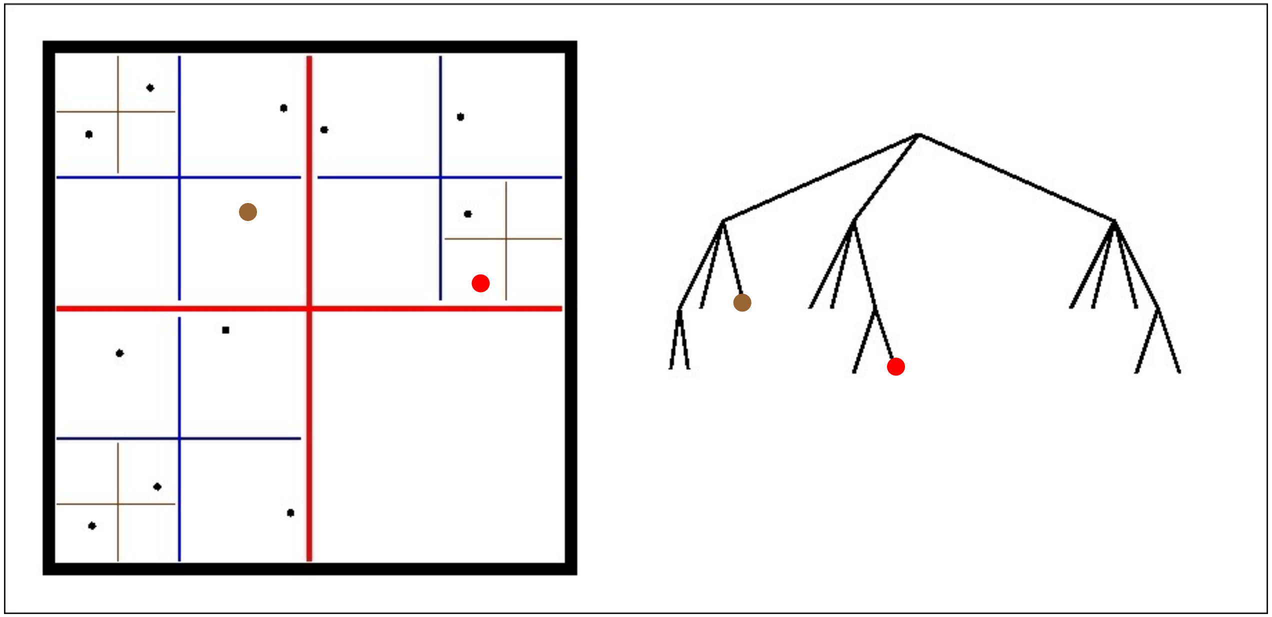

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Solving for Gravity

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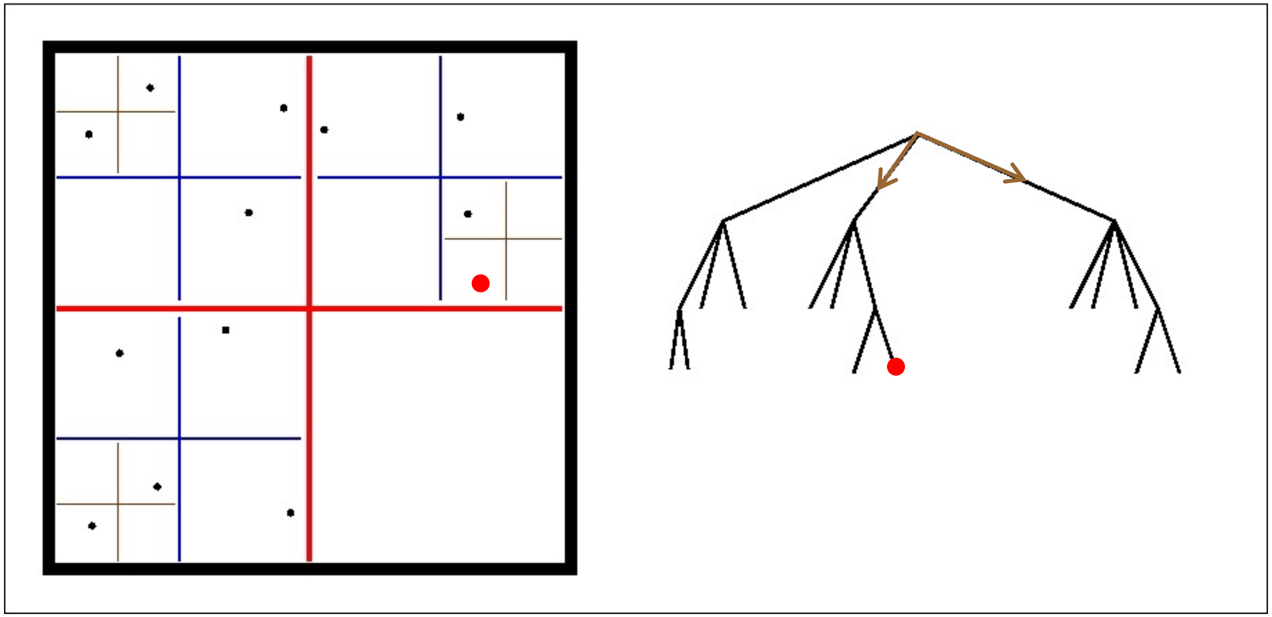


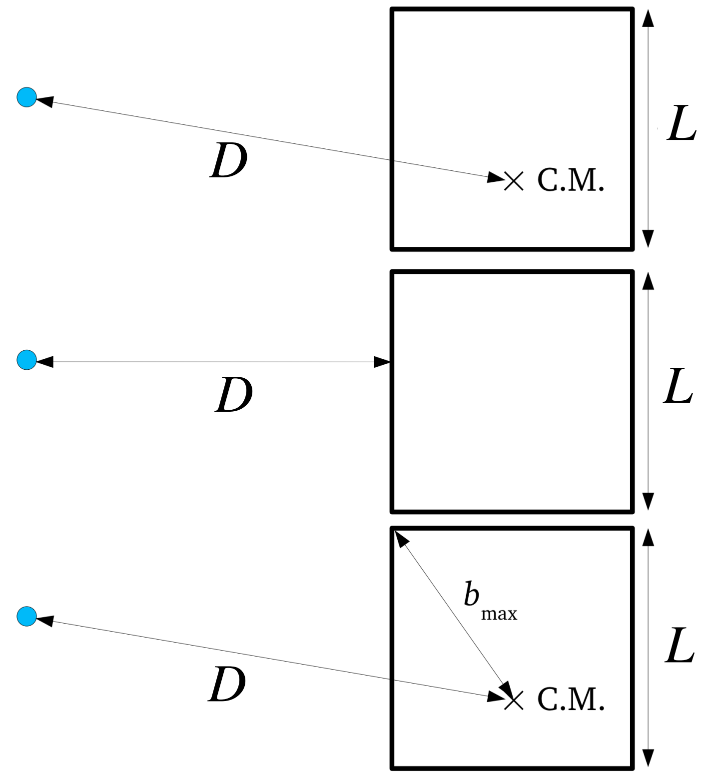
...as well as walking the other branches!

Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i\neq j} \frac{G m_i m_j}{(r_i - r_j)^3}(\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

- opening criteria:
  - Barnes-Hut $L/D < \theta$
  - Min-distance $L/D < \theta$
  - Bmax $b_{max}/D < \theta$



- direct particle-particle summation (PP)

other speed-ups?

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{(r_i - r_j)^3} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

GRAPE (GRAvity PipE):

- particle-particle summation hardwired into motherboard
- combination with tree possible

Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{(r_i - r_j)^3} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

CUDA:

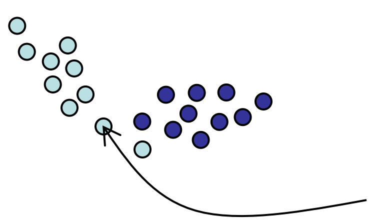
• use graphics board to perform calculations

- Poisson’s equation
- direct particle-particle summation
- the tree
- **force softening**
- periodic boundaries

- direct particle-particle summation (PP)

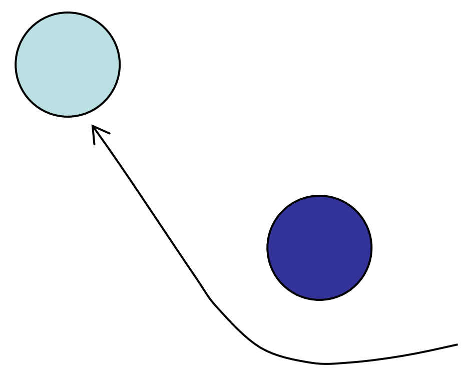
$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

we use (collisionless) particles to sample $f(x,v,t)$



the particles sampling the field adjust

vs.



the particle sampling the field bounces off

well sampled system

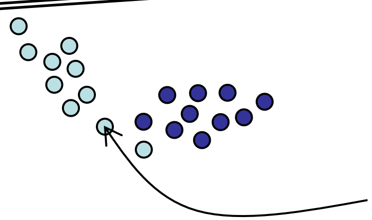
undersampled system

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

we use (collisionless) particles to sample $f(x,v,t)$

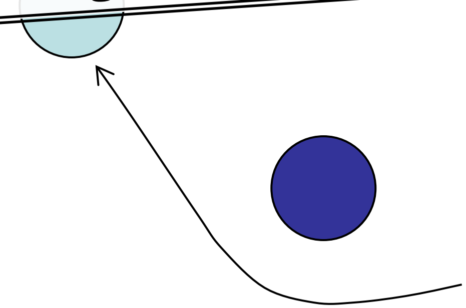
each particle should only feel the mean field generated by all particles!



the particles sampling the field adjust

well sampled system

vs.



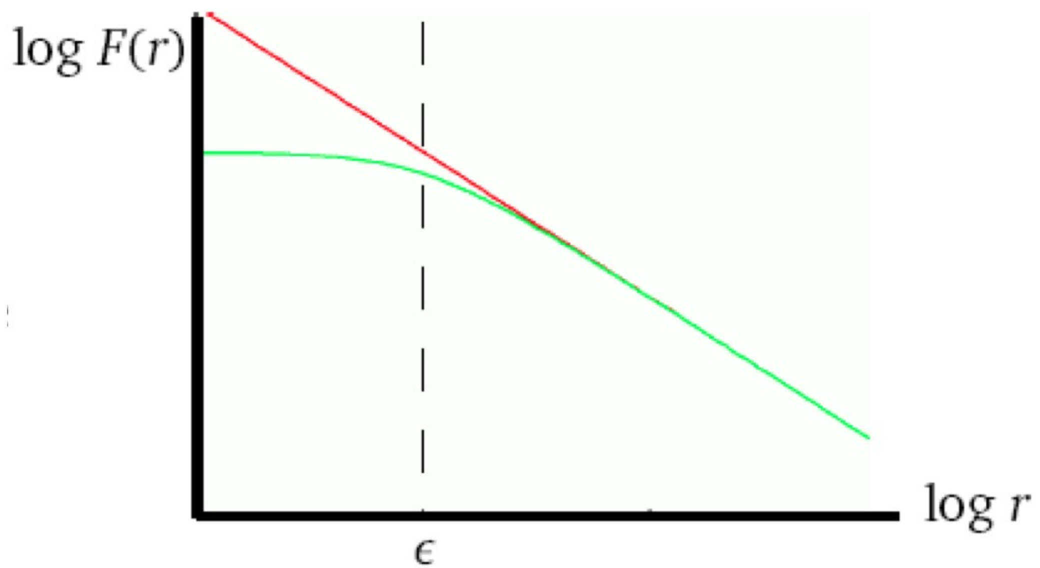
the particle sampling the field bounces off

undersampled system

Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$



Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i\neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

"soften" the force to…

1. avoid the singularity for $r_i = r_j$
2. smooth mass density on small scales

Solving for Gravity

- direct particle-particle summation (PP)

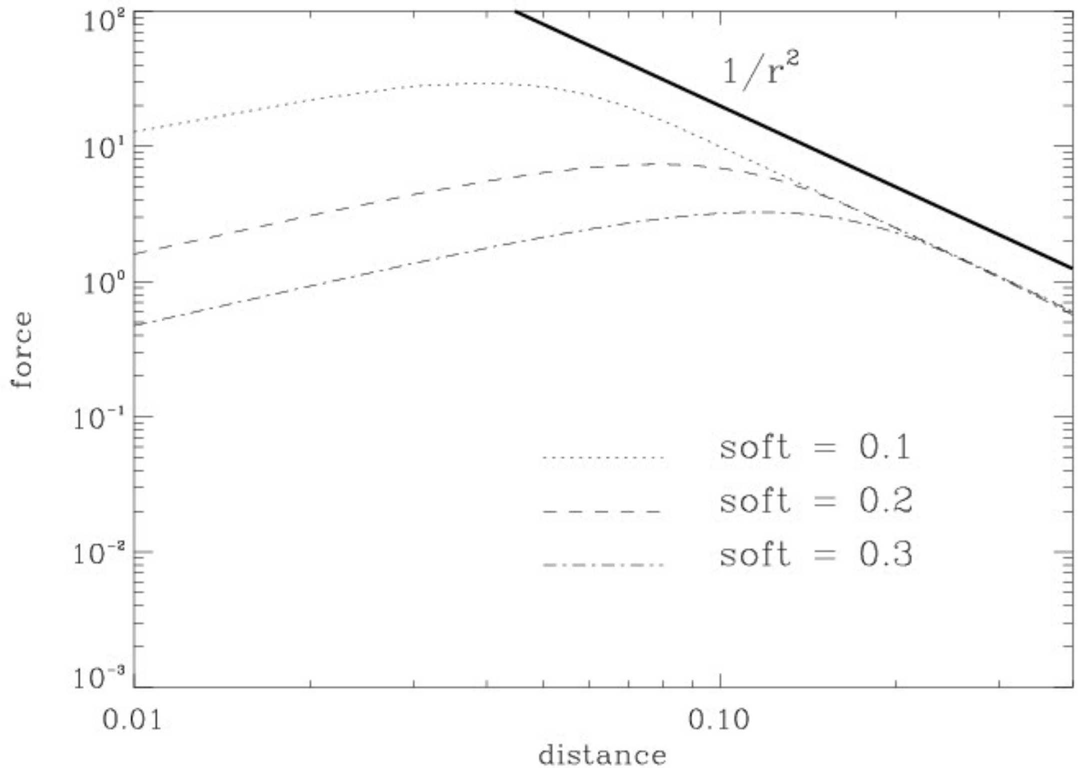
$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

ε determines the overall force resolution of the simulation

Solving for Gravity

- direct particle-particle summation (PP)

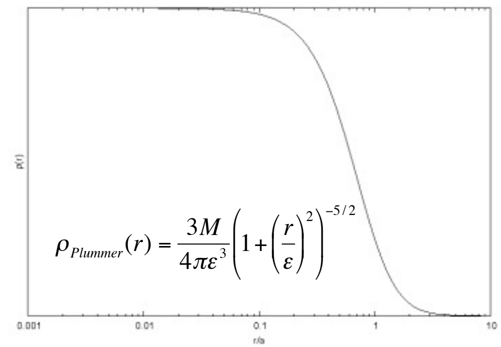
$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$



- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

- Plummer softening



- S2 softening
- spline softening
- …

for an exhaustive discussion please refer to Dehnen, MNRAS 324, 273 (2001)

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

error budget?

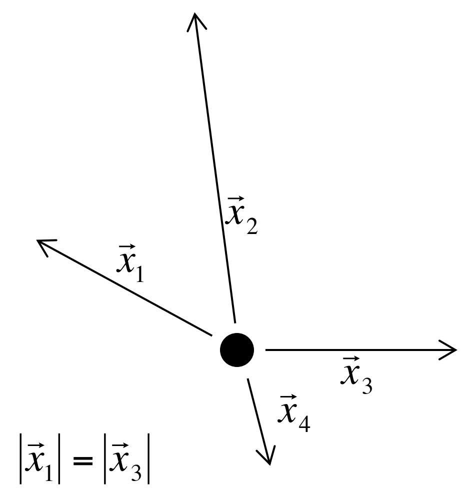
Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

error budget?

$$e^2 = \left\langle \left| \vec{F}(\vec{x}) - \vec{F}_{true}(\vec{x}) \right|^2 \right\rangle_{|\vec{x}|}$$



Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i\neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

error budget?

$$
\begin{aligned}
e^2 &= \left\langle \left| \vec{F}(\vec{x}) - \vec{F}_{true}(\vec{x}) \right|^2 \right\rangle \\
&= \left\langle \vec{F}^2(\vec{x}) \right\rangle - 2\left\langle \vec{F}(\vec{x}) \vec{F}_{true}(\vec{x}) \right\rangle + \left\langle \vec{F}^2{}_{true}(\vec{x}) \right\rangle \\
&= \left\langle \vec{F}^2(\vec{x}) \right\rangle - 2\left\langle \vec{F}(\vec{x}) \right\rangle \vec{F}_{true}(\vec{x}) + \vec{F}^2{}_{true}(\vec{x}) \\
&= \left\langle \vec{F}^2(\vec{x}) \right\rangle - 2\left\langle \vec{F}(\vec{x}) \right\rangle \vec{F}_{true}(\vec{x}) + \vec{F}^2{}_{true}(\vec{x}) + \left\langle \vec{F}(\vec{x}) \right\rangle^2 - \left\langle \vec{F}(\vec{x}) \right\rangle^2 \\
&= \left(\left\langle \vec{F}(\vec{x}) \right\rangle - \vec{F}_{true}(\vec{x}) \right)^2 + \left\langle \vec{F}^2(\vec{x}) \right\rangle - \left\langle \vec{F}(\vec{x}) \right\rangle^2
\end{aligned}
$$

Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i\neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

error budget?

$$\begin{aligned}
e^2 &= \left\langle \left| \vec{F}(\vec{x}) - \vec{F}_{true}(\vec{x}) \right|^2 \right\rangle \\
&= \left\langle \vec{F}^2(\vec{x}) \right\rangle - 2\left\langle \vec{F}(\vec{x}) \vec{F}_{true}(\vec{x}) \right\rangle + \left\langle \vec{F}^2{}_{true}(\vec{x}) \right\rangle \\
&= \left\langle \vec{F}^2(\vec{x}) \right\rangle - 2\left\langle \vec{F}(\vec{x}) \right\rangle \vec{F}_{true}(\vec{x}) + \vec{F}^2{}_{true}(\vec{x}) \\
&= \left\langle \vec{F}^2(\vec{x}) \right\rangle - 2\left\langle \vec{F}(\vec{x}) \right\rangle \vec{F}_{true}(\vec{x}) + \vec{F}^2{}_{true}(\vec{x}) + \left\langle \vec{F}(\vec{x}) \right\rangle^2 - \left\langle \vec{F}(\vec{x}) \right\rangle^2 \\
&= \left(\left\langle \vec{F}(\vec{x}) \right\rangle - \vec{F}_{true}(\vec{x}) \right)^2 + \left\langle \vec{F}^2(\vec{x}) \right\rangle - \left\langle \vec{F}(\vec{x}) \right\rangle^2
\end{aligned}$$

comparing numerical and analytical force

scatter of numerical force
(note, F only depends on $|x|$)

Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

error budget?

$$\begin{aligned}
e^2 &= \left\langle \left| \vec{F}(\vec{x}) - \vec{F}_{true}(\vec{x}) \right|^2 \right\rangle \\
&= \left\langle \vec{F}^2(\vec{x}) \right\rangle - 2\left\langle \vec{F}(\vec{x}) \vec{F}_{true}(\vec{x}) \right\rangle + \left\langle \vec{F}^2{}_{true}(\vec{x}) \right\rangle \\
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&= \left(\left\langle \vec{F}(\vec{x}) \right\rangle - \vec{F}_{true}(\vec{x}) \right)^2 + \left\langle \vec{F}^2(\vec{x}) \right\rangle - \left\langle \vec{F}(\vec{x}) \right\rangle^2 \\
&= \mathtt{bias}^2 + \mathtt{var}
\end{aligned}$$

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

error budget:

$$\texttt{bias} = \left(\left\langle \vec{F}(\vec{x}) \right\rangle - \vec{F}_{true}(\vec{x})\right)$$

$$\texttt{var} = \left\langle \vec{F}^2(\vec{x}) \right\rangle - \left\langle \vec{F}(\vec{x}) \right\rangle^2$$

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

error budget:

$$\texttt{bias} = \left(\left\langle \vec{F}(\vec{x}) \right\rangle - \vec{F}_{true}(\vec{x})\right) \propto \varepsilon^{\alpha}$$

$$\texttt{var} = \left\langle \vec{F}^2(\vec{x}) \right\rangle - \left\langle \vec{F}(\vec{x}) \right\rangle^2 \propto N^{-\beta}$$

α, β = non-trivial power-law indices…
$N\varepsilon^3$ = const.

Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

- error estimate:

$$MISE = \left\langle \iiint \rho(\vec{x}) \left| \vec{F}(\vec{x}) - \vec{F}_{true}(\vec{x}) \right|^2 d^3x \right\rangle$$

MISE = mean integrated square error

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

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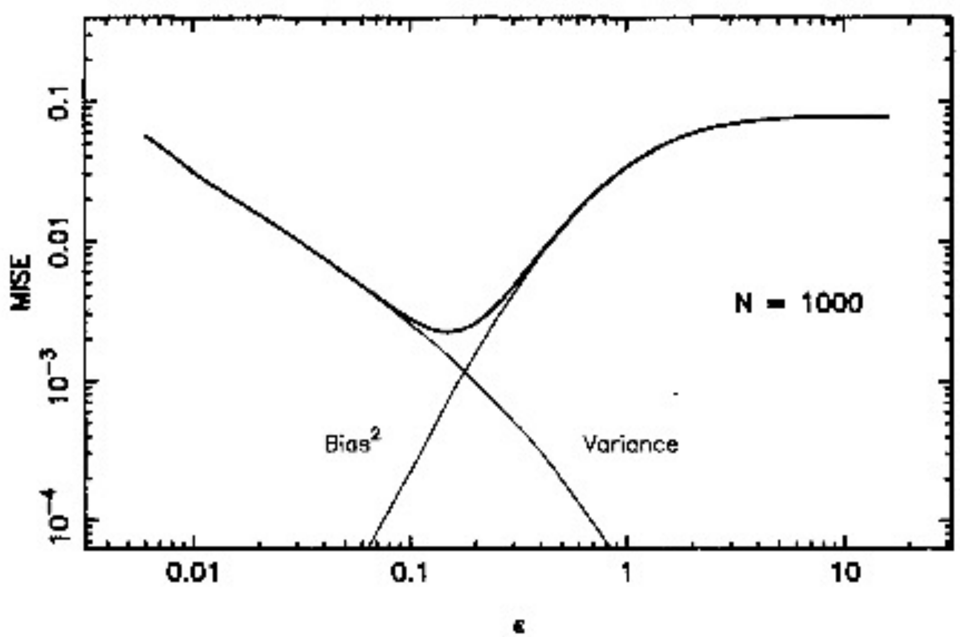


Merritt, ApJ 111, 2462 (1996)

interplay between N and ε: $N\varepsilon^3$=const.

const.=$\left(\frac{B}{30}\right)^3$ for cosmological simulations (where B is the size of the cubical domain in 1D)

MISE = mean integrated square error

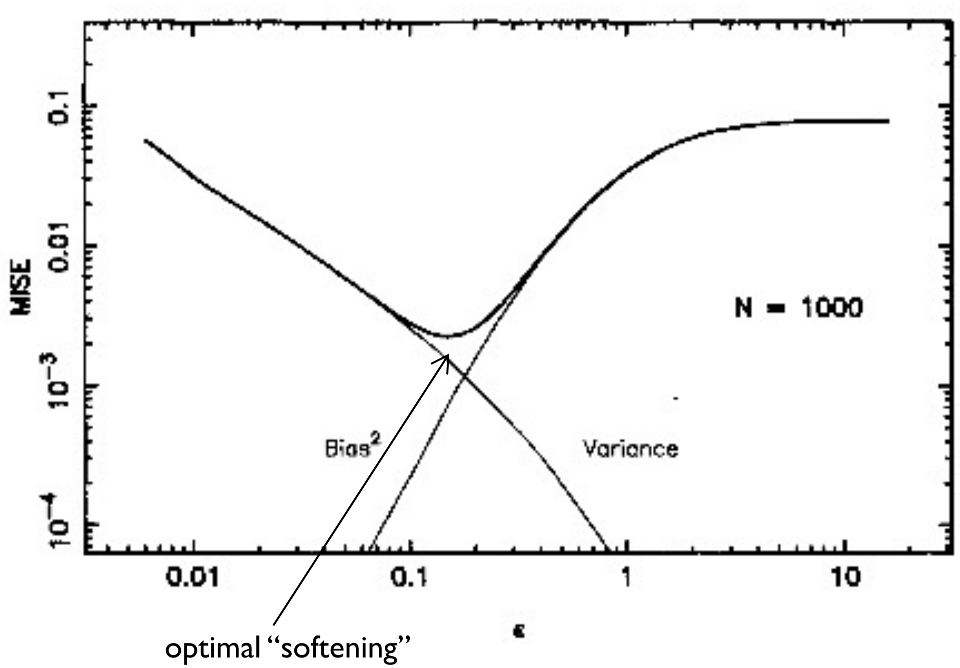
Solving for Gravity

- direct particle-particle summation (PP)

$$\vec{F}_i(\vec{r}_i) = -\sum_{i \neq j} \frac{G m_i m_j}{\left((r_i - r_j)^2 + \varepsilon^2\right)^{3/2}} (\vec{r}_i - \vec{r}_j) \quad \forall i \in N$$

- error estimate:

$$MISE = \left\langle \iiint \rho(\vec{x}) \left| \vec{F}(\vec{x}) - \vec{F}_{true}(\vec{x}) \right|^2 d^3x \right\rangle$$



Merritt, ApJ 111, 2462 (1996)

MISE = mean integrated square error

Solving for Gravity

- direct particle-particle summation (PP)

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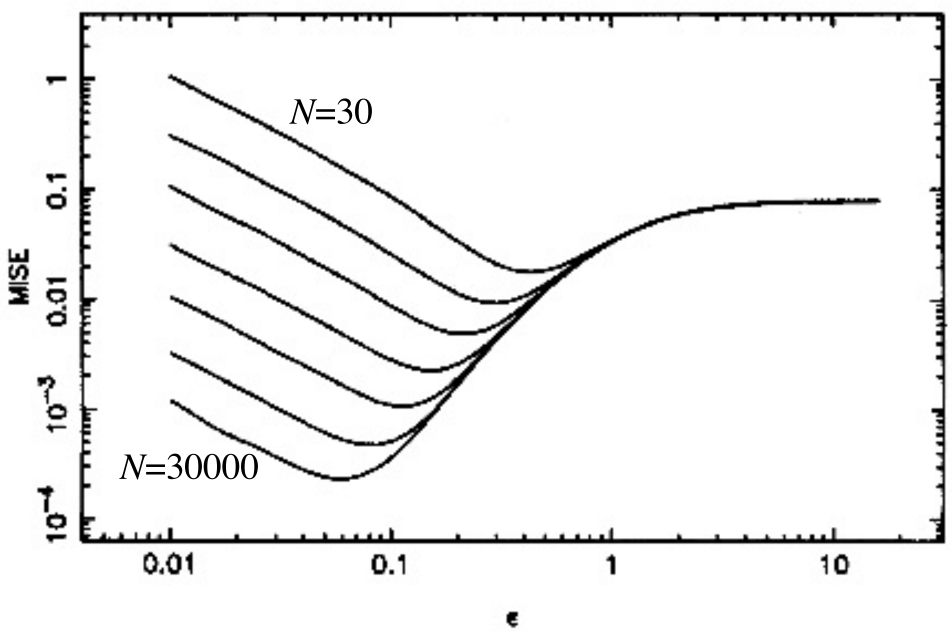

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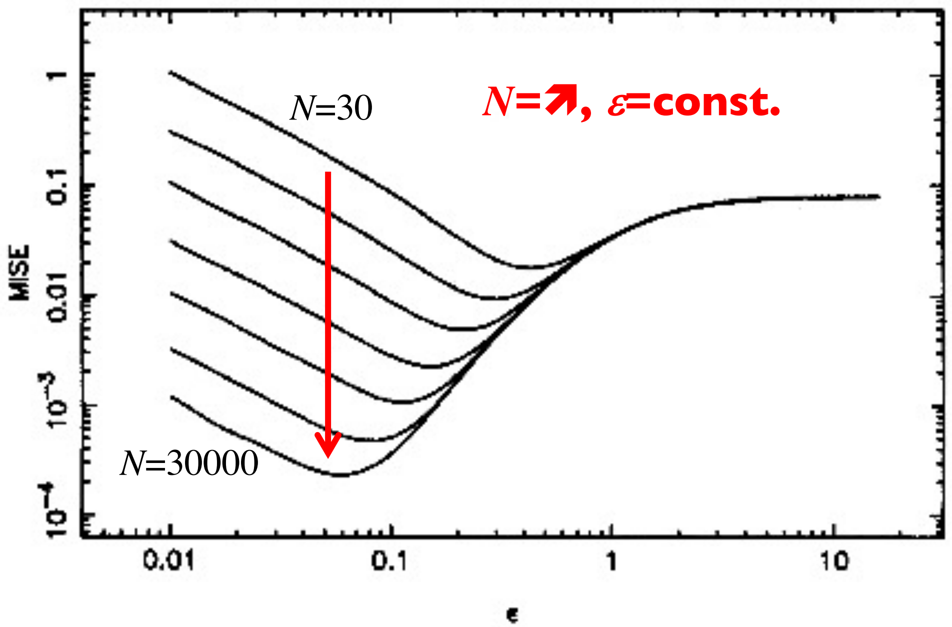

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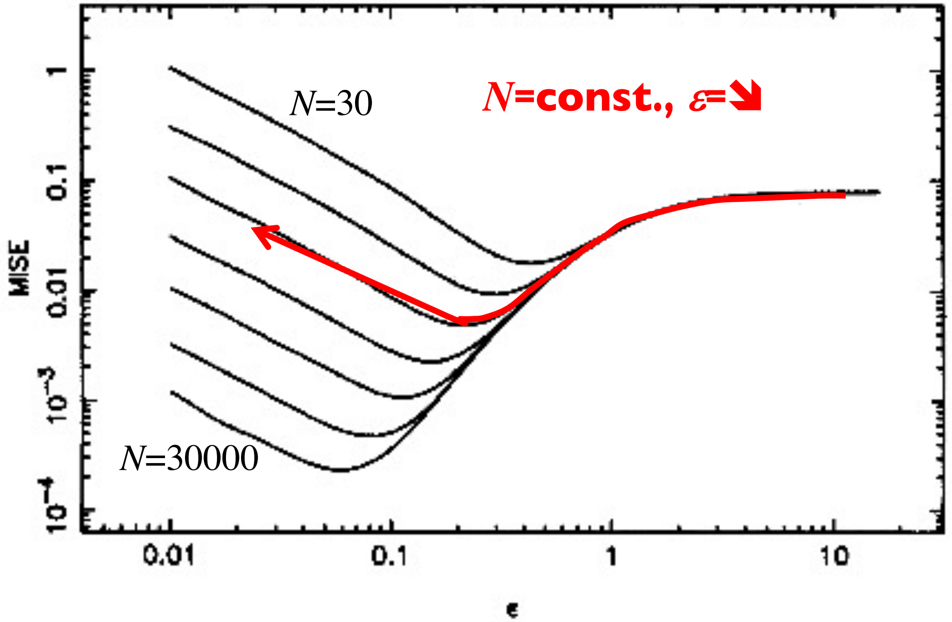

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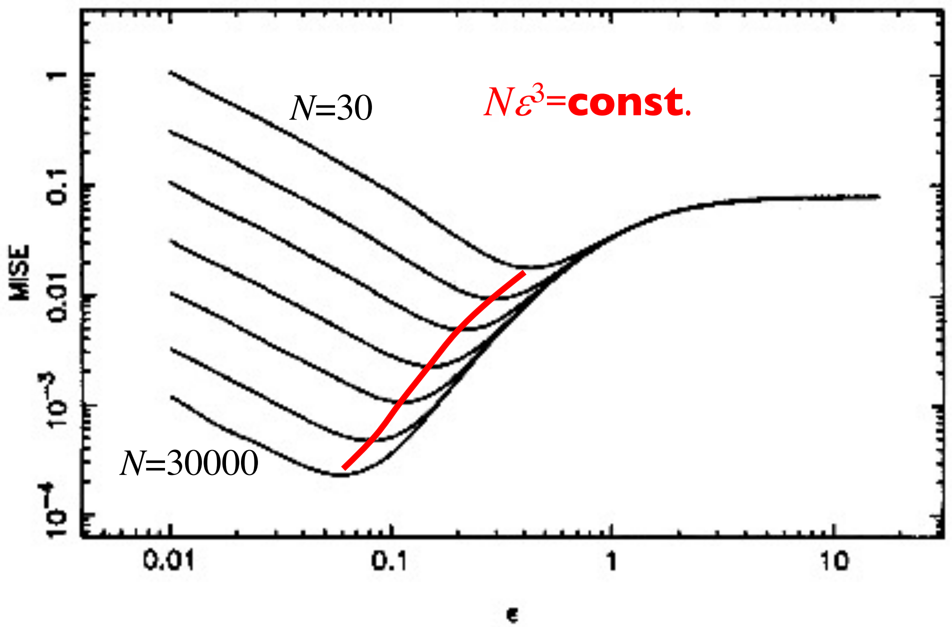


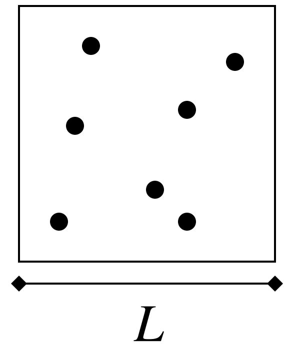
Merritt, ApJ 111, 2462 (1996)

MISE = mean integrated square error

- Poisson's equation
- direct particle-particle summation
- the tree
- force softening
- **periodic boundaries**

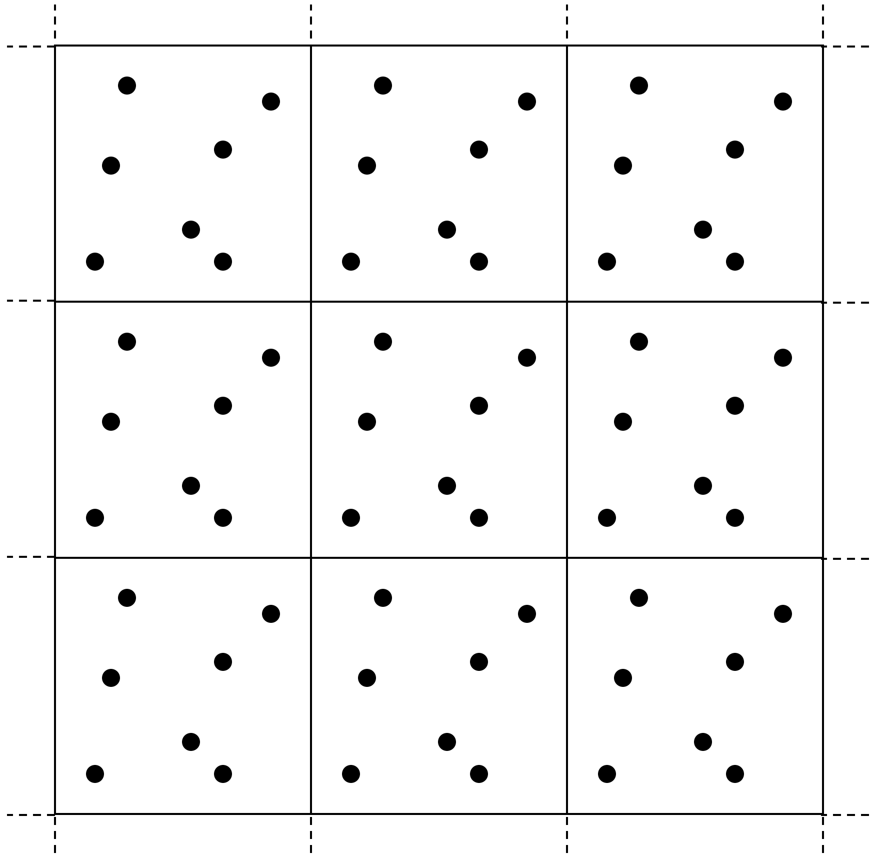
Solving for Gravity

- periodic boundary conditions



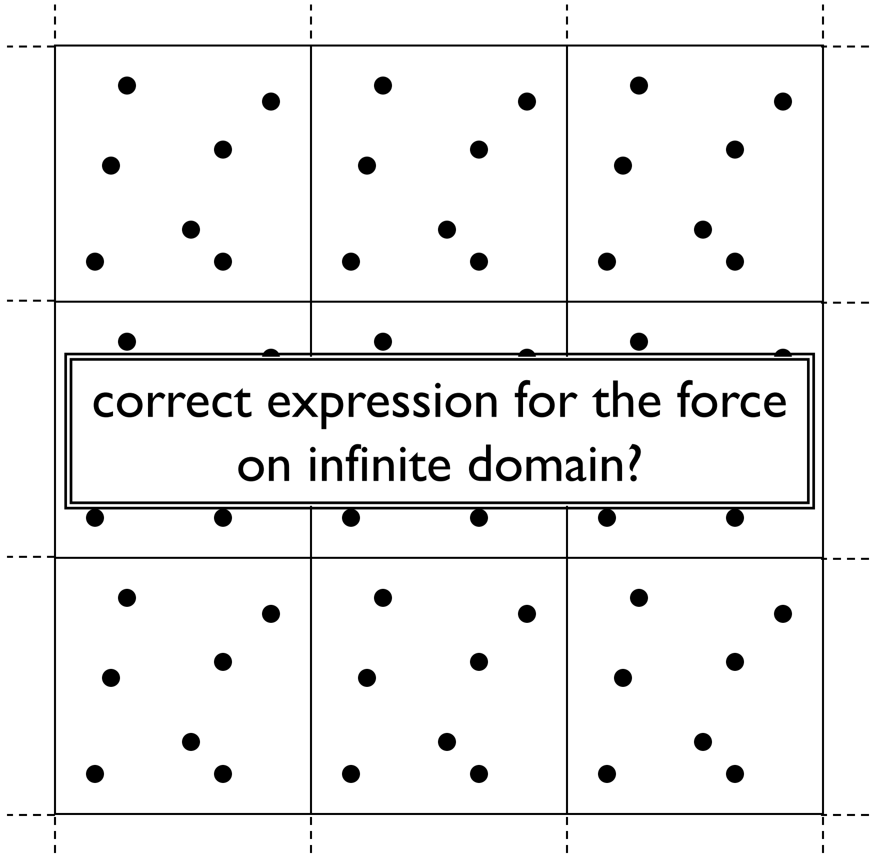
Solving for Gravity

- periodic boundary conditions



Solving for Gravity

- periodic boundary conditions



correct expression for the force on infinite domain?

Solving for Gravity

- periodic boundary conditions

$$\Delta_x \Phi(\vec{x}) = 4\pi G\left(\rho_x(\vec{x}) - \overline{\rho}\ \right)$$

Solving for Gravity

- periodic boundary conditions

$$\Delta_x \Phi(\vec{x}) = 4\pi G\left(\rho_x(\vec{x}) - \overline{\rho}\ \right)$$

we need to subtract the mean background density
in order for the solution to converge!*

*there is no solution to Poisson's equation in infinite space unless the source function averages to zero

Solving for Gravity

- periodic boundary conditions

fluctuates about zero!

$$\Delta_x \Phi(\vec{x}) = 4\pi G\left(\rho_x(\vec{x}) - \overline{\rho}\ \right)$$

Solving for Gravity

- periodic boundary conditions

$$\Delta_x \Phi(\vec{x}) = 4\pi G\left(\rho_x(\vec{x}) - \overline{\rho}\ \right)$$

$\downarrow$ general solution

$$\Phi(\vec{x}) = G \iiint \frac{\rho_x(\vec{x}') - \overline{\rho}_x}{\left|\vec{x} - \vec{x}'\right|} d^3x'$$

Solving for Gravity

- periodic boundary conditions

$$\Delta_x \Phi(\vec{x}) = 4\pi G\left(\rho_x(\vec{x}) - \overline{\rho}\right)$$

↓ general solution

$$\Phi(\vec{x}) = G \iiint \frac{\rho_x(\vec{x}\,') - \overline{\rho}}{\left|\vec{x} - \vec{x}\,'\right|} d^3x'$$

for tree codes: ↓ Poisson’s integral

$$\vec{F}(\vec{x}) = -G \iiint \frac{\rho_x(\vec{x}\,') - \overline{\rho}}{\left|\vec{x} - \vec{x}\,'\right|^3} (\vec{x} - \vec{x}\,') d^3x'$$

Solving for Gravity

- periodic boundary conditions for tree codes

$$\vec{F}(\vec{x}) = -G \iiint \frac{\rho_x(\vec{x}\,') - \overline{\rho}}{\left|\vec{x} - \vec{x}\,'\right|^3} (\vec{x} - \vec{x}\,') d^3 x'$$

Solving for Gravity

- periodic boundary conditions for tree codes

...but in the end it will not contribute to F!

$$\vec{F}(\vec{x}) = -G \iiint \frac{\rho_x(\vec{x}') - \bar{\rho}}{|\vec{x} - \vec{x}'|^3} (\vec{x} - \vec{x}') d^3x'$$

Solving for Gravity

- periodic boundary conditions for tree codes

...but in the end it will not contribute to F!

$$\vec{F}(\vec{x}) = -G \iiint \frac{\rho_x(\vec{x}\,') - \bar{\rho}}{|\vec{x}-\vec{x}\,'|^3} (\vec{x}-\vec{x}\,') d^3x'$$

$$\begin{aligned} i.e.,\ \vec{F}(\vec{0}) &= -G \iiint \frac{(\rho_x(\vec{x}) - \bar{\rho})}{|\vec{x}|^3} \vec{x} d^3x \\ &= -G \iiint \frac{\rho_x(\vec{x})}{|\vec{x}|^3} \vec{x} d^3x + G \iiint \frac{\bar{\rho}}{|\vec{x}|^3} \vec{x} d^3x \\ &= -G \iiint \frac{\rho_x(\vec{x})}{|\vec{x}|^3} \vec{x} d^3x + G\bar{\rho} \iiint \frac{\vec{x}}{|\vec{x}|^3} d^3x \end{aligned}$$

$$\iiint \frac{\vec{x}}{|\vec{x}|^3} d^3x = \iiint_{x,\vartheta,\varphi} \frac{1}{x^3} \begin{pmatrix} x\cos\varphi\sin\vartheta \\ x\sin\varphi\sin\vartheta \\ x\cos\vartheta \end{pmatrix} x^2 \sin\vartheta dx d\vartheta d\varphi = \iiint_{x,\vartheta,\varphi} \begin{pmatrix} \cos\varphi\sin\vartheta \\ \sin\varphi\sin\vartheta \\ \cos\vartheta \end{pmatrix} \sin\vartheta dx d\vartheta d\varphi = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- periodic boundary conditions for tree codes

$$\vec{F}(\vec{x}) = -G \iiint \frac{\rho_x(\vec{x}')}{\left|\vec{x}-\vec{x}'\right|^3}(\vec{x}-\vec{x}')d^3x'$$

- periodic boundary conditions for tree codes

$$\vec{F}(\vec{x}) = -G \iiint \frac{\rho_x(\vec{x}')}{\left|\vec{x}-\vec{x}'\right|^3}(\vec{x}-\vec{x}')d^3x'$$

↓ particle/discrete picture

$$\vec{F}(\vec{x}) = -G \sum_{i=1}^{N} \sum_{\vec{R}} \frac{m_i}{\left|\vec{x}-(\vec{x}_i+\vec{R})\right|^3}\left(\vec{x}-(\vec{x}_i+\vec{R})\right)$$

$$\vec{R} = \vec{n}L$$

Solving for Gravity

- periodic boundary conditions for tree codes

$$\vec{F}(\vec{x}) = -G \iiint \frac{\rho_x(\vec{x}')}{|\vec{x} - \vec{x}'|^3} (\vec{x} - \vec{x}') d^3 x'$$

↓ particle/discrete picture

$$\vec{F}(\vec{x}) = -G \sum_{i=1}^{N} \sum_{\vec{R}} \frac{m_i}{|\vec{x} - (\vec{x}_i + \vec{R})|^3} \left(\vec{x} - (\vec{x}_i + \vec{R})\right)$$

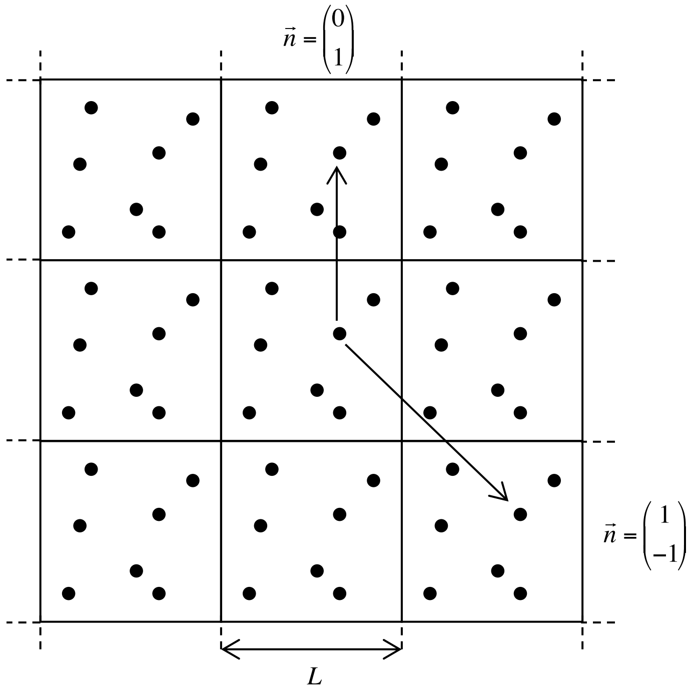
correct expression for the force on infinite domain!

$$\vec{R} = \vec{n} L$$

Solving for Gravity

- periodic boundary conditions for tree codes

$$\vec{F}(\vec{x}) = -G\sum_{i=1}^{N}\sum_{\vec{R}}\frac{m_i}{\left|\vec{x}-(\vec{x}_i+\vec{R})\right|^3}\left(\vec{x}-(\vec{x}_i+\vec{R})\right) \qquad \vec{R}=\vec{n}L$$



Solving for Gravity

- periodic boundary conditions for tree codes

$$\vec{F}(\vec{x}) = -G\sum_{i=1}^{N}\sum_{\vec{R}}\frac{m_i}{\left|\vec{x}-(\vec{x}_i+\vec{R})\right|^3}\left(\vec{x}-(\vec{x}_i+\vec{R})\right) \qquad \vec{R}=\vec{n}L$$

=> slow convergence and hence not feasible…

=> "Ewald summation" instead…

Solving for Gravity

- periodic boundary conditions for tree codes

$$\Delta\Phi(\vec{x}) = 4\pi G\left(\rho(\vec{x}) - \overline{\rho}\right)$$

Solving for Gravity

- periodic boundary conditions for tree codes

$$\Delta\Phi(\vec{x}) = 4\pi G\left(\rho(\vec{x}) - \bar{\rho}\right)$$

discrete particles ↓

$$\rho(\vec{x}) = \sum_{i=1}^{N} m_i \delta_{\text{Dirac}}(\vec{x} - \vec{x}_i)$$

Solving for Gravity

- periodic boundary conditions for tree codes

$$\Delta\Phi(\vec{x}) = 4\pi G\left(\rho(\vec{x}) - \overline{\rho}\right)$$

discrete particles $\downarrow$

$$\rho(\vec{x}) = \sum_{i=1}^{N} m_i \delta_{\text{Dirac}}(\vec{x} - \vec{x}_i)$$

$\downarrow$

"peculiar" density $$\rho_{peculiar}(\vec{x}) = \sum_{i=1}^{N} m_i \delta_{\text{Dirac}}(\vec{x} - \vec{x}_i) - \overline{\rho}$$

Solving for Gravity

- periodic boundary conditions for tree codes

$$\Delta\Phi(\vec{x}) = 4\pi G\left(\rho(\vec{x}) - \overline{\rho}\right)$$

discrete particles ↓

$$\rho(\vec{x}) = \sum_{i=1}^{N} m_i \delta_{\mathrm{Dirac}}(\vec{x} - \vec{x}_i)$$

↓

"peculiar" density $$\rho_{peculiar}(\vec{x}) = \sum_{i=1}^{N} m_i \delta_{\mathrm{Dirac}}(\vec{x} - \vec{x}_i) - \overline{\rho}$$

↓

"peculiar" and periodic density $$\rho_{periodic}(\vec{x}) = \sum_{i=1}^{N} \sum_{\vec{R}} m_i \delta_{\mathrm{Dirac}}(\vec{x} - (\vec{x}_i + \vec{R})) - \overline{\rho}$$

$\vec{R} = \vec{n}L$ ($\vec{n}$ = integer vector)

- periodic boundary conditions for tree codes

$$\rho_{periodic}(\vec{x}) = \sum_{i=1}^{N} \sum_{\vec{R}} m_i \delta_{\mathrm{Dirac}}(\vec{x} - \vec{x}_i - \vec{R}) - \overline{\rho}$$

- periodic boundary conditions for tree codes

$$\rho_{periodic}(\vec{x}) = \sum_{i=1}^{N} \sum_{\vec{R}} \underbrace{m_i \delta_{\text{Dirac}}(\vec{x} - \vec{x}_i - \vec{R})}_{\rho_1} \underbrace{- \overline{\rho}}_{\rho_2}$$

$$\rho_{periodic}(\vec{x}) = \sum_{i=1}^{N} \rho_1(\vec{x},\vec{x}_i) + \rho_2(\vec{x},\vec{x}_i)$$

Solving for Gravity

- periodic boundary conditions for tree codes

$$\rho_{periodic}(\vec{x}) = \sum_{i=1}^{N} \rho_1(\vec{x},\vec{x}_i) + \rho_2(\vec{x},\vec{x}_i)$$

$$\rho_1(\vec{x},\vec{x}_i) = \sum_{\vec{R}} m_i \delta_{\mathrm{Dirac}}(\vec{x} - \vec{x}_i - \vec{R})$$

$$\rho_2(\vec{x},\vec{x}_i) = -\overline{\rho}$$

Solving for Gravity

- periodic boundary conditions for tree codes

$$\rho_{periodic}(\vec{x}) = \sum_{i=1}^{N} \rho_1(\vec{x},\vec{x}_i) + \rho_2(\vec{x},\vec{x}_i)$$

$$\rho_1(\vec{x},\vec{x}_i) = -\sum_{\vec{R}} \frac{1}{\sqrt{\mu^2\pi}} e^{-\frac{(\vec{x}-\vec{x}_i-\vec{R})^2}{\mu^2}} + \sum_{\vec{R}} m_i \delta_{\mathrm{Dirac}}(\vec{x}-\vec{x}_i-\vec{R})$$

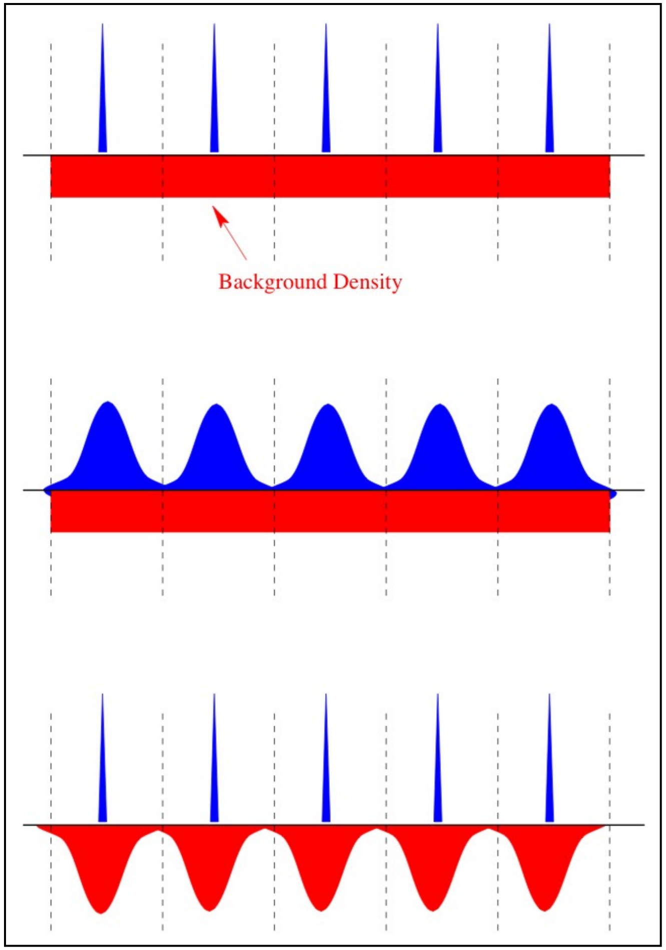
$$\rho_2(\vec{x},\vec{x}_i) = +\sum_{\vec{R}} \frac{1}{\sqrt{\mu^2\pi}} e^{-\frac{(\vec{x}-\vec{x}_i-\vec{R})^2}{\mu^2}} - \overline{\rho}$$

Ewald introduced (Gaussian) "screening charges":

- ρ_1 gives only a short-range contribution
- ρ_2 gives only a long-range contribution

Solving for Gravity

- periodic boundary conditions for tree codes



$$\rho_{periodic}(\vec{x}) = \sum_{i=1}^{N} \rho_1(\vec{x},\vec{x}_i) + \rho_2(\vec{x},\vec{x}_i)$$

potential obtained in…

$$\rho_2(\vec{x},\vec{x}_i) = \sum_{\vec{R}} \frac{1}{\sqrt{\mu^2\pi}} e^{-\frac{(\vec{x}-\vec{x}_i-\vec{R})^2}{\mu^2}} - \bar{\rho}$$

Fourier-space

$$\rho_1(\vec{x},\vec{x}_i) = \sum_{\vec{R}} m_i \delta_{\text{Dirac}}(\vec{x}-\vec{x}_i-\vec{R}) - \frac{1}{\sqrt{\mu^2\pi}} e^{-\frac{(\vec{x}-\vec{x}_i-\vec{R})^2}{\mu^2}}$$

real-space

$\Rightarrow$ exponential convergence and hence feasible!

(singularities are ‘screened’…)

- periodic boundary conditions for tree codes

detailed calculation…

Solving for Gravity

- periodic boundary conditions for tree codes

  - force due to particles in computational box:

$$\vec{F}(\vec{x}) = -G\sum_{i=1}^{N} \frac{m_i}{\left|\vec{x}-\vec{x}_i\right|^3}\left(\vec{x}-\vec{x}_i\right)$$

  - *additional* force due to periodic images:

$$\vec{F}_{Ewald}(\vec{x}) = \frac{\vec{x}}{x^3} - \sum_{\vec{R}} \frac{\vec{x}-\vec{R}}{\left|\vec{x}-\vec{R}\right|^3} \times \left[erfc\left(\frac{\left|\vec{x}-\vec{R}\right|}{\mu}\right) + \frac{2\left|\vec{x}-\vec{R}\right|}{\sqrt{\mu^2\pi}} e^{-\frac{\left|\vec{x}-\vec{R}\right|^2}{\mu^2}} \right] - \frac{2}{L^2}\sum_{\vec{n}\neq 0} \frac{\vec{n}}{n} \sin\left(\frac{2\pi}{L}\vec{n}\cdot\left(\vec{x}-\vec{R}\right)\right) e^{-\frac{(\mu\pi)^2 n^2}{L^2}}$$

- periodic boundary conditions for tree codes

  - in practice:

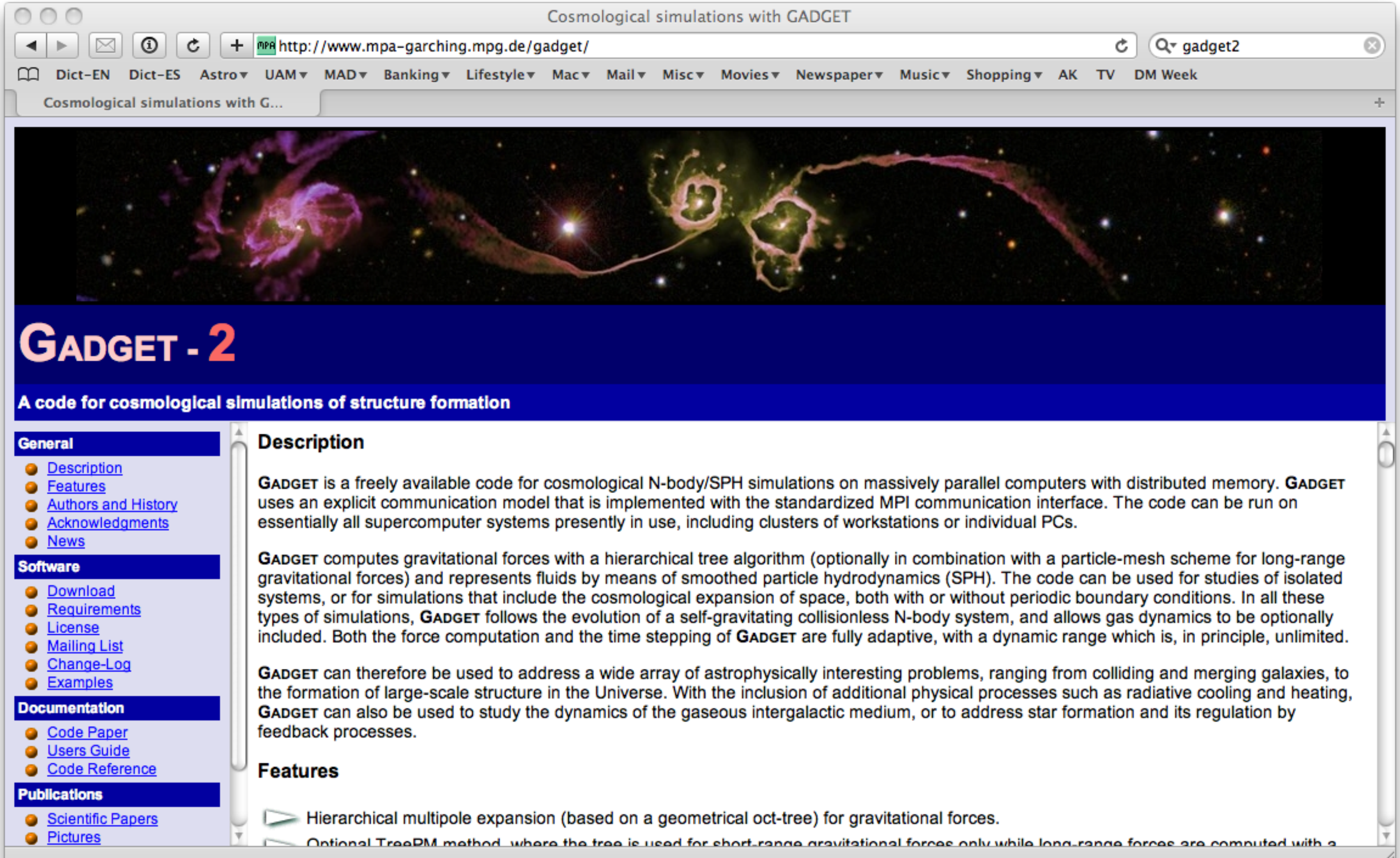
    1. $\mu = L/2, \quad |\vec{x} - \vec{R}| < 3L, \quad n^2 < 10$

    2. tabulate $F_{Ewald}(x)$ on a grid and interpolate…

  - *additional* force due to periodic images:

$$\vec{F}_{Ewald}(\vec{x}) = \frac{\vec{x}}{x^3} - \sum_{\vec{R}} \frac{\vec{x} - \vec{R}}{|\vec{x} - \vec{R}|^3} \times \left[erfc\left(\frac{|\vec{x} - \vec{R}|}{\mu} \right) + \frac{2|\vec{x} - \vec{R}|}{\sqrt{\mu^2 \pi}} e^{-\frac{|\vec{x} - \vec{R}|^2}{\mu^2}} \right] - \frac{2}{L^2} \sum_{\vec{n} \neq 0} \frac{\vec{n}}{n} \sin\left(\frac{2\pi}{L} \vec{n} \cdot (\vec{x} - \vec{R}) \right) e^{-\frac{(\mu\pi)^2 n^2}{L^2}}$$

Solving for Gravity



Cosmological simulations with GADGET

http://www.mpa-garching.mpg.de/gadget/

GADGET - 2

A code for cosmological simulations of structure formation

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Description

GADGET is a freely available code for cosmological N-body/SPH simulations on massively parallel computers with distributed memory. GADGET uses an explicit communication model that is implemented with the standardized MPI communication interface. The code can be run on essentially all supercomputer systems presently in use, including clusters of workstations or individual PCs.

GADGET computes gravitational forces with a hierarchical tree algorithm (optionally in combination with a particle-mesh scheme for long-range gravitational forces) and represents fluids by means of smoothed particle hydrodynamics (SPH). The code can be used for studies of isolated systems, or for simulations that include the cosmological expansion of space, both with or without periodic boundary conditions. In all these types of simulations, GADGET follows the evolution of a self-gravitating collisionless N-body system, and allows gas dynamics to be optionally included. Both the force computation and the time stepping of GADGET are fully adaptive, with a dynamic range which is, in principle, unlimited.

GADGET can therefore be used to address a wide array of astrophysically interesting problems, ranging from colliding and merging galaxies, to the formation of large-scale structure in the Universe. With the inclusion of additional physical processes such as radiative cooling and heating, GADGET can also be used to study the dynamics of the gaseous intergalactic medium, or to address star formation and its regulation by feedback processes.

Features

- Hierarchical multipole expansion (based on a geometrical oct-tree) for gravitational forces.
- Optional TreePM method, where the tree is used for short-range gravitational forces only while long-range forces are computed with a